Abstract Hardy inequalities and monotone functions

Workshop in Analysis and Probability

Alejandro Santacruz Hidalgo
Joint work with Gord Sinnamon

University of Western Ontario

July 17, 2024

Previous results

Hardy integral inequality (1925)

For $f \geq 0$, $p > 1$, $\left(\int_0^\infty \left(\frac{1}{x} \int_0^x f(s) ds \right)^p dx \right)^{\frac{1}{p}} \leq \frac{p}{p-1} \left(\frac{1}{x} \int_0^\infty f^p(s) ds \right)^{1/p}$.

Previous results

Hardy integral inequality (1925)

For $f \geq 0$, $p > 1$, $\left(\int_0^\infty \left(\frac{1}{x} \int_0^x f(s) ds \right)^p dx \right)^{\frac{1}{p}} \leq \frac{p}{p-1} \left(\frac{1}{x} \int_0^\infty f^p(s) ds \right)^{1/p}$.

One consequence: If $g(x) = \int_0^x f(t)\, dt$, then $\| \frac{g(\cdot)}{(\cdot)} \|_{L^p} \leq \frac{p}{p-1} \|g'\|_{L^p}$. Thus if $g \in W_0^1([0, \infty)$ then $x \mapsto g(x)/x \in L^p$.

Previous results

Hardy integral inequality (1925)

For $f \geq 0$, $p > 1$, $\left(\int_0^\infty \left(\frac{1}{x} \int_0^x f(s)ds \right)^p dx \right)^{\frac{1}{p}} \leq \frac{p}{p-1} \left(\frac{1}{x} \int_0^\infty f^p(s)ds \right)^{1/p}$.

One consequence: If $g(x) = \int_0^x f(t)\,dt$, then $\| \frac{g(\cdot)}{(\cdot)} \|_{L^p} \leq \frac{p}{p-1} \|g'\|_{L^p}$. Thus if $g \in W_0^1([0,\infty)$ then $x \mapsto g(x)/x \in L^p$.

Weighted Hardy inequality

Fix $p \in [1,\infty)$, $q \in (0,\infty)$, characterize the best constant C:

$$\left(\int_0^\infty \left(\int_0^x f(s)ds \right)^q v(x)\,dx \right)^{\frac{1}{q}} \leq C \left(\frac{1}{x} \int_0^\infty f^p(s)u(s)\,ds \right)^{1/p}, \quad \forall f \geq 0.$$

Previous results

Hardy integral inequality (1925)

For $f \geq 0$, $p > 1$, $\left(\int_0^\infty \left(\frac{1}{x} \int_0^x f(s)ds \right)^p dx \right)^{\frac{1}{p}} \leq \frac{p}{p-1} \left(\frac{1}{x} \int_0^\infty f^p(s)ds \right)^{1/p}$.

One consequence: If $g(x) = \int_0^x f(t)\,dt$, then $\| \frac{g(\cdot)}{(\cdot)} \|_{L^p} \leq \frac{p}{p-1} \|g'\|_{L^p}$. Thus if $g \in W_0^1([0,\infty)$ then $x \mapsto g(x)/x \in L^p$.

Weighted Hardy inequality

Fix $p \in [1,\infty)$, $q \in (0,\infty)$, characterize the best constant C:

$$\left(\int_0^\infty \left(\int_0^x f(s)ds \right)^q v(x)\,dx \right)^{\frac{1}{q}} \leq C \left(\frac{1}{x} \int_0^\infty f^p(s)u(s)\,ds \right)^{1/p}, \quad \forall f \geq 0.$$

Example: If $q < p$ and $p > 1$ then

$$C \approx \left(\int_0^\infty \left(\int_0^x u^{1-p'}(s)ds \right)^{r/p'} \left(\int_x^\infty v(s)ds \right)^{r/p} v(x)\,dx \right)^{\frac{1}{r}}.$$

More Hardy operators:

(Measures): $Hf(x) = \int_{[0,x]} f\,d\nu$, (Dual): $Hf(x) = \int_{[x,\infty)} f\,d\nu$.

(Sequences): $Hf(n) = \int_{[0,n]} f\,d\# = \sum_{k=0}^{n} f(n)$.

(Higher dimensions): $Hf(x) = \int_{B(|x|)} f(s)\,ds$, where

$B(|x|) = \{y \in \mathbb{R}^n : |y| \leq |x|\}$.

Abstract Hardy inequalities

More Hardy operators:

(Measures): $Hf(x) = \int_{[0,x]} f\, d\nu$, (Dual): $Hf(x) = \int_{[x,\infty)} f\, d\nu$.

(Sequences): $Hf(n) = \int_{[0,n]} f\, d\# = \sum_{k=0}^{n} f(n)$.

(Higher dimensions): $Hf(x) = \int_{B(|x|)} f(s)\, ds$, where

$B(|x|) = \{y \in \mathbb{R}^n : |y| \leq |x|\}$.

Definition (Abstract Hardy inequality) [Sinnamon 2022] Fix $p \geq 1$ and $q > 0$. Let $(Y, \mathcal{T}, \tau)$ and (U, Σ, μ) be σ-finite measure spaces. A map $B : Y \to \Sigma$ such that the range is **totally ordered** and $\mu(B(y)) < \infty$. Does there exist a constant C such that

$$\left(\int_Y \left(\int_{B(y)} f\, d\mu \right)^q d\tau(y) \right)^{1/q} \leq C \left(\int_U f^p\, d\eta \right)^{1/p}$$

holds for all $f \in L^+_\mu$?

Abstract Hardy inequalities

More Hardy operators:

(Measures): $Hf(x) = \int_{[0,x]} f\,d\nu$, (Dual): $Hf(x) = \int_{[x,\infty)} f\,d\nu$.

(Sequences): $Hf(n) = \int_{[0,n]} f\,d\# = \sum_{k=0}^{n} f(n)$.

(Higher dimensions): $Hf(x) = \int_{B(|x|)} f(s)\,ds$, where

$B(|x|) = \{y \in \mathbb{R}^n : |y| \leq |x|\}$.

Definition (Abstract Hardy inequality) [Sinnamon 2022] Fix $p \geq 1$ and $q > 0$. Let $(Y, \mathcal{T}, \tau)$ and (U, Σ, μ) be σ-finite measure spaces. A map $B : Y \to \Sigma$ such that the range is **totally ordered** and $\mu(B(y)) < \infty$. Does there exist a constant C such that

$$\left(\int_Y \left(\int_{B(y)} f\,d\mu \right)^q d\tau(y) \right)^{1/q} \leq C \left(\int_U f^p\,d\eta \right)^{1/p}$$

holds for all $f \in L^+_\mu$? Is $f \mapsto Hf$ where $Hf(y) = \int_{B(y)} f\,d\mu$ bounded from $L^p_\eta \to L^q_\tau$?

Monotone functions

Let (U, Σ, μ) be a $\sigma-$finite measure space. **We assume no order**.

Monotone functions

Let (U, Σ, μ) be a $\sigma-$finite measure space. **We assume no order**.

Definition: An ordered core is a totally ordered subset $\mathcal{A}$ of Σ (If $A, B \in \mathcal{A}$ then $A \subseteq B$ or $B \subseteq A$), containing $\emptyset$ and satisfying

Monotone functions

Let (U, Σ, μ) be a $\sigma-$finite measure space. **We assume no order**.

Definition: An ordered core is a totally ordered subset $\mathcal{A}$ of Σ (If $A, B \in \mathcal{A}$ then $A \subseteq B$ or $B \subseteq A$), containing $\emptyset$ and satisfying $\mu(E) < \infty$ for all $E \in \mathcal{A}$

Monotone functions

Let (U, Σ, μ) be a $\sigma-$finite measure space. **We assume no order**.

Definition: An ordered core is a totally ordered subset $\mathcal{A}$ of Σ (If $A, B \in \mathcal{A}$ then $A \subseteq B$ or $B \subseteq A$), containing $\emptyset$ and satisfying $\mu(E) < \infty$ for all $E \in \mathcal{A}$ and $\cup\mathcal{A} = \cup_n E_n$ for some $\{E_n\} \in \mathcal{A}$.

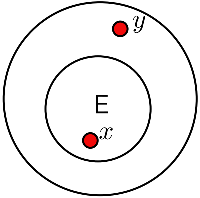
Monotone functions

Let (U, Σ, μ) be a $\sigma-$finite measure space. **We assume no order**.

Definition: An ordered core is a totally ordered subset $\mathcal{A}$ of Σ (If $A, B \in \mathcal{A}$ then $A \subseteq B$ or $B \subseteq A$), containing $\emptyset$ and satisfying $\mu(E) < \infty$ for all $E \in \mathcal{A}$ and $\cup\mathcal{A} = \cup_n E_n$ for some $\{E_n\} \in \mathcal{A}$.

Definition: A positive $\sigma(\mathcal{A})$-measurable function is core-decreasing if

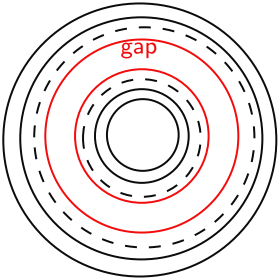
1.



if $x \in E$ and $y \notin E$, $f(x) \geq f(y)$

$\forall E \in \mathcal{A}$

2. f is constant in each gap



Examples

For an *abstract Hardy inequality*

$$\left(\int_Y \left(\int_{B(y)} f \, d\mu \right)^q d\tau(y) \right)^{1/q} \leq C \left(\int_U f^p \, d\eta \right)^{1/p},$$ the collection $\{B(y) : y \in Y\}$ induces an ordered core.

More examples

1. Let $U = [0, \infty)$, μ the Lebesgue measure. The core $\mathcal{A} = \{[0, x] : x > 0\}$. Note $\sigma(\mathcal{A})$ is the Borel σ-algebra. f is core decreasing if for each $E = [0, t]$, $x \in [0, t]$ and $y \notin [0, t]$ then $f(y) \leq f(x)$. (Hardy $B(x) = [0, x]$)

Examples

For an *abstract Hardy inequality*
$$\left(\int_Y \left(\int_{B(y)} f\,d\mu\right)^q d\tau(y)\right)^{1/q} \leq C\left(\int_U f^p\,d\eta\right)^{1/p}$$, the collection $\{B(y) : y \in Y\}$ induces an ordered core.

More examples

1. Let $U = [0, \infty)$, μ the Lebesgue measure. The core $\mathcal{A} = \{[0, x] : x > 0\}$. Note $\sigma(\mathcal{A})$ is the Borel σ-algebra. f is core decreasing if for each $E = [0, t]$, $x \in [0, t]$ and $y \notin [0, t]$ then $f(y) \leq f(x)$. (Hardy $B(x) = [0, x]$)

2. Let $U = [0, \infty)$, μ the Lebesgue measure. The core $\mathcal{A} = \{[0, n] : n \in \mathbb{N}^+\}$. Now $\sigma(\mathcal{A})$ is strictly smaller than the Borel σ-algebra. f is core decreasing if $x < y$ implies $f(y) \leq f(x)$ and f is constant on each set $(n-1, n]$. (Hardy $B(n) = [0, n]$)

Examples

For an *abstract Hardy inequality* $\left(\int_Y \left(\int_{B(y)} f\, d\mu\right)^q d\tau(y)\right)^{1/q} \leq C\left(\int_U f^p\, d\eta\right)^{1/p}$, the collection $\{B(y) : y \in Y\}$ induces an ordered core.

More examples

1. Let $U = [0, \infty)$, μ the Lebesgue measure. The core $\mathcal{A} = \{[0, x] : x > 0\}$. Note $\sigma(\mathcal{A})$ is the Borel σ-algebra. f is core decreasing if for each $E = [0, t]$, $x \in [0, t]$ and $y \notin [0, t]$ then $f(y) \leq f(x)$. (Hardy $B(x) = [0, x]$)

2. Let $U = [0, \infty)$, μ the Lebesgue measure. The core $\mathcal{A} = \{[0, n] : n \in \mathbb{N}^+\}$. Now $\sigma(\mathcal{A})$ is strictly smaller than the Borel σ-algebra. f is core decreasing if $x < y$ implies $f(y) \leq f(x)$ and f is constant on each set $(n-1, n]$. (Hardy $B(n) = [0, n]$)

3. Let $U = \mathbb{R}^d$, μ satisfying $\mu(B[0; r]) < \infty$ for each $r > 0$. The core $\mathcal{A} = \{B[0; r] : r > 0\}$. Again $\sigma(\mathcal{A})$ is strictly smaller than the Borel σ-algebra. f is core decreasing if it is radially decreasing. (Hardy $B(r) = B[0; r]$)

Maps between ordered cores

Definition: A map $r : \mathcal{A} \to \Sigma$ is a core morphism if: 1. $\tau(r(\emptyset)) = 0$.
2. There exists $c > 0$: $\tau(r(B) \setminus r(A)) \leq c\mu(B \setminus A)$ for all $A, B \in \mathcal{A}$.
3. If $A \subseteq B$ then $r(A) \subseteq r(B)$ for all $A, B \in \mathcal{A}$.

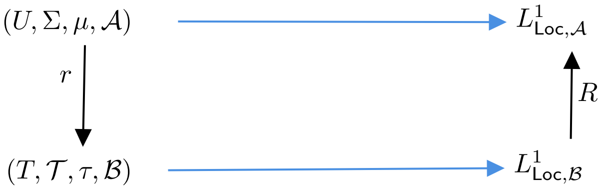
Maps between ordered cores

Definition: A map $r : \mathcal{A} \to \Sigma$ is a core morphism if: 1. $\tau(r(\emptyset)) = 0$.
2. There exists $c > 0$: $\tau(r(B) \setminus r(A)) \leq c\mu(B \setminus A)$ for all $A, B \in \mathcal{A}$.
3. If $A \subseteq B$ then $r(A) \subseteq r(B)$ for all $A, B \in \mathcal{A}$.

Theorem [Sinnamon, S.- 2024]

Given a core morphism $r : \mathcal{A} \to \Sigma$, there exists a linear map $R : L^1_{\text{Loc},\mathcal{B}} \to L^1_{\text{Loc},\mathcal{A}}$ such that

$$\int_A Rf \, d\mu = \int_{r(A)} f \, d\tau, \quad \forall A \in \mathcal{A}.$$



Induced measure on the half line

Every ordered core induces a Borel measure λ on $[0,\infty)$ and maps R, Q:

$$f \in L^1_{\mathrm{Loc},\mathcal{A}}$$

R

Q

$$g \in L^1_{loc}([0,\infty),\lambda)$$

Properties:

- $RQg = g$
- $QRf \neq f$ in general
- Q and R preserve monotonicity.
- $QRf = f$ if f is core-decreasing

Abstract Hardy inequality: The case $p \neq 1$

Recall, characterize the best C such that

$$\left(\int_Y \left(\int_{B(y)} f \, d\mu \right)^q d\tau(y) \right)^{1/q} \leq C \left(\int_U f^p \, d\eta \right)^{1/p} \text{ holds for all } f > 0.$$

Abstract Hardy inequality: The case $p \neq 1$

Recall, characterize the best C such that

$$\left(\int_Y \left(\int_{B(y)} f \, d\mu \right)^q d\tau(y) \right)^{1/q} \leq C \left(\int_U f^p \, d\eta \right)^{1/p}$$ holds for all $f > 0$.

Step 1: Use the Lebesgue decomposition and Radon-Nikodym to have the equivalent inequality $\left(\int_Y \left(\int_{B(y)} f \, d\mu \right)^q d\tau(y) \right)^{1/q} \leq C \left(\int_U f^p u \, d\mu \right)^{1/p}$.

Abstract Hardy inequality: The case $p \neq 1$

Recall, characterize the best C such that

$$\left(\int_Y \left(\int_{B(y)} f \, d\mu \right)^q d\tau(y) \right)^{1/q} \leq C \left(\int_U f^p \, d\eta \right)^{1/p} \text{ holds for all } f > 0.$$

Step 1: Use the Lebesgue decomposition and Radon-Nikodym to have the equivalent inequality $\left(\int_Y \left(\int_{B(y)} f \, d\mu \right)^q d\tau(y) \right)^{1/q} \leq C \left(\int_U f^p u \, d\mu \right)^{1/p}$.

Step 2: Substitute $f = gu^{1-p'}$ where $\frac{1}{p} + \frac{1}{p'} = 1$.

$$\left(\int_Y \left(\int_{B(y)} g u^{1-p'} \, d\mu \right)^q d\tau(y) \right)^{1/q} \leq C \left(\int_U g^p u^{1-p'} \, d\mu \right)^{1/p}.$$

Abstract Hardy inequality: The case $p \neq 1$

Recall, characterize the best C such that

$$\left(\int_Y \left(\int_{B(y)} f \, d\mu \right)^q d\tau(y) \right)^{1/q} \leq C \left(\int_U f^p \, d\eta \right)^{1/p}$$ holds for all $f > 0$.

Step 1: Use the Lebesgue decomposition and Radon-Nikodym to have the equivalent inequality $\left(\int_Y \left(\int_{B(y)} f \, d\mu \right)^q d\tau(y) \right)^{1/q} \leq C \left(\int_U f^p u \, d\mu \right)^{1/p}$.

Step 2: Substitute $f = g u^{1-p'}$ where $\frac{1}{p} + \frac{1}{p'} = 1$.

$$\left(\int_Y \left(\int_{B(y)} g u^{1-p'} \, d\mu \right)^q d\tau(y) \right)^{1/q} \leq C \left(\int_U g^p u^{1-p'} \, d\mu \right)^{1/p}.$$

Step 3: Use the maps R, Q to find an equivalent Hardy inequality on the half line.

[Sinnamon 2022]

There exists a nonincreasing function b with an equivalent inequality

$$\left(\int_0^\infty \left(\int_0^{b(y)} g(s) \, ds \right)^q dy \right)^{1/q} \leq C \left(\int_0^\infty g^p(s) \, ds \right)^{1/p}, \quad \forall g \in L^+.$$

Abstract Hardy inequality: The case $p = 1$

Characterize the best C such that $\left(\int_Y \left(\int_{B(y)} f \, d\mu \right)^q d\tau(y) \right)^{1/q} \leq C \int_U f \, d\eta$

holds for all $f > 0$.

Abstract Hardy inequality: The case $p = 1$

Characterize the best C such that $\left(\int_Y \left(\int_{B(y)} f \, d\mu \right)^q d\tau(y) \right)^{1/q} \leq C \int_U f \, d\eta$

holds for all $f > 0$.

Step 1: Use the Lebesgue decomposition and Radon-Nikodym to have the equivalent inequality $\left(\int_Y \left(\int_{B(y)} f \, d\mu \right)^q d\tau \right)^{1/q} \leq C \left(\int_U f^p u \, d\mu \right)^{1/p}$.

Abstract Hardy inequality: The case $p = 1$

Characterize the best C such that $\left(\int_Y \left(\int_{B(y)} f \, d\mu \right)^q d\tau(y) \right)^{1/q} \leq C \int_U f \, d\eta$ holds for all $f > 0$.

Step 1: Use the Lebesgue decomposition and Radon-Nikodym to have the equivalent inequality $\left(\int_Y \left(\int_{B(y)} f \, d\mu \right)^q d\tau \right)^{1/q} \leq C \left(\int_U f^p u \, d\mu \right)^{1/p}$.

Step 2 (?): Substitute $f = gu^{1-p'}$ where $\frac{1}{p} + \frac{1}{p'} = 1$. But $p' = \infty$.

Abstract Hardy inequality: The case $p = 1$

Characterize the best C such that $\left(\int_Y \left(\int_{B(y)} f \, d\mu \right)^q d\tau(y) \right)^{1/q} \leq C \int_U f \, d\eta$ holds for all $f > 0$.

Step 1: Use the Lebesgue decomposition and Radon-Nikodym to have the equivalent inequality $\left(\int_Y \left(\int_{B(y)} f \, d\mu \right)^q d\tau \right)^{1/q} \leq C \left(\int_U f^p u \, d\mu \right)^{1/p}$.

Step 2 (?): Substitute $f = gu^{1-p'}$ where $\frac{1}{p} + \frac{1}{p'} = 1$. But $p' = \infty$.
We need a different approach, we will show that the monotonicity of the function $y \to \int_{B(y)} f \, d\mu$ (**core increasing**) lets us replace u for a **core decreasing** function $\underline{u}$.

Abstract Hardy inequality: The case $p = 1$

Characterize the best C such that $\left(\int_Y \left(\int_{B(y)} f\,d\mu\right)^q d\tau(y)\right)^{1/q} \le C \int_U f\,d\eta$

holds for all $f > 0$.

Step 1: Use the Lebesgue decomposition and Radon-Nikodym to have the equivalent inequality $\left(\int_Y \left(\int_{B(y)} f\,d\mu\right)^q d\tau\right)^{1/q} \le C\left(\int_U f^p u\,d\mu\right)^{1/p}$.

Step 2 (?): Substitute $f = gu^{1-p'}$ where $\frac{1}{p} + \frac{1}{p'} = 1$. But $p' = \infty$.

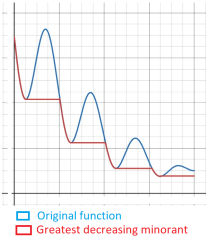
We need a different approach, we will show that the monotonicity of the function $y \to \int_{B(y)} f\,d\mu$ (**core increasing**) lets us replace u for a **core decreasing** function $\underline{u}$.

Step 2: Replace u by $\underline{u}$, $\left(\int_Y \left(\int_{B(y)} f\,d\mu\right)^q d\tau(y)\right)^{1/q} \le C \int_U f\underline{u}\,d\mu$.

Step 3: Use the maps R, Q to find an equivalent Hardy inequality on the half line.

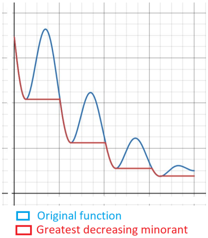
Greatest core decreasing minorant

Definition (pointwise): For $u \geq 0$, the function $\underline{u}$ is the greatest core decreasing minorant if $\underline{u}(s) \leq u(s)$ for almost all $s \in U$ and if $h(s) \leq u(s)$ and h is core decreasing, then $h \leq \underline{u}$.



Greatest core decreasing minorant

Definition (pointwise): For $u \geq 0$, the function $\underline{u}$ is the greatest core decreasing minorant if $\underline{u}(s) \leq u(s)$ for almost all $s \in U$ and if $h(s) \leq u(s)$ and h is core decreasing, then $h \leq \underline{u}$.



Theorem [S.-2024]

If $u \geq 0$,
$\int_U \underline{u} g \, d\mu = \inf \left\{ \int_U |u| \, h \, d\mu : h \in L^+(\Sigma) \text{ and } \int_E h \, d\mu \geq \int_E g \, d\mu \text{ for all } E \in \mathcal{A} \right\}.$

Transferring monotonicity

Lemma [S.-2024]

The best constant C in the inequality
$$\left(\int_Y \left(\int_{B(y)} f\,d\mu\right)^q d\tau(y)\right)^{1/q} \leq C \int_U f u\,d\mu$$
is the same as the best constant C in the inequality
$$\left(\int_Y \left(\int_{B(y)} f\,d\mu\right)^q d\tau\right)^{1/q} \leq C \int_U f \underline{u}\,d\mu$$

Transferring monotonicity

Lemma [S.-2024]

The best constant C in the inequality $\left(\int_Y \left(\int_{B(y)} f \, d\mu \right)^q d\tau(y) \right)^{1/q} \leq C \int_U f u \, d\mu$ is the same as the best constant C in the inequality $\left(\int_Y \left(\int_{B(y)} f \, d\mu \right)^q d\tau \right)^{1/q} \leq C \int_U f \underline{u} \, d\mu$

Important step: In general $R(fu) \neq R(f)R(u)$ but $R(f\underline{u}) = R(f)R(\underline{u})$ holds.

Transferring monotonicity

Lemma [S.-2024]

The best constant C in the inequality $\left(\int_Y \left(\int_{B(y)} f\,d\mu\right)^q d\tau(y)\right)^{1/q} \leq C\int_U fu\,d\mu$ is the same as the best constant C in the inequality $\left(\int_Y \left(\int_{B(y)} f\,d\mu\right)^q d\tau\right)^{1/q} \leq C\int_U f\underline{u}\,d\mu$

Important step: In general $R(fu) \neq R(f)R(u)$ but $R(f\underline{u}) = R(f)R(\underline{u})$ holds.

Lemma [S.-2024]

The best constant C in the inequality $\left(\int_Y \left(\int_{B(y)} f\,d\mu\right)^q d\tau\right)^{1/q} \leq C\int_U fu\,d\mu$ is the same as the best constant C in the inequality

$$\left(\int_{[0,\infty)} \left(\int_{[0,y]} Rf\,d\lambda\right)^q d\theta(y)\right)^{1/q} \leq C\int_{[0,\infty)} Rf\,R(\underline{u})\,d\lambda$$

Main result, case $p = 1$.

Theorem [S.2024]

For $(Y, \mathcal{T}, \tau), (U, \Sigma, \mu), (U, \Sigma, \nu)$ σ-finite and a core map $B : Y \to \Sigma$. Let $\eta = \eta_a + \eta_s$, where $d\eta_a = u d\mu$ and $\eta_s \perp \mu$. Then the best constant C in the inequality

$$\left(\int_Y \left(\int_{B(y)} f \, d\mu \right)^q d\tau(y) \right)^{1/q} \leq C \int_U f \, d\eta, \qquad (1)$$

satisfies

$$C \approx \left[\int_Y \left(\int_{\mu(B(z)) \leq \mu(B(y))} R\left(\frac{1}{\underline{u}} \right) \circ \mu \circ B(y) \, d\tau(y) \right)^{\frac{q}{1-q}} d\tau(z) \right]^{\frac{1-q}{q}}, \text{ for } q \in (0, 1),$$

and

$$C = \sup_{s \in U} \left(\frac{1}{\underline{u}}(s) \right) \tau\left(\{ y \in Y : s \in B(y) \} \right)^{1/q}, \text{ for } q \in [1, \infty).$$

Where $\underline{u}$ is the greatest core decreasing minorant.

Application: Hardy inequalities over metric measure spaces.

Previous results [Ruzhansky, Verma, Shriwastawa, Tiwari 2019-2024]

- $\mathbb{X}$ is a metric measure space, with a borel measure dx admitting a polar decomposition (λ), that is $\int\limits_{\mathbb{X}} f dx = \int_0^\infty \int_{\partial B[a;r]} f(r,\omega)\lambda(r,\omega)\, d\omega_r\, dr$.

Application: Hardy inequalities over metric measure spaces.

Previous results [Ruzhansky, Verma, Shriwastawa, Tiwari 2019-2024]
- $\mathbb{X}$ is a metric measure space, with a borel measure dx admitting a polar decomposition (λ), that is $\int_{\mathbb{X}} f dx = \int_0^{\infty} \int_{\partial B[a;r]} f(r,\omega)\lambda(r,\omega)\, d\omega_r\, dr$.
- Examples of spaces with polar decompositions: Euclidean spaces, Homogeneous Lie groups, Cartan-Hadamard manifolds (complete, simply connected, Riemannian manifold with non-positive sectional curvature).

Application: Hardy inequalities over metric measure spaces.

Previous results [Ruzhansky, Verma, Shriwastawa, Tiwari 2019-2024]

- $\mathbb{X}$ is a metric measure space, with a borel measure dx admitting a polar decomposition (λ), that is $\int\limits_{\mathbb{X}} f dx = \int_0^\infty \int_{\partial B[a;r]} f(r,\omega)\lambda(r,\omega)\, d\omega_r\, dr$.
- Examples of spaces with polar decompositions: Euclidean spaces, Homogeneous Lie groups, Cartan-Hadamard manifolds (complete, simply connected, Riemannian manifold with non-positive sectional curvature).
- Fix $u, v > 0$, then the inequality

$$\left(\int\limits_{\mathbb{X}} \left(\int_{B[a,x]} |f(y)|\, dy \right)^q u(x)\, dx \right)^{1/q} \le C \left(\int\limits_{\mathbb{X}} |f|^p\, v(x)\, dx \right)^{1/p}$$

holds for a finite C if and only if .... (finiteness of single parameter supremum on u, v or a single integral).

Application: Hardy inequalities over metric measure spaces.

Previous results [Ruzhansky, Verma, Shriwastawa, Tiwari 2019-2024]

- $\mathbb{X}$ is a metric measure space, with a borel measure dx admitting a polar decomposition (λ), that is $\int_{\mathbb{X}} f dx = \int_0^\infty \int_{\partial B[a;r]} f(r,\omega)\lambda(r,\omega)\, d\omega_r\, dr$.
- Examples of spaces with polar decompositions: Euclidean spaces, Homogeneous Lie groups, Cartan-Hadamard manifolds (complete, simply connected, Riemannian manifold with non-positive sectional curvature).
- Fix $u, v > 0$, then the inequality

$$\left(\int_{\mathbb{X}} \left(\int_{B[a,x]} |f(y)|\, dy \right)^q u(x)\, dx \right)^{1/q} \leq C \left(\int_{\mathbb{X}} |f|^p\, v(x)\, dx \right)^{1/p}$$

holds for a finite C if and only if .... (finiteness of single parameter supremum on u, v or a single integral).

- Applications to weights (u, v) related to Riezs potentials (inverses of the fractional Laplacian) to get Hardy-Sobolev inequalities.

Our contribution. $p = 1$

$\mathcal{A} = \{B[a; r] : r > 0\}$ is an ordered core, the map $B : \mathbb{X} \to \Sigma$ defined by $B(x) = B[a; |x|]$ has totally ordered range.

Our contribution. $p = 1$

$\mathcal{A} = \{B[a; r] : r > 0\}$ is an ordered core, the map $B : \mathbb{X} \to \Sigma$ defined by $B(x) = B[a; |x|]$ has totally ordered range.

Theorem [S.2024]

With hypothesis only on the weights. The best constant in the Hardy inequality

$$\left(\int_{\mathbb{X}} \left(\int_{B[a;|x|_a]} f(y)\, d\mu(y) \right)^q \omega(x)\, d\mu(x) \right)^{\frac{1}{q}} \leq C \int_{\mathbb{X}} f(x) v(x)\, d\mu(x), \ \forall f \in L^+_\mu$$

satisfies $C \approx \left(\int_{\mathbb{X}} \left(\int_{z \leq_{\mathcal{A}} x} \frac{1}{\underline{v}}(x) \omega(x)\, d\mu(x) \right)^{\frac{q}{1-q}} \omega(z)\, d\mu(z) \right)^{\frac{1-q}{q}}$, for $q \in (0, 1)$,

and $C = \sup_{x \in X} \left(\frac{1}{\underline{v}}(x) \right) \left(\int_{x \leq_{\mathcal{A}} t} \omega(t)\, d\mu(t) \right)^{1/q}$, for $q \in [1, \infty)$.

Where $\underline{v}(x) = \text{ess inf}\{v(t) : t \in B[a; |x|_a]\}$, $x \leq_{\mathcal{A}} t$ means $B[a; |x|_a] \subseteq B(a, |t|)$ and $B[a; |x|_a] = \{z \in \mathbb{X} : \text{dist}(a, z) \leq \text{dist}(a, x)\}$.

Our contribution, the case $p > 1$

Theorem [S.2024] Generalizes the previous results in MMS

With hypothesis only on the weights. The best constant in the Hardy inequality the Hardy inequality

$$\left(\int_{\mathbb{X}}\left(\int_{B[a;|x|_a]} f(y)\,d\mu(y)\right)^q \omega(x)\,d\mu(x)\right)^{\frac{1}{q}} \leq C\left(\int_{\mathbb{X}} f(x)^p v(x)\,d\mu(x)\right)^{\frac{1}{p}}$$

holds if and only if $p \leq q$ and:

$$\sup_{x\neq a}\left\{\left(\int_{\mathbb{X}\setminus B[a;|x|_a]} \omega\,d\mu\right)^{\frac{1}{q}}\left(\int_{B[a;|x|_a]} v^{1-p'}\,d\mu\right)^{\frac{1}{p'}}\right\} < \infty,$$

$0 < q < 1 < p$ and

$$\int_{\mathbb{X}}\left(\int_{\mathbb{X}\setminus B[a;|x|_a]} \omega\,d\mu\right)^{\frac{r}{p}}\left(\int_{B[a;|x|_a]} v^{1-p'}\,d\mu\right)^{\frac{r}{p'}} u(s)\,d\mu(s) < \infty,$$

or $1 < q < p$ and

$$\int_{\mathbb{X}}\left(\int_{\mathbb{X}\setminus B[a;|x|_a]} \omega\,d\mu\right)^{\frac{r}{q}}\left(\int_{B[a;|x|_a]} v^{1-p'}\,d\mu\right)^{\frac{r}{q'}} v^{1-p'}(s)\,d\mu(s) < \infty.$$

Here $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$.

Ideas for future work.

Sobolev inequalities

We have Hardy inequalities for more metric spaces (e.g. Riemannian manifolds instead of Cartan-Hadamard manifolds), does it lead to Hardy-Sobolev type inequalities?

Ideas for future work.

Sobolev inequalities

We have Hardy inequalities for more metric spaces (e.g. Riemannian manifolds instead of Cartan-Hadamard manifolds), does it lead to Hardy-Sobolev type inequalities? If not for all metric measure space, what structure is needed? (manifold?, group action?).

Ideas for future work.

Sobolev inequalities

We have Hardy inequalities for more metric spaces (e.g. Riemannian manifolds instead of Cartan-Hadamard manifolds), does it lead to Hardy-Sobolev type inequalities? If not for all metric measure space, what structure is needed? (manifold?, group action?).

Function spaces

How do we exploit the flexibility of Hardy inequalities in core sets? (e.g. using diadic cubes, Morrey spaces (?)).

Ideas for future work.

Sobolev inequalities

We have Hardy inequalities for more metric spaces (e.g. Riemannian manifolds instead of Cartan-Hadamard manifolds), does it lead to Hardy-Sobolev type inequalities? If not for all metric measure space, what structure is needed? (manifold?, group action?).

Function spaces

How do we exploit the flexibility of Hardy inequalities in core sets? (e.g. using diadic cubes, Morrey spaces (?)).

Operator theory

Can we extend monotone 'functions' to the noncommutative setting? (e.g. von Neumann algebras with semifinite trace).

Bibliography

1. A. Kufner, L. Maligranda, and L. Persson. **The Hardy inequality: about its history and some related results**. Vydavatelsky Servis Publishing House, Pilsen, 2007.

2. Ruzhansky, M., and Verma, D. **Hardy inequalities on metric measure spaces**. Proc. A. 475, 2223 (2019), 20180310, 15.

3. Santacruz Hidalgo, A. **Abstract hardy inequalities: The case p=1**. (submitted)

4. Santacruz Hidalgo, A., and Sinnamon, G. **Core decreasing functions**. (to appear in Journal of Functional Analysis)

5. Sinnamon, G. **Hardy inequalities in normal form**. Trans. Amer. Math. Soc. 375, 2 (2022), 961–995.

Thank you!