CARLEMAN APPROXIMATION ON CERTAIN CLASS OF SETS IN $\mathbb{C}^n$

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1. Introduction and the statements of the results

A closed subset S of $\mathbb{C}^n$ is said to be a Carleman set if for $f \in \mathcal{C}(S, \mathbb{C})$ and strictly positive real valued continuous function ϵ, there exists $g \in \mathcal{O}(\mathbb{C}^n)$ such that

$$|f(z) - g(z)| < \epsilon(z) \ \ \forall z \in S.$$

In 1927 Carleman showed that the set $\mathbb{R}$ allows Carleman approximation as a subset of $\mathbb{C}$ [1]. In one variable such sets are well understood. A set $S \subset \mathbb{C}$ is Carleman set if and only if it is polynomially convex, locally connected at infinity and $IntS = \varnothing$. In higher dimensions: Totally real affince subspaces of $\mathbb{C}^n$ are Carleman sets (Hoischen [**?**], Scheinberg [**?**]). A necessary condition for Carleman approximation is polynomial convexity. A compact subset K is said to be polynomially convex if $K = \widehat{K} := \{z \in \mathbb{C}^n : |p(z)| \leq \sup_K |p| \ \ \forall p \in \mathbb{C}[z_1, \ldots, z_n]\}$. A closed subset $M \subset \mathbb{C}^n$ is called *polynomially convex* if there is a sequence $\{K_m\}$ of polynomially convex compact subsets such that $K_m \subset K_{m+1}$, $\cup K_m = M$. Magnusson and Wold [5] proved that a stratified totally real set allows Carleman approximation if and only if it is polynomially convex and has bounded E-hulls.

- A closed set $M \subset \mathbb{C}^n$ is said to have bounded E-hull if for any compact set $K \subset \mathbb{C}^n$ the set $\widehat{K \cup M} \setminus (K \cup M)$ is bounded.
- A subset of $\mathbb{C}$ is said to be a stratified totally real set if there are closed sets $X_0 \subset X_0 \subset \ldots \subset X_n = X$ such that $X_j \setminus X_{j-1}$ is totally real.

For a given closed set, it is extremely difficult to show whether it is polynomially convex or whether it has bounded E-hull. In this paper we consider two different setting for proving Carleman approximation directly:

(i) Finite union of maximal totally real subspaces in $\mathbb{C}^n$. Here we will assume polynomial convexity.
(ii) Union of two Lipschitz graphs.

We now describe both of them in details.

1.1. The union of finitely many totally real subspaces.

Let $L_1, ..., L_m$ be maximal totally real subspaces such that $L_k \cap L_j = \{0\}$ for all $k \neq j \in \{1, ..., m\}$ and $\cup_{j=1}^m L_j$ is polynomially convex. One of the aim of this article is to show that $\cup_{j=1}^m L_j$ allows Carleman approximation by entire functions in $\mathbb{C}^n$. Clearly, $\cup_{j=1}^m L_j$ is a stratified totally real set. It is not clear and very difficult to determine if it has bounded E-hull. Hence the question makes sense. For the union of two totally real maximal subspaces the answer was given by Manne [7]. The union of two totally real subspaces L_1 and L_2 of real dimension n which intersect only at the origin allows Carleman approximation when $L_1 \cup L_2$ is polynomially convex [7]. The polynomial convexity of $L_1 \cup L_2$ is completely classified [1]. Let L_1 and L_2 be totally real subspaces of real dimension n

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such that they intersect only at the origin. First, take $L_1 = \mathbb{R}^n$. Let L_2 have a basis $(w_1, \ldots, w_n) = (u_1 + iv_1, \ldots, u_n + iv_n)$, where u_js and v_js are real, and let $(a_1, \ldots, a_n)$ be a basis of $\mathbb{R}^n$. So, the set $(w_1, \ldots, w_n, a_1, \ldots, a_n)$ is a basis of $\mathbb{C}^n$ as $L_1 \cap L_2 = \{0\}$. It follows that $(v_1, \ldots, v_n)$ spans $\mathbb{R}^n$. Find the real matrix A such that $u_j = Av_j$. Finally,

$$L_2 = (A + iI)\mathbb{R}^n$$

Let $M(A) = (A + iI)\mathbb{R}^n$. It follows that $A + iI$ is invertible. For the general case, find an invertible $\mathbb{C}$-linear map that maps L_1 to $\mathbb{R}^n$. Two totally real subspaces of dimension n that intersect only at the origin can be seen as $\mathbb{R}^n$ and $M(A)$ under an invertible $\mathbb{C}$-linear map.

Weinstock [1] showed that $\mathbb{R}^n \cup M(A)$ is polynomially convex if, and only if, A does not have an eigenvalue of the form it where $t \in \mathbb{R}$ and $|t| > 1$. Carleman approximation has been studied on the union of two maximal totally real subspaces in [7] when they are polynomially convex. We show a result for the union of multiple totally real subspaces:

Theorem 1.1. *Let $L_1, ..., L_m \subset \mathbb{C}^n$ be maximal totally real subspaces such that $L_k \cap L_j = \{0\}$ for all $k \neq j \in \{0, ..., m\}$ and $\cup L_j$ is polynomially convex. Then $\cup L_j$ allows Carleman approximation by entire functions in $\mathbb{C}^n$.*

In general determining polynomial convexity of union of finitely many totally real subspaces is very difficult. However, an open set of three tuples of totally real planes whose union is polynomially convex is given by Gorai [3]. For union of finitely many totally real planes there are some classes where polynomial convexity is shown [4].

To investigate this deeper for three maximal totally real subspaces: We will look at the polynomial convexity of three totally real planes in $\mathbb{C}^2$ in Theorem 5.1 and Theorem 5.2.

1.2. **The union of two Lipschitz graphs.** The other problem we study in this paper is Carleman approximation for union of two Lipschitz graphs. Our initial interest was to look at union of two totally real submanifolds but the polynomial convexity becomes much tougher compared to the local polynomial convexity.

Let $\psi : \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$ follows the Lipschitz condition, ie, for all $x_1, x_2 \in \mathbb{R}^n$

$$|\psi(x_1) - \psi(x_2)| < \alpha |x_1 - x_2|$$

where $\alpha \in (0, 1)$. Let $\Gamma_\psi = \{x + i\psi(x) : x \in \mathbb{R}^n\}$, the graph of ψ over the imaginary coordinates.

We say a set $\mathcal{L}$ is a Lipschitz graph if $\mathcal{L}$ is related to Γ_ψ under an invertible $\mathbb{C}$-linear map for some ψ as defined above. We can identify a totally real submanifold locally with a Lipschitz graph. Lipschitz graphs have interesting properties:

- (Thm 1.6.9 [9]) Given $K \subset \Gamma_\psi$ compact, $\mathcal{P}(K) = \mathcal{C}(K)$.
- (Section 5 [6]) Γ_ψ allows Carleman approximation as a subset of $\mathbb{C}^n$.

This article will study Carleman approximation on the union of two Lipschitz graphs $\mathcal{L}_1$ and $\mathcal{L}_2$ which intersect only at the origin such that $\mathcal{L}_1 \cup \mathcal{L}_2$ is polynomially convex. The article also discusses the polynomial convexity of $\mathcal{L}_1 \cup \mathcal{L}_2$.

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two Lipschitz graphs, that intersect only at the origin, given by

$$\mathcal{L}_1 = \{x + i\psi_1(x) : x \in \mathbb{R}^n\} \tag{1.1}$$

$$\begin{aligned} \mathcal{L}_2 &= (A + iI)\{x + i\psi_2(x) : x \in \mathbb{R}^n\} \\ &= \{(A + iI)x + i(A + iI)\psi_2(x) : x \in \mathbb{R}^n\} \end{aligned} \tag{1.2}$$

Just like before, A is a real matrix with no eigenvalue which is purely imaginary and has modulus greater than 1 and, let $\psi_1, \psi_2 \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$ which follow the Lipschitz condition, ie, for all $x_1, x_2 \in \mathbb{R}^n$

$$|\psi_j(x_1) - \psi_j(x_2)| < \alpha_j |x_1 - x_2|$$

where $\alpha_j \in (0,1)$ for $j = 1,2$ and $\psi_j(0) = 0$.

Further, If $D\psi_1(0) = 0$ and $D\psi_2(0) = 0$, then the above represents two Lipschitz graphs whose tangent spaces at the origin is polynomially convex.

- When is $\mathcal{L}_1 \cup \mathcal{L}_2$ polynomially convex?
- Can we choose appropriate α_j to ensure this?

Proposition 1.2. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be Lipschitz graphs on $\mathbb{C}^2$ such that they only intersect at the origin and be given by*

$$\mathcal{L}_1 = \{x + i\psi_1(x) : x \in \mathbb{R}^n\}$$

$$\mathcal{L}_2 = \{(A + iI)x + i(A + iI)\psi_2(x) : x \in \mathbb{R}^n\}$$

where

- *The matrix A is a real 2×2 matrix, that doesn't have an eigenvalue which is purely imaginary and has modulus greater than one. (ie, $(A + iI)\mathbb{R}^2 \cup \mathbb{R}^2$ is polynomially convex - [1])*
- *For $j = 1,2$, the function ψ_j is Lipschitz with coefficient α_j such that α_j is less than a constant that depends on A.*

Then $\mathcal{L}_1 \cup \mathcal{L}_2$ is polynomially convex.

Suppose we take $\mathcal{L}_1$ and $\mathcal{L}_2$ as in Proposition 1.2. Then by the Lipschitz condition, setting one point to 0, we get $\|\psi_j(x)\| \leq \alpha_j \|x\|$, for $x \in \mathbb{R}^n$. So we have

$$\begin{aligned} \mathcal{L}_1 &= \{x + i\psi_1(x) : x \in \mathbb{R}^n\} \\ &\subset \{x + iy \in \mathbb{C}^n : \|y\| \leq \alpha_1 \|x\|\} \end{aligned}$$

Similarly,

$$\begin{aligned} \mathcal{L}_2 &= \{(A + iI)(x + i\psi_2(x)) : x \in \mathbb{R}^n\} \\ &\subset (A + iI)\{x + iy \in \mathbb{C}^n : \|y\| \leq \alpha_2 \|x\|\} \end{aligned}$$

As $(A + iI)\mathbb{R}^n \cap \mathbb{R}^n = \{0\}$, we can choose $\alpha_1 > 0$ and $\alpha_2 > 0$ small enough such that

$$\{x + iy \in \mathbb{C}^n : \|y\| \leq \alpha_1 \|x\|\} \bigcap (A + iI)\{x + iy \in \mathbb{C}^n : \|y\| \leq \alpha_2 \|x\|\} = \{0\} \quad (1.3)$$

Theorem 1.3. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two Lipschitz graphs as in Proposition 1.2. Then $\mathcal{L}_1 \cup \mathcal{L}_2$ allows Carleman approximation by entire functions in $\mathbb{C}^n$ when $\alpha_1 > 0$ and $\alpha_2 > 0$ are chosen such that*

- *$\alpha_1 < 1/3$ and $\alpha_2 < 1/3$*
- *α_1 and α_2 small enough such that (1.3) holds.*
- *α_1 and α_2 small enough such that Proposition 1.2 holds (for the polynomial convexity of $\mathcal{L}_1 \cup \mathcal{L}_2$)*

2. Technical preliminaries

Theorem 2.1. *Let $\mathcal{L}$ be a Lipschitz graph as follows*

$$\mathcal{L} = \{x + i\psi(x) : x \in \mathbb{R}^n\}$$

where $\psi : \mathbb{R}^n \to \mathbb{R}^n$ is $\mathcal{C}^1$ with the Lipschitz condition $\|\psi(x) - \psi(y)\| \leq \alpha \|x - y\|$ for all $x, y \in \mathbb{R}$ and $0 < \alpha < 1$. Then $\mathcal{L}$ allows Carleman approximation.

This is proved in section 5 of [6]. The following proposition from this proof would aid us

Proposition 2.2. *Let $\mathcal{L} = \{(x+i\psi(x)) : x \in \mathbb{R}^n\}$, where $\psi \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$. Let $f \in \mathcal{C}(\mathcal{L})$ have compact support and $\epsilon > 0$. Define*

$$h_t(z) := \left(\frac{1}{t\sqrt{\pi}}\right)^n \int_{\mathcal{L}} f(u) \exp(-\frac{(z-u)^2}{t^2}) du$$

Then $h_t \in \mathcal{O}(\mathbb{C}^n)$ and $h_t \to f$ uniformly on $\mathcal{L}$ as $t \to 0^+$.

Further, for $\delta > 0$, define

$$A_\delta := \{z \in \mathbb{C}^n : Re(z-w)^2 \geq \delta \quad \forall w \in supp(f)\}$$

Then $h_t \to 0$ uniformly on A_δ.

This also hints the polynomial convexity of $\mathcal{L}$. The polynomial convexity of $\mathcal{L}$ follows Theorem 1.6.9 of [9]:

Theorem 2.3. *Let $K \subset \mathcal{L}$ be compact. Then K is polynomially convex.*

This can also be proved using Proposition 2.2 as follows:

Proof. To show polynomial convexity it is enough to check $\mathcal{P}(K) = \mathcal{C}(K)$ (Theorem 1.2.10 [9]). Take $f \in \mathcal{C}(K)$, we can extend f to a compactly supported continuous function on whole of R^n using Tietze's Extension Theorem (Theorem 20.4 [8]). Then we may apply Proposition 2.2 to show f is a limit of polynomials. □

A compact set $K \subset \mathcal{L}_1 \cup \mathcal{L}_2$ can be seen as $K_1 \cup K_2$ where $K_j \subset \mathcal{L}_j$ is compact. As K_1 and K_2 are polynomially convexity, Kallin's lemma could be used to check if $K = K_1 \cup K_2$ is polynomially convex.

The following is the Kallin's lemma (Theorem 1.6.19 [9]).

Lemma 2.4 (Kallin)**.** *Let K_1 and K_2 be two polynomially convex compact set in $\mathbb{C}^n$. Let P be a holomorphic polynomial on $\mathbb{C}^n$ such that $Y_j = (\widehat{P(K_j)})$ for $j = 1, 2$ meet at at most at the origin, which is a boundary point for both the sets. If the set $P^{-1}(0) \cap (K_1 \cup K_2)$ is polynomially convex, then $K_1 \cup K_2$ is polynomially convex.*

The following version follows from the previous lemma

Lemma 2.5 (Kallin)**.** *Let K_1 and K_2 be two polynomially convex compact set in $\mathbb{C}^n$. Let P be a holomorphic polynomial on $\mathbb{C}^n$ such that:*

- *$P^{-1}(0) \cap (K_1 \cup K_2)$ is polynomially convex*
- *$P(K_1) \subset \{z \in \mathbb{C} : Im(z) < 0\} \cup \{0\}$*
- *$P(K_2) \subset \{z \in \mathbb{C} : Im(z) > 0\} \cup \{0\}$*

Then $K_1 \cup K_2$ is polynomially convex.

As L_j is a maximal totally real subspace, we can find a invertible $\mathbb{C}$-linear map ϕ_j such that $\phi_j(L_j) = \mathbb{R}^n$. Maximal totally real subspace are like $\mathbb{R}^n$. Here is an approximation result for compactly supported continuous functions on $\mathbb{R}^n$.

Proposition 2.6. *Let $f \in \mathcal{C}(\mathbb{R}^n)$ have compact support and $\epsilon > 0$. Define*

$$h_t(z) := \left(\frac{1}{t\sqrt{\pi}}\right)^n \int_{\mathbb{R}^n} f(u) \exp(-\frac{(z-u)^2}{t^2}) du$$

Then $h_t \in \mathcal{O}(\mathbb{C}^n)$ and $h_t \to f$ uniformly on $\mathbb{R}^n$ as $t \to 0^+$.

Further, for $\delta > 0$, define

$$A_\delta := \{z \in \mathbb{C}^n : Re(z-w)^2 \geq \delta \quad \forall w \in supp(f)\}$$

Then $h_t \to 0$ uniformly on A_δ.

Proof. For $t > 0$, the following holds

$$\left(\frac{1}{t\sqrt{\pi}}\right)^n \int_{\mathbb{R}^n} \exp(-\frac{u^2}{t^2})du = 1$$

Take $\epsilon > 0$. The function f is uniformly continuous as f is continuous and compactly supported. Say, it is bounded by $M > 0$. Find $\delta > 0$ such that $d(f(x), f(y)) < \epsilon/2$ for $x, y \in \mathbb{R}^n$ such that $d(x, y) < \delta$.

$$\begin{aligned}
|h_t(x) - f(x)| &= \left(\frac{1}{t\sqrt{\pi}}\right)^n \left|\int_{\mathbb{R}^n} f(u)\exp(-\frac{(u-x)^2}{t^2})du - \int_{\mathbb{R}^n} f(x)\exp(-\frac{(u-x)^2}{t^2})du\right| \\
&\leq \left(\frac{1}{t\sqrt{\pi}}\right)^n \int_{\mathbb{R}^n} |(f(u) - f(x))|\exp(-\frac{(u-x)^2}{t^2})du \\
&= \left(\frac{1}{t\sqrt{\pi}}\right)^n \int_{B(x,\delta_o)} + \int_{B(x,\delta_o)^c} |(f(u) - f(x))|\exp(-\frac{(u-x)^2}{t^2})du \\
&< \epsilon/2 + \left(\frac{1}{t\sqrt{\pi}}\right)^n \exp(-\frac{\delta_o^2}{t^2})2M|\mathrm{supp}(f)|
\end{aligned}$$

We can choose $t > 0$ small enough such that the second term is less than $\epsilon/2$. This shows $h_t \to f$ uniformly as $t \to 0^+$. Suppose $u \in A_\delta$, then

$$\begin{aligned}
|h_t(u)| &= \left|\left(\frac{1}{t\sqrt{\pi}}\right)^n \int_{\mathbb{R}^n} f(u)\exp(-\frac{(z-u)^2}{t^2})du\right| \\
&\leq \left(\frac{1}{t\sqrt{\pi}}\right)^n \int_{\mathrm{supp}(f)} M\exp(-\frac{\delta}{t^2})du \\
&\leq \left(\frac{1}{t\sqrt{\pi}}\right)^n \exp(-\frac{\delta}{t^2})M|\mathrm{supp}(f)|
\end{aligned}$$

We can choose $t > 0$ small enough such that the above is less than ϵ. This shows $h_t \to 0$ uniformly on A_δ.

□

Lemma 2.7. *Let $a, b, c \in \mathbb{R}$ such that $ab > c^2$. Then $ax^2 + by^2 + 2cxy = 0$ if, and only if, $(x, y) = (0, 0)$. Where $x, y \in \mathbb{R}$.*

Proof. The proof is an application of AM-GM inequality.

Assume $a > 0$. Then $b > 0$ and $\sqrt{ab} > |c|$.

- Case 1: $x \neq 0, y \neq 0$:
  By AM-GM inequality on ax^2 and by^2, we get:
  $$ax^2 + by^2 \geq 2\sqrt{ab}|xy| > 2|c||xy|$$
- Case 2: either $x = 0$ or $y = 0$ but not both zero:
  Say $x = 0$ and $y > 0$,
  $$ax^2 + by^2 = by^2 > 0 = 2|c||xy|$$

If we have $ax^2 + by^2 + 2cxy = 0$ then $ax^2 + by^2 = 2|c||xy|$ which only happens when $(x, y) = (0, 0)$.

When $a < 0$, we replace a with $-a$, b with $-b$ and c with $-c$ and the proof follows.

□

3. The union of finitely many totally real planes

Proposition 3.1. *Let $L_1, ..., L_m$ be as in Theorem 1.1. Fix $R > 1$, and let*

$$K = \{z \in (\cup L_j) : |z| \leq R\} \text{ and } K_j = \{z \in L_j : 1 \leq |z| \leq R\}$$

Then there exists $\delta > 0$ such that for any continuous function $f \in \mathcal{C}(\cup L_j)$ with compact support in $\cup K_j$ and any $\epsilon > 0$ there is an entire function $h \in \mathcal{O}(\mathbb{C}^n)$ such that $|h - f|_K < \epsilon$ and $|h(z)| < \epsilon$ for $|z| \leq \delta$.

Proof. For $0 \leq j \leq m$, we will map K_j to a subset of $\mathbb{R}^n$ via ϕ_j to use Proposition 2.6 for approximating f on K_j. Before doing so, we will define some neighbourhoods of K_j.

We can choose $\gamma > 0$ such that

$$d(\phi_j(K_j), \phi_j(L_k)) > 3\gamma$$

for $0 \leq i, j \leq m$ where $i \neq j$ as ϕ_j is a homeomorphism, K_j is compact, L_k is closed and they are disjoint.

We define a neighbourhood of K

$$\Omega_o = \{z \in \mathbb{C}^n : d(z, K) < \eta\}$$

where $\eta > 0$ is taken so small that

$$\phi_j(\Omega_o) \subset \{z \in \mathbb{C}^n : d(z, \phi_j(K)) < \gamma\}$$

for $0 \leq j \leq m$. It is possible to do so as ϕ_js are continuous.

For $0 \leq j \leq m$, define two open sets $\tilde{U}_j$ and $\tilde{V}_j$ such that they cover $\phi_j(K)$.

$$\tilde{U}_j = \{z \in \mathbb{C}^n : d\left(z, \phi_j(K)\right) < \gamma \text{ and } d\left(z, \phi_j(K_j)\right) < 2\gamma\}$$

$$\tilde{V}_j = \left\{z \in \mathbb{C}^n : d\left(z, \phi_j(K)\right) < \gamma \text{ and } d\left(z, \phi_j(K_j)\right) > \frac{3}{2}\gamma\right\}$$

Observe that, $\phi_j(K_j) \subset \tilde{U}_j$, $\phi_j(L_k \cap K) \subset \tilde{V}_j$ for $k \neq j$ and $0 \in \tilde{V}_j$.

In the lines of Proposition 2.6, define

$$\tilde{h}_j^{(t)}(z) := \left(\frac{1}{t\sqrt{\pi}}\right)^n \int_{\mathbb{R}^n} f \circ \phi_j^{-1}(u) e^{-\frac{(z-u)^2}{t^2}} du$$

Consider all limits as $t \to 0^+$. Now by Proposition 2.6 we have $\tilde{h}_j^{(t)} \in \mathcal{O}(C^n)$, and $\tilde{h}_j^{(t)} \to f \circ \phi_j^{-1}$ uniformly on $\phi_j(L_j) = \mathbb{R}^n$. Let $\tilde{W}_j = \tilde{V}_j \cap \tilde{U}_j$, we will show $\tilde{W}_j \subset A_{\frac{\gamma^2}{4}}$ with the notation of Proposition 2.6; this gives us $\tilde{h}_j^{(t)} \to 0$ uniformly on $\tilde{W}_j$. Let $z \in \tilde{W}_j$, $u \in \phi_j(K_j) \subset \mathbb{R}^n$, say $z = (x_1 + iy_1, ..., x_n + iy_n)$ and $u = (u_1, ..., u_n) \in \mathbb{R}^n$.

$$\begin{aligned} Re(z-u)^2 &= Re\sum_{l=1}^{n}(z_l - u_l)^2 \\ &= \sum_{l=1}^{n}(x_l - u_l)^2 - y_l^2 \\ &= \sum_{l=1}^{n}(x_l - u_l)^2 - \sum_{l=1}^{n} y_l^2 \end{aligned}$$

As $z \in \tilde{W}_j \subset \tilde{V}_j$, $d(z,u) > \frac{3}{2}\gamma$ which gives

$$\sum_{l=1}^{n}(x_l - u_l)^2 + \sum_{l=1}^{n} y_l^2 > \frac{9}{4}\gamma^2$$

By definition of U_j, $d(z, \phi_j(K)) < \gamma$ and $d(z, \phi_j(K_j)) < 2\gamma$. By the choice of γ, $d(\phi_j(K_j), \phi_j(L_k)) > 3\gamma$ for $k \neq j$; this implies $d(z, \phi_j(L_k)) > \gamma$. That leaves us with $d(z, \phi_j(K_j)) < \gamma$.

$$\sum_{l=1}^{n} y_l^2 = d(z, \mathbb{R}^n) = d(z, \phi_j(L_j)) < d(z, \phi_j(K_j)) < \gamma$$

Combining both the inequalities, $\sum_{l=1}^{n}(x_l - u_l)^2 > \frac{5}{4}\gamma^2$. Finally,

$$Re(z-u)^2 = \sum_{l=1}^{n}(x_l - u_l)^2 - \sum_{l=1}^{n} y_l^2 > \frac{5}{4}\gamma^2 - \gamma^2 = \frac{\gamma^2}{4}$$

This proves $\tilde{W}_j \subset A_{\gamma^2/4}$ and $\tilde{h}_j^{(t)} \to 0$ uniformly (Proposition 2.6).

The set K is polynomially convex as it is a compact subset of $\cup L_j$. We can find a pseudoconvex domain Ω such that $K \subset \Omega \subset \Omega_o$ by a polynomial polyhedra.

Let $0 \leq j \leq m$. Set $U_j = \Omega \cap \tilde{U}_j$, $V_j = \Omega \cap \tilde{V}_j$ and $W_j = V_j \cap U_j = \Omega \cap \phi_j^{-1}(\tilde{W}_j)$. Then $U_j \cup V_j = \Omega$ and $\tilde{h}_j^{(t)} \circ \phi_j \to 0$ uniformly on W_j.

Consider the linear operator $T : \mathcal{O}(U_j) \bigoplus \mathcal{O}(V_j) \to \mathcal{O}(W_j)$ given by

$$T(\alpha, \beta) = (\alpha - \beta)|_{W_j}$$

The Cousin I problem can be solved on the pseudoconvex set Ω with $\{U_j, V_j\}$; This shows the surjectivity of T. Which allows us to use the open mapping theorem for Fréchet spaces. As $\tilde{h}_j^{(t)} \circ \phi_j \to 0$ in $\mathcal{O}(W_j)$, we can solve for $(\tilde{\alpha}_j^{(t)}, \tilde{\beta}_j^{(t)}) \in \mathcal{O}(U_j) \bigoplus \mathcal{O}(V_j)$ with

$$T(\tilde{\alpha}_j^{(t)}, \tilde{\beta}_j^{(t)}) = (\tilde{\alpha}_j^{(t)} - \tilde{\beta}_j^{(t)})|_{W_j} = \tilde{h}_j^{(t)} \circ \phi_j$$

such that $(\tilde{\alpha}_j^{(t)}, \tilde{\beta}_j^{(t)}) \to 0$, ie, $\tilde{\alpha}_j^{(t)} \to 0$ in $\mathcal{O}(U_j)$ and $\tilde{\beta}_j^{(t)} \to 0$ in $\mathcal{O}(V_j)$.

We have seen that $0 \in V_j$ for all j. Take $\delta > 0$ such that the closed ball $B(0,\delta) \subset\subset V_j$ for all j. Define $h_j^{(t)} \in \mathcal{O}(\Omega)$ with $h_j^{(t)} = \tilde{h}_j^{(t)} \circ \phi_j - \tilde{\alpha}_j^{(t)}$ on U_j and $h_j^{(t)} = -\tilde{\beta}_j^{(t)}$ on V_j. This is well defined due to the Cousin's I problem.

Then $h_j^{(t)} \to 0$ in $\mathcal{O}(V_j)$; this gives: $h_j^{(t)} \to 0$ uniformly on $L_k \cap K \subset\subset V_j$ for $k \neq j$, and $h_j^{(t)} \to 0$ uniformly on $B(0,\delta) \subset\subset V_j$ for $k \neq j$. Further, $h_j^{(t)} \to f$ uniformly on compact subsets of $L_j \cap V_j$ as $f = 0$ on $L_j - K_j$.

The entire function $\tilde{h}_j^{(t)} \circ \phi \to f$ uniformly on L_j and $\tilde{\alpha}_j^{(t)} \to 0$ in $\mathcal{O}(U_j)$ which implies $h_j^{(t)} \to f$ uniformly on compact subsets of $L_j \cap U_j$. Combining with the previous argument, $h_j^{(t)} \to f$ uniformly on $L_j \cap K$, $h_j^{(t)} \to 0$ uniformly on $L_k \cap K$ for $k \neq j$, and $h_j^{(t)} \to 0$ uniformly on $B(0,\delta)$.

Define $h^{(t)} := \sum_{j=1}^{n} h_j^{(t)}$. Then $h^{(t)} \to f$ uniformly on $L_j \cap K$ for all j. Hence, $h^{(t)} \to f$ uniformly on K and $h^{(t)} \to 0$ uniformly on $B(0,\delta)$.

□

Corollary 3.2. *For $R_o > 1$, we can choose δ for which the following holds. If $f_o \in \mathcal{C}(\cup L_j)$ such that $f_o(z) = 0$ for $|z| \leq r$ for some $r > 0$. Then there exists $h_o \in \mathcal{O}(\mathbb{C}^n)$*

such that $|h_o(z) - f_o(z)| < \epsilon$ *for* $z \in (\cup L_j) \cap \{z \in \mathbb{C}^n : |z| \leq rR_o\}$*, and* $|h_o(z)| < \epsilon$ *for* $|z| \leq r\delta$*.*

Proof. We define

$$f(z) := f_o(z/r)\rho_{R_o}(|z|)$$

where $\rho_{R_o} : [0, +\infty) \to [0, 1]$ is a continuous function that is 1 on $[0, R_o]$ and 0 on $[R_o + 1, +\infty)$. Let $R = R_o + 1$; it follows that $\text{supp}(f_o) \subset \cup K_j$ with the notations of Proposition 3.1. By the proposition, we have $\delta > 0$ corresponding to R. Then there is an entire function $h \in \mathcal{O}(\mathbb{C}^n)$ such that $|f - h|_K < \epsilon$ and $|h|_{B(0,\delta)} < \epsilon$. Define entire function $h_o(z) := h(rz)$. We have $|f(z) - h(z)| < \epsilon$ for $z \in (\cup L_j) \cap \{z \in \mathbb{C}^n : |z| \leq R\}$. Replacing f with f_o, $|f_o(z/r) - h_o(z/r)| < \epsilon$ for $z \in (\cup L_j) \cap \{z \in \mathbb{C}^n : |z| \leq R_o\}$. Which is $|f_o(z) - h_o(z)| < \epsilon$ for $z \in (\cup L_j) \cap \{z \in \mathbb{C}^n : |z| \leq rR_o\}$ and $|h_o(z)| < \epsilon$ for $|z| \leq r\delta$. $\square$

Proof of Theorem 1.1. :

We are given $f \in \mathcal{C}(\cup L_j)$ and $\epsilon \in \mathcal{C}(\cup L_j, \mathbb{R}_{>0})$. Let $R = R_o$ and δ as chosen in Corollary 3.2. Take an increasing sequence of positive reals $\{r_\eta\}$ such that $r_{\eta+2} = Rr_\eta$. We define continuous functions $\rho_\eta : [0, +\infty) \to [0, 1]$ such that $\rho_\eta(t)$ is 0 for $t \in [0, r_\eta]$ and 1 for $t \in [r_{\eta+1}, +\infty)$.

Construct an exhaustion of $\cup L_j$ by compacts

$$K_\eta = \{z \in \cup L_j : |z| \leq r_\eta R\}$$

for which the corresponding closed ball as in Corollary 3.2 will be

$$B_\eta = B(0, r_\eta \delta) \text{ and } \epsilon_\eta = \min_{t \in K_\eta} \epsilon(t)$$

As K_0 is polynomially convex, we can approximation f on K_0 by an entire function h_0 such that

$$|h_0(z) - f(z)| < \epsilon_0/2 \quad \forall z \in K_0$$

We want an entire function h_1 such that

$$|h_1(z) - \rho_0(|z|)(f(z) - h_0(z))| < \epsilon_1/2^2 \quad \forall z \in K_1$$

and $|h_1|_{B_1} \leq \epsilon_1/2^2$. Such a h_1 can be found using Corollary 3.2 as $f_o(z) = \rho_0(|z|)(f(z) - h_0(z))$ satisfies the conditions with $r = r_0$.

This process can be done inductively to get h_η an entire functions such that

$$|h_\eta(z) - \rho_\eta(|z|)(f(z) - \sum_{l=1}^{\eta-1} h_l(z))| < \epsilon_\eta/2^{\eta+1} \quad \forall z \in K_\eta$$

and $|h_\eta|_{B_\eta} < \epsilon_\eta/2^{\eta+1}$.

Define $h = \sum_{\eta \in \mathbb{N}} h_\eta$. Given any compact set $A \subset \mathbb{C}^n$, we can find η such that $A \subset B_\eta$. So, h converges uniformly on A as $|\sum_{j>\eta} h_j|_A < \epsilon_\eta/2^\eta$. Which mean, h is an entire function.

We are done if we show $|h(z) - f(z)| < \epsilon(z)$ for $z \in \cup \mathcal{L}_j$. Take any $z \in \cup L_j$, say $r_{\eta_o} < |z| \leq r_{\eta_o+1}$.

$$\begin{aligned}
|h(z)-f(z)| &= \left|\sum_{\eta=0}^{\infty} h_\eta(z) - f(z)\right| \\
&\leq \sum_{\eta>\eta_o}^{\infty} |h_\eta(z)| + \left|\sum_{\eta=0}^{\eta_o} h_\eta - f(z)\right| \\
&\leq \epsilon/2^{\eta_o+1} + \left|h_{\eta_o}(z) - \rho_{\eta_o}(z)\left(f(z) - \sum_{\eta=0}^{\eta_o-1} h_\eta(z)\right)\right| \\
&\quad + (1-\rho_{\eta_o}(z))\left|\sum_{\eta=0}^{\eta_o-1} h_\eta(z) - f(z)\right| \\
&\leq \epsilon(z)/2^{\eta_o+1} + \epsilon(z)/2^{\eta_o+1} \\
&\quad + (1-\rho_{\eta_o}(z))\left|h_{\eta_o-1} - \rho_{\eta_o-1}(z)\left(f(z) - \sum_{\eta=0}^{\eta_o-2} h_\eta(z)\right)\right| \\
&\leq \epsilon(z)/2^{\eta_o+1} + \epsilon(z)/2^{\eta_o+1} + \epsilon(z)/2^{\eta_o} \leq \epsilon(z)
\end{aligned}$$

□

4. The union of two Lipschitz graphs

4.1. **polynomial convexity of the union of two Lipschitz graphs.** In this section, the polynomial convexity of $\mathcal{L}_1 \cup \mathcal{L}_2$ will be studied where $\mathcal{L}_1$ and $\mathcal{L}_2$ are Lipschitz graphs as given in **??**. Restricting the study to $\mathbb{C}^2$, gives the following proposition:

The proof of this proposition generalizes the calculation done in Theorem 1.2 of [2].

Proof. (Proposition 1.2) The matrix A is 2×2 real valued. Then A would be similar to one of the following real matrix:

$$\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \qquad \lambda \in \mathbb{R} \tag{4.1}$$

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \qquad \lambda_1, \lambda_2 \in \mathbb{R} \tag{4.2}$$

$$\begin{bmatrix} s & -t \\ t & s \end{bmatrix} \qquad s, t \in \mathbb{R} \tag{4.3}$$

This can be showed by looking at the different possible Jordan forms for a 2×2 matrix with real entries. If the roots for the characteristic polynomial are imaginary then we get (4.3). If the roots are real and A is not diagonizable then we get (4.1). Finally, if the roots are real and A is diagonizable we get (4.2).

So, there is an invertible complex 2×2 matrix such that SAS^{-1} has one of the above forms. First, we look at what happens to $(A+iI)\mathbb{R}^n$ under the action of S. Say $x \in \mathbb{R}^n$, then $S(A+iI)x = SAS^{-1}y + iSIS^{-1}y = (SAS^{-1} + iI)y$, where $y = S^{-1}x$. Which gives $(SAS^{-1} + iI)\mathbb{R}^n = S((A+iI)\mathbb{R}^n)$. Let us look at what happens to $\mathcal{L}_1$ and

$\mathcal{L}_2$ under the action of S.

$$\begin{aligned} S\mathcal{L}_1 &= \{S(x+i\psi_1(x)) : x \in \mathbb{R}^n\} \\ &= \{S(x)+iS(\psi_1(x)) : x \in \mathbb{R}^n\} \\ &= \{y+iS(\psi_1(S^{-1}y)) : y \in \mathbb{R}^n\} \end{aligned}$$

Similarly for $\mathcal{L}_2$

$$\begin{aligned} S\mathcal{L}_2 &= \{S((A+iI)(x+i\psi_2(x))) : x \in \mathbb{R}^n\} \\ &= \{S((A+iI)(S^{-1}y+i\psi_2(S^{-1}y))) : y \in \mathbb{R}^n\} \\ &= \{S(A+iI)S^{-1}y+iS(A+iI)\psi_2(S^{-1}y) : y \in \mathbb{R}^n\} \\ &= \{(SAS^{-1}+iI)y+iS(A+iI)S^{-1}S\psi_2(S^{-1}y) : y \in \mathbb{R}^n\} \\ &= \{(SAS^{-1}+iI)y+i(SAS^{-1}+iI)S\psi_2(S^{-1}y) : y \in \mathbb{R}^n\} \end{aligned}$$

If we set $A' = SAS^{-1}$, $\psi'_j(x) = S\psi_j(S^{-1}x)$, and $\alpha'_j = \|S\|\|S^{-1}\|\alpha_j$. Then the above looks like

$$\begin{aligned} \mathcal{L}'_1 &= \{(x+i\psi'_1(x))) : x \in \mathbb{R}^n\} \\ \mathcal{L}'_2 &= \{(A'+iI)(x+i\psi'_2(x))) : x \in \mathbb{R}^n\} \end{aligned}$$

The Lipschitz condition will still be followed with the constants α'_j

$$\begin{aligned} |\psi'_j(x)-\psi'_j(y)| &= |S\psi_j(S^{-1}x)-S\psi_j(S^{-1}y)| \\ &= |S(\psi_j(S^{-1}x)-\psi_j(S^{-1}y))| \\ &\leq \|S\||\psi_j(S^{-1}x)-\psi_j(S^{-1}y)| \\ &\leq \|S\|\alpha_j|S^{-1}x-S^{-1}y| \\ &\leq \|S\|\alpha_j\|S^{-1}\||x-y| \\ &\leq \alpha_j\|S\|\|S^{-1}\||x-y| \\ &\leq \alpha'_j|x-y| \end{aligned}$$

Hence, we could consider A to be of the three forms given above. We will have to work cases for each of (4.1), (4.2), and (4.3).

Case 1: Let A be of type (4.1).

Let $\lambda \in \mathbb{R}$

$$A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$

We would like to show any compact set $K_1 \cup K_2 = K \subset \mathcal{L}_1 \cup \mathcal{L}_2$ is polynomially convex for compacts sets $K_1 \subset \mathcal{L}_1$ and $K_2 \subset \mathcal{L}_2$. We would apply the Kallin's lemma as stated in Lemma 2.5.

Define $G : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}$

$$G(z,w) = z_1w_1 + z_2w_2$$

We claim that the suitable polynomial for applying Lemma 2.5 is $p : \mathbb{C}^2 \to \mathbb{C}$

$$p(z) := G((A-iI)z, z)$$

Or explicitly, if $z = (z_1, z_2) \in \mathbb{C}^2$ then computing $p(z)$ we get

$$p(z) = z_1\left(z_1\left(\lambda-i\right)+z_2\right)+z_2^2\left(\lambda-i\right)$$

We would like to show

$$p(K_1) \subset \{z \in \mathbb{C} : Im(z) < 0\} \cup \{0\} \tag{4.4}$$
$$p(K_2) \subset \{z \in \mathbb{C} : Im(z) > 0\} \cup \{0\} \tag{4.5}$$

for an appropriate choice of $\alpha_1 > 0$ and $\alpha_2 > 0$. This would , by Lemma 2.5, show that any $K_1 \cup K_2$ is polynomially convex.

Let $z \in \mathcal{L}_1$, then $z = (x, y) + i\psi_1(x, y)$ for some $(x, y) \in \mathbb{R}^2$. If we solve for $im(p(z))$ we get

$$\begin{aligned} im(p(z)) = \left(\psi_1^{(1)}(x,y)\right)^2 + \left(\psi_1^{(2)}(x,y)\right)^2 + 2\psi_1^{(1)}(x,y)\lambda x \\ + 2\psi_1^{(2)}(x,y)\lambda y + \psi_1^{(1)}(x,y)y + \psi_1^{(2)}(x,y)x - x^2 - y^2 \end{aligned} \tag{4.6}$$

Where $\psi_1(x, y) = (\psi_1^{(1)}(x, y), \psi_1^{(2)}(x, y))$.

By the Lipschitz condition on ψ_1, we have

$$\begin{aligned} \|\psi_1((x,y)) - \psi_1((0,0))\| &\leq \alpha_1\|(x,y) - (0,0)\| \\ \|\psi_1((x,y))\| &\leq \alpha_1\|(x,y)\| \end{aligned}$$

We will set $v = \|(x, y)\|$, then $(\psi_1^{(1)}(x, y))^2 + (\psi_1^{(2)}(x, y))^2 \leq \alpha_1^2 v^2$. Further, $|x|\leq v$, $|y|\leq v$, $|\psi_1^{(1)}(x, y)|\leq \|\psi_1(x, y)\|\leq \alpha_1 v$. , and $|\psi_1^{(2)}(x, y)|\leq \|\psi_1(x, y)\|\leq \alpha_1 v$.

Now we make use of this inequality in (4.6). Giving us:

$$im(p(z)) \leq v^2(\alpha_1^2 + 4\alpha_1\lambda + 2\alpha_1 - 1)$$

Choose α_1 small enough such that:

$$\alpha_1^2 + 4|\lambda|\alpha_1 + 2\alpha_1 < 1$$

Then $im(p(z)) < 0$ for $z \neq 0 \in \mathcal{L}_1$.

Now we look at the other Lipschitz graph, let $z \in \mathcal{L}_2$, say $z = (A + iI)(x, y) + i(A + iI)\psi_2(x, y)$ for some $(x, y) \in \mathbb{R}^2$. If we solve for the explicit form of z:

$$z = \begin{bmatrix} i\psi^{(2)}(x,y) + y + (\lambda + i)\left(i\psi^{(1)}(x,y) + x\right) \\ (\lambda + i)\left(i\psi^{(2)}(x,y) + y\right) \end{bmatrix}$$

Now solving for $im(p(z))$:

$$\begin{aligned} im(p(z)) = &-\left(\psi_2^{(1)}(x,y)\right)^2(1+\lambda^2) - \left(\psi_2^{(2)}(x,y)\right)^2(1+\lambda^2) \\ &- 2\psi_2^{(1)}(x,y)\psi_2^{(2)}(x,y)\lambda + 2\psi_2^{(1)}(x,y)\lambda^3 x + 3\psi_2^{(1)}(x,y)\lambda^2 y \\ &+ 2\psi_2^{(1)}(x,y)\lambda x + \psi_2^{(1)}(x,y)y + 2\psi_2^{(2)}(x,y)\lambda^3 y + 3\psi_2^{(2)}(x,y)\lambda^2 x \\ &+ 6\psi_2^{(2)}(x,y)\lambda y + \psi_2^{(2)}(x,y)x + \lambda^2x^2 + \lambda^2y^2 + 2\lambda xy + x^2 + y^2 \end{aligned} \tag{4.7}$$

Again we obtain the inequalities.

Set $v = \|(x, y)\|$, then $(\psi_2^{(1)}(x, y))^2 + (\psi_2^{(2)}(x, y))^2 \leq \alpha_2^2 v^2$. Further, $|x|\leq v$, $|y|\leq v$, $|\psi_2^{(1)}(x, y)|\leq \|\psi_2(x, y)\|\leq \alpha_2 v$. , and $|\psi_2^{(2)}(x, y)|\leq \|\psi_2(x, y)\|\leq \alpha_2 v$.

$$im(p(z)) \geq -[\alpha_2^2(|\lambda|+1)^2 + \alpha_2(4|\lambda|^3+6|\lambda|^2+10|\lambda|+2) - |\lambda|^2]v^2 + (|\lambda|x+y)^2 + (1-|\lambda|^2)x^2$$

Take $\alpha_2 > 0$ so small that

$$\alpha_2^2(|\lambda|+1)^2 + \alpha_2(4|\lambda|^3+6|\lambda|^2+10|\lambda|+2) < |\lambda|^2$$

so that $im(p(z)) > 0$ when $z \neq 0 \in \mathcal{L}_2$.

Moreover, both these results show that $p^{-1}(0) \cap (K_1 \cup K_2) = \{0\}$ which is polynomially convex. Hence, $K_1 \cup K_2$ is polynomially convex by Kallins Lemma 2.5. This would work for any $K_1 \subset \mathcal{L}_1$ compact and $K_2 \subset \mathcal{L}_2$ compact.

Case 2: Let A be of type (4.2).
Let $\lambda_1, \lambda_2 \in \mathbb{R}$

$$A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

We will mimic the same steps as in Case 1. Take the same polynomial p.

Again like last time, take compacts $K_1 \subset \mathcal{L}_1$ and $K_2 \subset \mathcal{L}_2$

$$p(K_1) \subset \{z \in \mathbb{C} : Im(z) < 0\} \cup \{0\}$$
$$p(K_2) \subset \{z \in \mathbb{C} : Im(z) > 0\} \cup \{0\}$$

Let $z \in \mathcal{L}_1$, then $z = (x, y) + i\psi_1(x, y)$ for some $(x, y) \in \mathbb{R}^2$. If we solve for $im(p(z))$ we get

$$im(p(z)) = \left(\psi_1^{(1)}(x,y)\right)^2 + \left(\psi_1^{(2)}(x,y)\right)^2 + 2\psi_1^{(1)}(x,y)\lambda_1 x + 2\psi_1^{(2)}(x,y)\lambda_2 y - x^2 - y^2 \tag{4.8}$$

The same inequality shown in case 1 will work here. Using them here:

$$im(p(z)) \leq (\alpha_1^2 + 2\alpha_1|\lambda_1| + 2\alpha_1|\lambda_2| - 1)v^2$$

Choose $\alpha_1 > 0$ small enough such that:

$$\alpha_1^2 + 2\alpha_1|\lambda_1| + 2\alpha_1|\lambda_2| < 1$$

Then $im(p(z)) < 0$ for $z \neq 0 \in \mathcal{L}_1$.

Now let $z \in \mathcal{L}_2$, say $z = (A + iI)(x, y) + i(A + iI)\psi_2(x, y)$ for some $(x, y) \in \mathbb{R}^2$. If we solve for the explicit form of z:

$$z = \begin{bmatrix} (\lambda_1 + i)\left(i\psi_2^{(1)}(x,y) + x\right) \\ (\lambda_2 + i)\left(i\psi_2^{(2)}(x,y) + y\right) \end{bmatrix}$$

$$\begin{aligned} im(p(z)) = &- \left(\psi_2^{(1)}(x,y)\right)^2 \lambda_1^2 - \left(\psi_2^{(1)}(x,y)\right)^2 + 2\psi_2^{(1)}(x,y)\lambda_1^3 x \\ &+ 2\psi_2^{(1)}(x,y)\lambda_1 x - \left(\psi_2^{(2)}(x,y)\right)^2 \lambda_2^2 - \left(\psi_2^{(2)}(x,y)\right)^2 \\ &+ 2\psi_2^{(2)}(x,y)\lambda_2^3 y + 2\psi_2^{(2)}(x,y)\lambda_2 y + \lambda_1^2 x^2 + \lambda_2^2 y^2 + x^2 + y^2 \end{aligned}$$

With the inequalities we get

$$im(p(z)) \geq -[\alpha^2(1 + |\lambda_1|^2 + |\lambda_2|^2) + 2\alpha(|\lambda_1|^3 + |\lambda_2|^3 + |\lambda_1| + |\lambda_2|)]v^2 + v^2 + |\lambda_1|x^2 + |\lambda_2|y^2$$

If we choose $\alpha > 0$ small enough such that

$$\alpha^2(1 + |\lambda_1|^2 + |\lambda_2|^2) + 2\alpha(|\lambda_1|^3 + |\lambda_2|^3 + |\lambda_1| + |\lambda_2|) < 1$$

Then $im(p(z)) > 0$ when $z \neq 0 \in \mathcal{L}_2$.

Again, both these results show that $p^{-1}(0) \cap (K_1 \cup K_2) = \{0\}$ which is polynomially convex. Hence, $K_1 \cup K_2$ is polynomially convex by Kallins Lemma 2.5. This would work for any $K_1 \subset \mathcal{L}_1$ compact and $K_2 \subset \mathcal{L}_2$ compact.

Case 3: Let A be of type (4.3), ie, say $s, t \in \mathbb{R}$

$$A = \begin{bmatrix} s & -t \\ t & s \end{bmatrix}$$

Take compacts $K_1 \subset \mathcal{L}_1$ and $K_2 \subset \mathcal{L}_2$. We wish to show $K_1 \cup K_2$ is polynomially convex.

Consider the polynomial

$$p(z) = z_1^2 + z_2^2$$

Let $z \in \mathcal{L}_1$, then $z = (x,y) + i\psi_1(x,y)$ for some $(x,y) \in \mathbb{R}^2$. If we solve for $im(p(z))$ we get

$$im(p(z)) = 2\psi_1^{(1)}(x,y)x + 2\psi_1^{(2)}(x,y)y$$

and solving for $re(p(z))$ we get

$$re(p(z)) = -\left(\psi_1^{(1)}(x,y)\right)^2 - \left(\psi_1^{(2)}(x,y)\right)^2 + x^2 + y^2$$

Using the inequalities here, we get

$$re(p(z)) \geq (1-\alpha_1^2)v^2$$

If we choose $\alpha_1 > 0$ small enough such that

$$\alpha_1^2 < 1$$

Then $re(p(z)) > 0$ when $z \neq 0 \in \mathcal{L}_1$.

Let $z \in \mathcal{L}_2$, say $z = (A+iI)(x,y) + i(A+iI)\psi_2(x,y)$ for some $(x,y) \in \mathbb{R}^2$. If we solve for the explicit form of z:

$$z = \begin{bmatrix} -t\left(i\psi_2^{(2)}(x,y)+y\right) + (s+i)\left(i\psi_2^{(1)}(x,y)+x\right) \\ t\left(i\psi_2^{(1)}(x,y)+x\right) + (s+i)\left(i\psi_2^{(2)}(x,y)+y\right) \end{bmatrix}$$

Now we solve for $im(p(z))$ and $re(p(z))$:

$$\begin{aligned} im(p(z)) = &-2\left(\psi_2^{(1)}(x,y)\right)^2 s + 2\psi_2^{(1)}(x,y)s^2x + 2\psi_2^{(1)}(x,y)t^2x \\ &- 2\psi_2^{(1)}(x,y)x - 2\left(\psi_2^{(2)}(x,y)\right)^2 s + 2\psi_2^{(2)}(x,y)s^2y \\ &+ 2\psi_2^{(2)}(x,y)t^2y - 2\psi_2^{(2)}(x,y)y + 2sx^2 + 2sy^2 \end{aligned}$$

$$\begin{aligned} re(p(z)) = &-\left(\psi_2^{(1)}(x,y)\right)^2 s^2 - \left(\psi_2^{(1)}(x,y)\right)^2 t^2 + \left(\psi_2^{(1)}(x,y)\right)^2 \\ &- 4\psi_2^{(1)}(x,y)sx - \left(\psi_2^{(2)}(x,y)\right)^2 s^2 - \left(\psi_2^{(2)}(x,y)\right)^2 t^2 \\ &+ \left(\psi_2^{(2)}(x,y)\right)^2 - 4\psi_2^{(2)}(x,y)sy + (t^2+s^2-1)x^2 + (t^2+s^2-1)y^2 \end{aligned}$$

We consider two subcases now based on if $t^2 + s^2 < 1$.

Case 3.1 Say $t^2 + s^2 < 1$. Using the inequalities on $re(p(z))$:

$$re(p(z)) \leq [(t^2+s^2-1)(1-\alpha_2^2) - 8\alpha_2|s|]v^2$$

Then if choose $\alpha_2 > 0$ so small that

$$(t^2+s^2-1)(1-\alpha_2^2) - 8\alpha_2|s| < 0$$

Then $re(p(z)) < 0$ when $z \neq 0 \in \mathcal{L}_2$.

Again it is clear that $p^{-1}(0) \cap (K_1 \cup K_2) = \{0\}$. By Kallin's Lemma 2.5 we get $K_1 \cup K_2$ is polynomially convex.

Case 3.2 Say $t^2 + s^2 \geq 1$.

We will show that $p(K_1)$ and $p(K_2)$ lie in different sectors which only intersect at the origin. Then we claim $K_1 \cup K_2$ is polynomially convex by Kallin's Lemma 2.5.

Consider the following angular sectors

$$\begin{aligned} V_1 &= \{x+iy \in \mathbb{C} : |y| \leq \epsilon|x|\} \\ V_2 &= \{x+iy \in \mathbb{C} : |(t^2+s^2-1)y - 2sx| \leq \epsilon|y|\} \end{aligned}$$

Where $\epsilon > 0$ small enough such that $V_1 \cup V_2 = \{0\}$.

We claim for suitable choice of α_1 and α_2 we get $p(\mathcal{L}_1) \subset V_1$ and $p(\mathcal{L}_2) \subset V_2$.

- Let $z \in \mathcal{L}_1$. Apply the inequalities on $|im(p(z))|$ which we had calculated
$$|im(p(z))| \leq 2\alpha_1 v^2$$
and on $|re(p(z))|$
$$|re(p(z))| \geq (1-\alpha_1^2)v^2$$
If we choose α_1 small enough such that such that $2\alpha_1 < \epsilon(1-\alpha_1^2)$ then
$$|im(p(z))| \leq \epsilon |re(p(z))|$$
Hence, $p(\mathcal{L}_1) \subset V_1$.
- Let $z \in \mathcal{L}_2$. Calculating $(t^2+s^2-1)(re(p(z))) - 2s(im(p(z))) =$
$$\begin{aligned} &2\psi^{(1)}(x,y)s^4x + 4\psi^{(1)}(x,y)s^2t^2x + 4\psi^{(1)}(x,y)s^2x + 2\psi^{(1)}(x,y)t^4x \\ &- 4\psi^{(1)}(x,y)t^2x + 2\psi^{(1)}(x,y)x + 2\psi^{(2)}(x,y)s^4y + 4\psi^{(2)}(x,y)s^2t^2y \\ &+ 4\psi^{(2)}(x,y)s^2y + 2\psi^{(2)}(x,y)t^4y - 4\psi^{(2)}(x,y)t^2y + 2\psi^{(2)}(x,y)y \end{aligned}$$
Using inequalities, we get
$$|(t^2+s^2-1)(re(p(z))) - 2s(im(p(z)))| \leq \alpha_2(4s^4 + 8s^2t^2 + 8s^2 + 4t^4 + 8t^2)v^2$$
Now we find a lower bound for $|im(p(z))|$. First note that $|s| \neq 0$; to ensure A has no eigenvalue which is purely imaginary of modulus greater than 1 we must have $|s| \neq 0$. Now we use the inequalities on $|im(p(z))|$:
$$|im(p(z))| \geq 2|s|v^2 - \alpha_2[2\alpha_2|s| + 4s^2 + t^2 + 4]v^2$$
We can choose $\alpha_2 > 0$ small such that
$$\epsilon(2|s| - \alpha_2[2\alpha_2|s| + 4s^2 + t^2 + 4]) \geq \alpha_2(4s^4 + 8s^2t^2 + 8s^2 + 4t^4 + 8t^2)$$
This ensures
$$|(t^2+s^2-1)(re(p(z))) - 2s(im(p(z)))| \leq \epsilon |im(p(z))|$$
Hence, $p(\mathcal{L}_2) \subset V_2$.

Again, $p^{-1}(0) \cap (K_1 \cup K_2) = \{0\}$. The angular sectors intersected with a closed ball is polynomially convex. And $A_1 \cap A_2 = \{0\}$. Hence $K_1 \cup K_2$ is polynomially convex by Kallin's Lemma 2.4. □

The proof of Theorem 1.3 will follow similar arguments to that of the union of two totally real subspaces [7]. Before this, we show the following lemma:

Proposition 4.1. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be as in Theorem 1.3. Fix $R > 1$, and let*
$$K = \{z \in \mathcal{L}_1 \cup \mathcal{L}_2 : |z| \leq R\} \text{ and } K_j = \{z \in \mathcal{L}_j : 1 \leq |z| \leq R\} \quad j = 1,2$$
Then there exists $\delta > 0$ such that for any continuous function $f \in \mathcal{C}(\mathcal{L}_1 \cup \mathcal{L}_2)$ with compact support in $K_1 \cup K_2$ and
any $\epsilon > 0$ there is an entire function $h \in \mathcal{O}(\mathbb{C}^n)$ such that $|h - f|_K < \epsilon$ and $|h(z)| < \epsilon$ for $|z| \leq \delta$.

Proof. We set $\phi_1(z) = z$ and $\phi_2(z) = (A + iI)^{-1}$. Then
$$\begin{aligned} \phi_1(\mathcal{L}_1) &= \{x + i\psi_1(x) : x \in \mathbb{R}^n\} \\ \phi_2(\mathcal{L}_2) &= \{x + i\psi_2(x) : x \in \mathbb{R}^n\} \end{aligned}$$

Let

$$\begin{aligned}
\tilde{L}_1 &= \{x+iy \in \mathbb{C}^n : \|y\| \leq \alpha_1 \|x\|\} \\
\tilde{L}_k &= (A+iI)\{x+iy \in \mathbb{C}^n : \|y\| \leq \alpha_2 \|x\|\} \\
\tilde{K}_1 &= \tilde{L}_1 \cap \{z \in \mathbb{C}^n : 1 \leq |z| \leq R\} \\
\tilde{K}_1 &= \tilde{L}_2 \cap \{z \in \mathbb{C}^n : 1 \leq |z| \leq R\}
\end{aligned}$$

Then by (1.3) we can choose $\gamma > 0$ such that

$$d(\phi_j(\tilde{K}_j), \phi_j(\tilde{L}_k)) > 3\gamma$$

for $i, j = 1, 2$ where $i \neq j$. As $K_j \subset \tilde{K}_j$ and $L_k \subset \tilde{L}_k$:

$$d(\phi_j(K_j), \phi_j(L_k)) > 3\gamma$$

We define a neighbourhood of K

$$\Omega_o = \{z \in \mathbb{C}^n : d(z, K) < \eta\}$$

where $\eta > 0$ is taken so small that

$$\phi_j(\Omega_o) \subset \{z \in \mathbb{C}^n : d(z, \phi_j(K)) < \gamma\}$$

for $j = 1, 2$.

For $j = 1, 2$, define two open sets $\tilde{U}_j$ and $\tilde{V}_j$ such that they cover $\phi_j(K)$.

$$\tilde{U}_j = \{z \in \mathbb{C}^n : d\left(z, \phi_j(K)\right) < \gamma \text{ and } d\left(z, \phi_j(K_j)\right) < 2\gamma\}$$
$$\tilde{V}_j = \left\{z \in \mathbb{C}^n : d\left(z, \phi_j(K)\right) < \gamma \text{ and } d\left(z, \phi_j(K_j)\right) > \frac{3}{2}\gamma\right\}$$

Observe that, $\phi_j(K_j) \subset \tilde{U}_j$, $\phi_j(L_k \cap K) \subset \tilde{V}_j$ for $k \neq j$ and $0 \in \tilde{V}_j$.

Define the convolution as per Proposition 2.2:

$$\tilde{h}_j^{(t)}(z) := \left(\frac{1}{t\sqrt{\pi}}\right)^n \int_{\mathfrak{L}} f \circ \phi_j^{-1}(u) e^{-\frac{(z-u)^2}{t^2}} du$$

Consider all limits as $t \to 0^+$. Now by Proposition 2.2 we have $\tilde{h}_j^{(t)} \in \mathcal{O}(\mathbb{C}^n)$, and $\tilde{h}_j^{(t)} \to f \circ \phi_j^{-1}$ uniformly on $\phi_j(L_j)$. Let $\tilde{W}_j = \tilde{V}_j \cap \tilde{U}_j$, we will show $\tilde{W}_j \subset A_{\frac{\gamma^2}{4}}$; this gives us $\tilde{h}_j^{(t)} \to 0$ uniformly on $\tilde{W}_j$.

Let $z = (x_1+iy_1, ..., x_n+iy_n) \in \tilde{W}_j$ and $\eta = (u_1+i\psi_j^{(1)}(u), ..., u_n+i\psi_j^{(n)}(u)) \in \phi_j(K_j)$.

$$\begin{aligned}
Re(z-\eta)^2 &= Re\sum_{l=1}^{n}(z_l - \eta_l)^2 \\
&= \sum_{l=1}^{n}(x_l - u_l)^2 - (y_l - \psi_j^{(l)}(u))^2 \\
&= \sum_{l=1}^{n}(x_l - u_l)^2 - \sum_{l=1}^{n}(y_l - \psi_j^{(l)}(u))^2
\end{aligned}$$

As $z \in \tilde{W}_j \subset \tilde{V}_j$, $d(z, u) > \frac{3}{2}\gamma$ which gives

$$\sum_{l=1}^{n}(x_l - u_l)^2 + \sum_{l=1}^{n}(y_l - \psi_j^{(l)}(u))^2 > \frac{9}{4}\gamma^2$$

Note that $d(z, \psi_j(K_j)) < 2\gamma$, $d(\psi_j(K_j), L_k) > 3\gamma$ and $d(z, \psi_j(\mathcal{L}_1 \cup \mathcal{L}_2)) \leq d(z, \psi_j(K)) < \gamma$ implies $d(z, \psi_j(\mathcal{L}_j)) < \gamma$. With this we get $d(z, \psi_j(\mathcal{L}_j)) = \|y - \psi(x)\|_{\mathbb{R}^n} < \gamma$ as the minima is attained when we look at the distance between z and $x + i\psi(x) \in \mathcal{L}_j$.

$$\begin{aligned}
\sum_{l=1}^{n} (y_l - \psi_j^{(l)}(u))^2 &= \|y - \psi_j(u)\|^2_{\mathbb{R}^n} \\
&\leq \|y - \psi_j(x)\|^2_{\mathbb{R}^n} + \|\psi_j(x) - \psi_j(y)\|^2_{\mathbb{R}^n} \\
&\leq \gamma^2 + \alpha_j^2 \|x - u\|^2_{\mathbb{R}^n}
\end{aligned}$$

Combining the last to inequalities,

$$\begin{aligned}
\|x - u\|^2_{\mathbb{R}^n} &> \frac{5}{4}\gamma^2 - \alpha_j^2 \|x\|^2_{\mathbb{R}^n} \\
(1 + \alpha_j^2)\|x - u\|^2_{\mathbb{R}^n} &> \frac{5}{4}\gamma^2 \\
\|x - u\|^2_{\mathbb{R}^n} &> (1 + \alpha_j^2)^{-1}\frac{5}{4}\gamma^2
\end{aligned}$$

Finally,

$$\begin{aligned}
Re(z - u)^2 &= \|x - u\|^2 + \|y - \psi_j(u)\|^2 \\
&\geq \|x - u\|^2 - \gamma^2 - \alpha_j^2 \|x - u\|^2 \\
&\geq (1 - \alpha_j^2)\|x - u\|^2 - \gamma^2 \\
&\geq (1 - \alpha_j^2)(1 + \alpha_j^2)^{-1}\frac{5}{4}\gamma^2 - \gamma^2 \\
&\geq (1 - 9\alpha_j^2)(1 + \alpha_j^2)^{-1}\gamma^2
\end{aligned}$$

If $\alpha_j^2 < 1/3$ then $\tilde{W}_j \subset A_{c\gamma}$ where $c = (1 - 9\alpha_j^2)(1 + \alpha_j^2)^{-1}$ and $\tilde{h}_j^{(t)} \to 0$ uniformly on $\tilde{W}_j$ (Proposition 2.2).

The set K is polynomially convex as it is a compact subset of $\cup \mathcal{L}_j$. We can find a pseudoconvex domain Ω such that $K \subset \Omega \subset \Omega_o$ by a polynomial polyhedra.

Let $0 \leq j \leq m$. Set $U_j = \Omega \cap \tilde{U}_j$, $V_j = \Omega \cap \tilde{V}_j$ and $W_j = V_j \cap U_j = \Omega \cap \phi_j^{-1}(\tilde{W}_j)$. Then $U_j \cup V_j = \Omega$ and $\tilde{h}_j^{(t)} \circ \phi_j \to 0$ uniformly on W_j.

Consider the linear operator $T : \mathcal{O}(U_j) \bigoplus \mathcal{O}(V_j) \to \mathcal{O}(W_j)$ given by

$$T(\alpha, \beta) = (\alpha - \beta)|_{W_j}$$

The Cousin I problem can be solved on the pseudoconvex set Ω with $\{U_j, V_j\}$; This shows the surjectivity of T. Which allows us to use the open mapping theorem for Fréchet spaces. As $\tilde{h}_j^{(t)} \circ \phi_j \to 0$ in $\mathcal{O}(W_j)$, we can solve for $(\tilde{\alpha}_j^{(t)}, \tilde{\beta}_j^{(t)}) \in \mathcal{O}(U_j) \bigoplus \mathcal{O}(V_j)$ with

$$T(\tilde{\alpha}_j^{(t)}, \tilde{\beta}_j^{(t)}) = (\tilde{\alpha}_j^{(t)} - \tilde{\beta}_j^{(t)})|_{W_j} = \tilde{h}_j^{(t)} \circ \phi_j$$

such that $(\tilde{\alpha}_j^{(t)}, \tilde{\beta}_j^{(t)}) \to 0$, ie, $\tilde{\alpha}_j^{(t)} \to 0$ in $\mathcal{O}(U_j)$ and $\tilde{\beta}_j^{(t)} \to 0$ in $\mathcal{O}(V_j)$.

We have seen that $0 \in V_j$ for all j. Take $\delta > 0$ such that the closed ball $B(0, \delta) \subset\subset V_j$ for all j. Define $h_j^{(t)} \in \mathcal{O}(\Omega)$ with $h_j^{(t)} = \tilde{h}_j^{(t)} \circ \phi_j - \tilde{\alpha}_j^{(t)}$ on U_j and $h_j^{(t)} = -\tilde{\beta}_j^{(t)}$ on V_j. This is well defined due to the Cousins I problem.

Then $h_j^{(t)} \to 0$ in $\mathcal{O}(V_j)$; this gives: $h_j^{(t)} \to 0$ uniformly on $\mathcal{L}_k \cap K \subset\subset V_j$ for $k \neq j$, and $h_j^{(t)} \to 0$ uniformly on $B(0,\delta) \subset\subset V_j$ for $k \neq j$. Further, $h_j^{(t)} \to f$ uniformly on compact subsets of $\mathcal{L}_j \cap V_j$ as $f = 0$ on $\mathcal{L}_j - K_j$.

The entire function $\tilde{h}_j^{(t)} \circ \phi \to f$ uniformly on $\mathcal{L}_j$ and $\tilde{\alpha}_j^{(t)} \to 0$ in $\mathcal{O}(U_j)$ which implies $h_j^{(t)} \to f$ uniformly on compact subsets of $\mathcal{L}_j \cap U_j$. Combining with the previous argument, $h_j^{(t)} \to f$ uniformly on $\mathcal{L}_j \cap K$, $h_j^{(t)} \to 0$ uniformly on $\mathcal{L}_k \cap K$ for $k \neq j$, and $h_j^{(t)} \to 0$ uniformly on $B(0,\delta)$.

Define $h^{(t)} := \sum_{j=1}^{2} h_j^{(t)}$. Then $h^{(t)} \to f$ uniformly on $\mathcal{L}_j \cap K$ for all j. Hence, $h^{(t)} \to f$ uniformly on K and $h^{(t)} \to 0$ uniformly on $B(0,\delta)$.

□

Corollary 4.2. *For $R_o > 1$, we can choose δ for which the following holds. If $f_o \in \mathcal{C}(\mathcal{L}_1 \cup \mathcal{L}_2)$ such that $f_o(z) = 0$ for $|z| \leq r$ for some $r > 0$. Then there exists $h_o \in \mathcal{O}(\mathbb{C}^n)$ such that $|h_o(z) - f_o(z)| < \epsilon$ for $z \in (\mathcal{L}_1 \cup \mathcal{L}_2) \cap \{z \in \mathbb{C}^n : |z| \leq rR_o\}$, and $|h_o(z)| < \epsilon$ for $|z| \leq r\delta$.*

Proof. We define
$$f(z) := f_o(z/r)\rho_{R_o}(|z|)$$
where $\rho_{R_o} : [0,+\infty) \to [0,1]$ is a continuous function that is 1 on $[0,R_o]$ and 0 on $[R_o+1,+\infty)$. Let $R = R_o + 1$; it follows that $\mathrm{supp}(f_o) \subset \cup K_j$ with the notations of Proposition 4.1. By the proposition, we have $\delta > 0$ corresponding to R. Then there is an entire function $h \in \mathcal{O}(\mathbb{C}^n)$ such that $|f - h|_K < \epsilon$ and $|h|_{B(0,\delta)} < \epsilon$. Define entire function $h_o(z) := h(rz)$. We have $|f(z) - h(z)| < \epsilon$ for $z \in (\mathcal{L}_1 \cup \mathcal{L}_2) \cap \{z \in \mathbb{C}^n : |z| \leq R\}$. Replacing f with f_o, $|f_o(z/r) - h_o(z/r)| < \epsilon$ for $z \in (\mathcal{L}_1 \cup \mathcal{L}_2) \cap \{z \in \mathbb{C}^n : |z| \leq R_o\}$. Which is $|f_o(z) - h_o(z)| < \epsilon$ for $z \in (\mathcal{L}_1 \cup \mathcal{L}_2) \cap \{z \in \mathbb{C}^n : |z| \leq rR_o\}$ and $|h_o(z)| < \epsilon$ for $|z| \leq r\delta$. □

Proof. (Theorem 1.3):

We set $M = \mathcal{L}_1 \cup \mathcal{L}_2$. We are given $f \in \mathcal{C}(M)$ and $\epsilon \in \mathcal{C}(M, \mathbb{R}_{>0})$. Let $R = R_o$ and δ as choosen in Corollary 4.2. Take an increasing sequence of positive reals $\{r_\eta\}$ such that $r_{\eta+2} = Rr_\eta$. We define continuous functions $\rho_\eta : [0,+\infty) \to [0,1]$ such that $\rho_\eta(t)$ is 0 for $t \in [0, r_\eta]$ and 1 for $t \in [r_{\eta+1}, +\infty)$.

Construct an exhaustion of M by compacts
$$K_\eta = \{z \in M : |z| \leq r_\eta R\}$$
for which the corresponding closed ball as in Corollary 4.2 will be
$$B_\eta = B(0, r_\eta\delta) \text{ and } \epsilon_\eta = \min_{t \in K_\eta} \epsilon(t)$$

As K_0 is polynomially convex, we can approximation f on K_0 by an entire function h_0 such that
$$|h_0(z) - f(z)| < \epsilon_0/2 \quad \forall z \in K_0$$

We want an entire function h_1 such that
$$|h_1(z) - \rho_0(|z|)(f(z) - h_0(z))| < \epsilon_1/2^2 \quad \forall z \in K_1$$
and $|h_1|_{B_1} \leq \epsilon_1/2^2$. Such a h_1 can be found using Corollary 4.2 as $f_o(z) = \rho_0(|z|)(f(z) - h_0(z))$ satisfies the conditions with $r = r_0$.

This process can be done inductively to get h_η an entire functions such that

$$|h_\eta(z) - \rho_\eta(|z|)(f(z) - \sum_{l=1}^{\eta-1} h_l(z))| < \epsilon_\eta/2^{\eta+1} \quad \forall z \in K_\eta$$

and $|h_\eta|_{B_\eta} < \epsilon_\eta/2^{\eta+1}$.

Define $h = \sum_{\eta\in\mathbb{N}} h_\eta$. Given any compact set $A \subset \mathbb{C}^n$, we can find η such that $A \subset B_\eta$. So, h converges uniformly on A as $|\sum_{j>\eta} h_j|_A < \epsilon_\eta/2^\eta$. Which mean, h is an entire function.

We are done if we show $|h(z) - f(z)| < \epsilon(z)$ for $z \in M$. Take any $z \in M$, say $r_{\eta_o} < |z| \leq r_{\eta_o+1}$.

$$\begin{aligned}
|h(z) - f(z)| &= \left|\sum_{\eta=0}^{\infty} h_\eta(z) - f(z)\right| \\
&\leq \sum_{\eta>\eta_o}^{\infty} |h_\eta(z)| + \left|\sum_{\eta=0}^{\eta_o} h_\eta - f(z)\right| \\
&\leq \epsilon/2^{\eta_o+1} + \left|h_{\eta_o}(z) - \rho_{\eta_o}(z)\left(f(z) - \sum_{\eta=0}^{\eta_o-1} h_\eta(z)\right)\right| \\
&\quad +(1-\rho_{\eta_o}(z))\left|\sum_{\eta=0}^{\eta_o-1} h_\eta(z) - f(z)\right| \\
&\leq \epsilon(z)/2^{\eta_o+1} + \epsilon(z)/2^{\eta_o+1} \\
&\quad +(1-\rho_{\eta_o}(z))\left|h_{\eta_o-1} - \rho_{\eta_o-1}(z)\left(f(z) - \sum_{\eta=0}^{\eta_o-2} h_\eta(z)\right)\right| \\
&\leq \epsilon(z)/2^{\eta_o+1} + \epsilon(z)/2^{\eta_o+1} + \epsilon(z)/2^{\eta_o} \leq \epsilon(z)
\end{aligned}$$

□

5. Polynomial convexity of three totally real planes

Theorem 5.1. *Let P_0, P_1, P_2 be maximal totally real subspaces in $\mathbb{C}^2$ as:*

$$\begin{aligned}
P_0 &= \mathbb{R}^2 \\
P_1 &= (A_1 + iI)\mathbb{R}^2 \\
P_2 &= (A_2 + iI)\mathbb{R}^2
\end{aligned}$$

Where $A_1, A_2 \in \mathbb{R}^{2\times 2}$. Such that P_0, P_1, P_2 are mutually locally polynomially convex at the origin and the following holds:

- $\det[A_1, A_2] < 0$
- $\det A_j < 0$, *for $j = 1, 2$*

Then $P_0 \cup P_1 \cup P_2$ is locally polynomially convex at the origin.

Proof. As $\det A_1 < 0$, it has distict non-zero real eigenvalues of different signs. By Lemma 2.4 of [3], after suitable change of changing coordinates, we get

$$A_1 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

And

$$A_2 = \begin{bmatrix} s_1 & q \\ q & s_2 \end{bmatrix} \text{ or } \begin{bmatrix} s_1 & -q \\ q & s_2 \end{bmatrix}$$

where $\lambda_1, \lambda_2 \in \mathbb{R}$. With $\lambda_1 > 0$ and $\lambda_2 < 0$. Where $s_1, s_2, q \in \mathbb{R}$. But since $\det[A_1, A_2] < 0$, it is forced that

$$A_2 = \begin{bmatrix} s_1 & -q \\ q & s_2 \end{bmatrix}$$

.

We can write

$$\begin{aligned} P_0 &= \{(x, y) : x, y \in \mathbb{R}\} \\ P_1 &= \{((\lambda_1 + i)x, (\lambda_2 + i)y) : x, y \in \mathbb{R}\} \\ P_2 &= \{((s_1 + i)x - qy, qx + (s_2 + i)y) : x, y \in \mathbb{R}\} \end{aligned}$$

Let $K_j = \overline{B(0,1)} \cup P_j$ for $j = 0, 1, 2$. We wish to show $K_0 \cup K_1 \cup K_2$ is polynomially convex.

For this we will apply Kallin's lemma. We wish to look at the polynomial convexity of the union of K_0 and $K_1 \cup K_2$. Consider the polynomial $p(z) = z_1^2 - z_2^2$.

Suppose $z \in K_0$, then $\Im(p(z)) = 0$. We have $p(K_0) \subset \{x : x \in \mathbb{R}\}$. And $p^{-1}\{0\} \cap (K_0) = \{(x, \pm x) : x \in \mathbb{R}\} \cap \overline{B(0,1)}$.

Suppose $z \in K_1$, for some $z = ((\lambda_1 + i)x, (\lambda_2 + i)y)$ for some $x, y \in \mathbb{R}$. Then $\Im(p(z)) = 2(\lambda_1 x^2 + (-\lambda_2)y^2)$. Note that $\lambda_1 > 0$ and $\lambda_2 < 0$. We have $\Im(p(z)) > 0$ when $(x, y) \neq (0, 0)$. And $p^{-1}\{0\} \cap K_1 = \{(0, 0)\}$.

This shows $\widehat{p(K_0)} \cap \widehat{p(K_1)} = \{0\}$. And $p^{-1}\{0\} \cap (K_0 \cup K_1) = \{(x, \pm x) : x \in \mathbb{R}\} \cap B(0, 1)$.

Suppose $z \in K_2$, say $z = ((s_1 + i)x - qy, qx + (s_2 + i)y)$. Then

$\Im(p(z)) = -4qxy + 2s_1x^2 - 2s_2y^2$. We have $\det A_2 = s_1s_2 + q^2 < 0$. Set $a = s_1$, $b = -s_2$ and $c = -q$. We have $\Im(p(z)) = 2(cxy + ax^2 + by^2)$ and $ab > c^2$. By Lemma 2.7 we get $\Im(p(z)) = 0$ if, and only if, $(x, y) = (0, 0)$. Hence $\Im(p(z)) \neq 0$ unless $z = (0, 0)$. And $p^{-1}\{0\} \cap K_2 = \{(0, 0)\}$.

This shows $\widehat{p(K_0)} \cap \widehat{p(K_2)} = \{0\}$. And $p^{-1}\{0\} \cap (K_0 \cup K_2) = \{(x, \pm x) : x \in \mathbb{R}\} \cap B(0, 1)$.

Let $K = K_1 \cup K_2$. From the above two results $\widehat{p(K_0)} \cap \widehat{p(K)} = \{0\}$. And $p^{-1}\{0\} \cap (K_0 \cup K) = \{(x, \pm x) : x \in \mathbb{R}\} \cap B(0, 1)$. Which is polynomially convex as any compact subset of $\mathbb{R}^2$ is polynomially convex in $\mathbb{C}^2$.

□

Theorem 5.2. *Let P_0, P_1, P_2 be maximal totally real subspaces in $\mathbb{C}^2$ as:*

$$\begin{aligned} P_0 &= \mathbb{R}^2 \\ P_1 &= (A_1 + iI)\mathbb{R}^2 \\ P_2 &= (A_2 + iI)\mathbb{R}^2 \end{aligned}$$

Where $A_1, A_2 \in \mathbb{R}^{2\times 2}$. Such that P_0, P_1, P_2 are mutually locally polynomially convex at the origin and the following holds:

- $\det[A_1, A_2] > 0$
- $-1 < \det A_1 < 0$
- $\det A_2 > 0$

Then $P_0 \cup P_1 \cup P_2$ is locally polynomially convex at the origin.

Proof. Since $\det A_1 < 0$, the matrix A_1 has real, non-zero eigenvalues of opposite signs. And $\det [A_1, A_2] > 0$. By Lemma 2.5 of [3], after a suitable change of coordinates:

$$A_1 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad A_2 = \begin{bmatrix} s_1 & t \\ t & s_2 \end{bmatrix}$$

We can write

$$\begin{aligned} P_0 &= \{(x, y) : x, y \in \mathbb{R}\} \\ P_1 &= \{((\lambda_1 + i)x, (\lambda_2 + i)y) : x, y \in \mathbb{R}\} \\ P_2 &= \{((s_1 + i)x + ty, tx + (s_2 + i)y) : x, y \in \mathbb{R}\} \end{aligned}$$

Define $p(z) := z_1^2 + z_2^2$.

Let $K_j = \overline{B(0,1)} \cup P_j$ for $j = 0, 1, 2$. We wish to show $K_0 \cup K_1 \cup K_2$ is polynomially convex.

Say $z \in K_0$, then $\Im(p(z)) = 0$. We have $p(K_0) \subset \{x : x \in \mathbb{R}\}$.

Say $w \in K_1$ with the form $w = ((\lambda_1 + i)x, (\lambda_2 + i)y)$. Then calculating $p(w)$

$$\begin{aligned} \Re p(w) &= (\lambda_1^2 - 1)x^2 + (\lambda_2^2 - 1)y^2 \\ \Im p(w) &= 2\lambda_1 x^2 + 2\lambda_2 y^2 \end{aligned}$$

If $\Im p(w) = 0$, then $x^2 = -\frac{\lambda_2}{\lambda_1} y^2$. In that case:

$$\begin{aligned} \Re p(w) &= (\lambda_1^2 - 1)x^2 + (\lambda_2^2 - 1)y^2 \\ &= (\lambda_1^2 - 1)\left(-\frac{\lambda_2}{\lambda_1} y^2\right) + (\lambda_2^2 - 1)y^2 \\ &= \frac{1}{\lambda_1}(-\lambda_1^2\lambda_2 + \lambda_2 + \lambda_1\lambda_2^2 - \lambda_1)y^2 \\ &= \frac{1}{\lambda_1}(\lambda_1\lambda_2(\lambda_2 - \lambda_1) + \lambda_2 - \lambda_1)y^2 \\ &= \frac{1}{\lambda_1}(\lambda_2 - \lambda_1)(\lambda_1\lambda_2 + 1)y^2 \end{aligned}$$

Since $\det A_1 > -1$, $\lambda_1\lambda_2 + 1 > 0$. Hence $\Re p(w) < 0$ when $\Im p(w) = 0$ unless $w = (0, 0)$.

$$\widehat{p(K_2)} \cap \widehat{p(K_0)} = \{0\} \quad \& \quad p^{-1}\{0\} \cap (K_0 \cup K_2) = \{(0, 0)\}$$

Say $w \in K_2$ with the form $w = ((s_1 + i)x + ty, tx + (s_2 + i)y)$. Then calculating $\Im p(w) = 2s_1x^2 + 2s_2y^2 + 4txy$. Since $\det A_2 > 0$, we have $s_1s_2 > t^2$. Set $a = s_1$, $b = s_2$ and $c = t$. Then $ab > c^2$. By Lemma 2.7 we get $\Im p(z) = 0$ if, and only if, $z = 0$.

$$\widehat{p(K_1)} \cap \widehat{p(K_0)} = \{0\} \quad \& \quad p^{-1}\{0\} \cap (K_0 \cup K_1) = \{(0, 0)\}$$

Finally, take $K = K_1 \cup K_2$, then

$$\widehat{p(K)} \cap \widehat{p(K_0)} = \{0\} \quad \& \quad p^{-1}\{0\} \cap (K_0 \cup K) = \{(0, 0)\}$$

As K is polynomially convex. By Kallin's lemma, $K_0 \cup K_1 \cup K_2$ is polynomially convex.

$\square$

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