Cartan-Hadamard Theorem

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1 Introduction

There is no metric S^2 which has non-positive sectional curvature. There are topological restrictions on which manifolds can carry metrics with non-positive sectional curvature.

Theorem 1.1 (The Cartan-Hadamard Theorem)**.** *If M is a connected manifold that is complete and all of whose sectional curvatures are non-positive, then the exponent map*

$$\exp_p : T_pM \to M$$

is a covering map for any point $p \in M$. In particular, the universal covering space of M is diffeomorphic to $\mathbb{R}^n$.

Further, if M is simply connected, then M is diffeomorphic to $\mathbb{R}^n$.

2 Jacobi Fields

A Jacobi field is a vector field along a geodesic that characterizes a one parameter family of geodesics. We will see that a one parameter family of geodesics satisfies the Jacobi equation (2), and conversely, any vector field that satisfies the Jacobi equation (2) will correspond to a one parameter family of geodesics.

2.1 Preliminaries

We will begin by defining family of curves and its variation.

Definition 2.1. A one parameter family of curves

$$\Gamma : (-\epsilon, \epsilon) \times [a, b] \to M$$

is called a smooth *admissible family* if Γ is smooth.

Given an admissible curve $\gamma : [a, b] \to M$, a *variation* of γ is an admissible family Γ such that $\Gamma_0(t) = \Gamma(0, t) = \gamma(t)\ \forall t \in [a, b]$. Then, the *variation field* of $\Gamma(s, t)$ is the vector field $V(t) = \partial_s \Gamma(0, t)$ along γ.

We say Γ is a *variation through geodesics* if each of the curves $\Gamma_s(t) = \Gamma(s, t)$ is a geodesic for $s \in (-\epsilon, \epsilon)$.

Define the following vector fields on Γ:

$$T(s,t) = \partial_t \Gamma(s,t),$$

$$S(s,t) = \partial_s \Gamma(s,t).$$

If Γ is a variation through geodesics, the geodesic equations gives

$$D_t T \equiv 0.$$

Further, taking the covariant derivative along s gives

$$D_s D_t T \equiv 0. \tag{1}$$

Lemma 2.2 (Symmetry Lemma)**.** *Let $\Gamma : (-\epsilon, \epsilon) \times [a,b] \to M$ be a smooth admissible family of curves. Then*

$$D_s \partial_t \Gamma = D_t \partial_s \Gamma.$$

Proof. We will compute this using coordinates (x^k) and let $\partial_k = \frac{\partial}{\partial x^k}$. In this coordinates, let $\Gamma(s,t) = (f^1(s,t), ..., f^n(s,t))$. Then

$$\partial_t \Gamma = \sum_k \frac{\partial f^k}{\partial t} \partial_k,$$

$$\partial_s \Gamma = \sum_k \frac{\partial f^k}{\partial s} \partial_k.$$

Applying D_s and using the Christoffel symbols identity $\Gamma^i_{jk} = \Gamma^i_{kj}$:

$$\begin{aligned}
D_s \partial_t \Gamma &= \sum_k D_s \left(\frac{\partial f^k}{\partial t} \partial_k \right) \\
&= \sum_k \frac{\partial^2 f^k}{\partial s \partial t} \partial_k + \sum_k \frac{\partial f^k}{\partial t} D_s \left(\partial_k \right) \\
&= \sum_k \frac{\partial^2 f^k}{\partial s \partial t} \partial_k + \sum_{ijk} \frac{\partial f^k}{\partial t} \frac{\partial f^j}{\partial s} \Gamma^i_{jk} \partial_i \\
&= \sum_k \frac{\partial^2 f^k}{\partial s \partial t} \partial_k + \sum_{ijk} \frac{\partial f^j}{\partial s} \frac{\partial f^k}{\partial t} \Gamma^i_{kj} \partial_i \\
&= \sum_j \frac{\partial^2 f^j}{\partial t \partial s} \partial_j + \sum_j \frac{\partial f^j}{\partial s} D_t \left(\partial_j \right) \\
&= \sum_j D_t \left(\frac{\partial f^j}{\partial t} \partial_j \right) = D_t \partial_s \Gamma.
\end{aligned}$$

□

We will look at the commutation of covariant derivatives. Since these are double derivatives, we get a relation with curvature:

Lemma 2.3. *Let Γ be a smooth admissible family of curves, and V is a smooth vector field along Γ, then*

$$D_s D_t V - D_t D_s V = R(S,T)V.$$

Proof. We will compute this using coordinates (x^k) and let $\partial_k = \frac{\partial}{\partial x^k}$. Write $V(s,t) = \sum_i V^i(s,t)\partial_i$. Then

$$D_t V = \sum_i \frac{\partial V^i}{\partial t}\partial_i + V^i D_t \partial_i.$$

Applying D_s:

$$D_s D_t V = \sum_i \frac{\partial^2 V^i}{\partial s \partial t}\partial_i + \frac{\partial V^i}{\partial t} D_s \partial_i + \frac{\partial V^i}{\partial t} D_t \partial_i + V^i D_s D_t \partial_i.$$

Similarly,

$$D_t D_s V = \sum_i \frac{\partial^2 V^i}{\partial s \partial t}\partial_i + \frac{\partial V^i}{\partial t} D_t \partial_i + \frac{\partial V^i}{\partial t} D_s \partial_i + V^i D_t D_s \partial_i.$$

The difference gives only the last terms:

$$D_s D_t V - D_t D_s V = \sum V^i (D_s D_t \partial_i - D_t D_s \partial_i).$$

Let $\Gamma(s,t) = (f^1(s,t), ..., f^n(s,t))$. Then

$$T = \sum_k \frac{\partial f^k}{\partial t}\partial_k; \quad S = \sum_k \frac{\partial f^k}{\partial s}\partial_k.$$

and

$$D_t \partial_i = \sum_k \frac{\partial f^j}{\partial t}\nabla_{\partial_j}\partial_i; \quad D_s \partial_i = \sum_k \frac{\partial f^j}{\partial s}\nabla_{\partial_j}\partial_i.$$

We can take the second covariant now:

$$D_s D_t \partial_i = \sum_k \frac{\partial^2 f^j}{\partial s \partial t}\nabla_{\partial_j}\partial_i + \sum_{kj} \frac{\partial f^j}{\partial t}\frac{\partial f^k}{\partial s}\nabla_{\partial_k}\nabla_{\partial_j}\partial_i$$

Taking the difference, the first terms cancel:

$$\begin{aligned} D_s D_t \partial_i - D_t D_s \partial_i &= \sum_{kj} \frac{\partial f^j}{\partial t}\frac{\partial f^k}{\partial s}(\nabla_{\partial_k}\nabla_{\partial_j}\partial_i - \nabla_{\partial_j}\nabla_{\partial_k}\partial_i) \\ &= \sum_{kj} \frac{\partial f^j}{\partial t}\frac{\partial f^k}{\partial s} R(\partial_k, \partial_j)\partial_i \\ &= R(S,T)\partial_i \end{aligned}$$

Hence,

$$D_s D_t V - D_t D_s V = R(S,T)V.$$

□

2.2 The Jacobi Equation

Theorem 2.4. *(The Jacobi Equation) Let γ be a geodesic and V be a vector field along γ. If V is the variation field of a variation through geodesics, then V satisfies*

$$D_t^2 V + R(V, \dot{\gamma})\dot{\gamma} = 0. \tag{2}$$

Proof. Recall equation (1) for a variation through geodesics: $D_s D_t T = 0$. Apply lemma 1 to get:

$$0 = D_s D_t T = D_t D_s T + R(S,T)T$$

Now by symmetry lemma 2.2, we have $D_s T = D_t S$. Therefore,

$$D_t^2 S + R(S,T)T = 0.$$

Evaluate at $s = 0$. Then, $S(0,t) = V(t)$ and $T(0,t) = \dot{\gamma}(t)$. This gives the Jacobi equation:

$$D_t^2 V + R(V, \dot{\gamma})\dot{\gamma} = 0.$$

□

Definition 2.5 (Jacobi field)**.** A vector field X along a geodesic γ is called a *Jacobi field* if it satisfies the Jacobi equation:

$$D_t^2 V + R(V, \dot{\gamma})\dot{\gamma} = 0.$$

We have seen that every variation field of a variation through geodesics satisfies the Jacobi equation (2). Conversely, we have:

Proposition 2.6. *Every Jacobi field along a geodesic γ is the variation field of some variation of γ through geodesics.*

Before we look at the proof of the lemma, we need some machinery: A Jacobi field is completely determined by its value and covariant derivative a point of the geodesic.

Proposition 2.7 (Existence and Uniqueness of Jacobi Fields)**.** *Let $\gamma : I \to M$ be a geodesic, $a \in I$ and $p = \gamma(a)$. For any pair of vectors $X, Y \in T_pM$, there is a unique Jacobi field J along γ satisfying the initial conditions*

$$J(a) = X; \quad D_t J(a) = Y.$$

Proof. Let $\{E_i\}$ be an orthonormal frame for T_pM. Write $J(t) = \sum_i J^i(t)E_i$. We can express the Jacobi equation with respect to this orthonormal frame as

$$\ddot{J}^i + \sum_{jkl} R^i_{jkl} J^j \dot{\gamma}^k \dot{\gamma}^l = 0.$$

Set $V^i = \dot{J}^i$ to get a system of $2n$ unknowns $\{J^i, V^i\}$. And $2n$ first-order linear equations: for $i = 1, ..., n$

$$\dot{V}^i + \sum_{jkl} R^i_{jkl} J^j \dot{\gamma}^k \dot{\gamma}^l = 0.$$

and $V^i = \dot{J}^i$. Then by the existence and uniqueness of first-order linear system, we have a solution on the whole interval I with any initial conditions $J^i(a) = X^i$ and $V^i(a) = Y^i$. □

Proof of Proposition 2.6. Let $\gamma : [a,b] \to M$ be a geodesic segment joining p and q. And J be a Jacobi field along γ. Let $\sigma : (-\epsilon, \epsilon) \to M$ be the geodesic with $\sigma(0) = p$ and $\dot{\sigma}(0) = J(0)$. Let $X(s)$ be the parallel vector field along σ with $X(0) = \dot{\gamma}(0)$, and let $Y(s)$ be the parallel vector field along σ with $Y(0) = D_t J(0)$. Consider the following variation through geodesics

$$\Gamma(s,t) = \exp_{\sigma(s)}(t(X(s) + sY(s))).$$

Fixing s produces a geodesic. And $\Gamma(0,t) = \gamma(t)$ as $\Gamma(0,t) = \exp_{\sigma(0)} tX(0)$ is a geodesic starting at p in the direction of $\partial_t \Gamma(0,t) = X(0) = \dot{\gamma}(0)$.

Let V be the variation of Γ, i.e., $V(t) = \partial_s \Gamma|_{(0,t)}$. We want to show $V = J$. We show this using the uniqueness of Jacobian field (Proposition 2.7). To this end, we will show $J(0) = V(0)$ and $D_t J(0) = D_t V(0)$.

The variation V at 0 will be

$$V(0) = \partial_s \Gamma(s,0)|_{s=0} = \partial_s \exp_{\sigma(s)}(0)|_{s=0} = \partial_s \sigma(s)|_{s=0} = \dot{\sigma}(0) = J(0)$$

By the symmetry lemma 2.2, we have $D_t V = D_s T$ where $T = \partial_t \Gamma(s,t)$. We compute

$$T(s,0) = \partial_t \Gamma(0,s) = (\exp_{\sigma(s)})_{*,0} \frac{d}{dt}(t(X(s) + sY(s)))|_{t=0} = X(s) + sY(s),$$

by using the fact that $(\exp_{\sigma(s)})_{*,0}$ is identity (with the usual identification of the vector space of T_pM with itself.

$$(D_s T)(0,0) = (D_s T(s,0))_{t=0} = (D_s(X(s) + sY(s)))_{t=0} = Y(0) = D_t J(0)$$

as X and Y are parallel transports along $\sigma(s)$, we get $D_s X \equiv D_s Y \equiv 0$.

Therefore, $D_t V(0) = D_s T(0,0) = D_t J(0)$. And by uniqueness of Jacobi field, $V = J$ along γ. □

3 Conjugate Points

Definition 3.1 (Conjugate points)**.** Let γ be a geodesic segment joining $p, q \in M$. We say q is *conjugate* to p along γ if there is a nontrivial Jacobi field along γ that vanishes at p and q.

Geodesics meet at a conjugate point. There exists a one parameter family of geodesic between conjugate points.

Conjugate points are precisely the images of singularities of the exponential map:

Proposition 3.2. *Suppose $p \in M$, $V \in T_pM$, and $q = \exp_p V$. Then $\exp_p$ is a local diffeomorphism in a neighbourhood of V if (and only if) q is not conjugate to p along the geodesic $\gamma(t) = \exp_p tV$, $t \in [0,1]$.*

Proof. By the inverse function theorem, the map $\exp_p$ is a local diffeomorphism near V if its derivative $(\exp_p)_*$ is an isomorphism at V. As the dimension of the spaces are the same, it suffices for $(\exp_p)_*$ to be injective at V.

As $T_pM \cong \mathbb{R}^n$, we can identify $T_V(T_pM)$ with T_pM. We can compute the derivative $(\exp_p)_{*,V}$ at V with the curve $s \mapsto V + sW$:

$$(\exp_p)_{*,V} W = \left.\frac{d}{ds}\right|_{s=0} \exp_p(V + sW),$$

where $W \in T_V(T_pM)$.

Consider the following variation of γ through geodesics:

$$\Gamma_W(s,t) = \exp_p t(V + sW).$$

By Theorem 2.4, the variation field $J_W(t) = \partial_s \Gamma_W(0,t)$ is a Jacobi field along γ.

We can check

$$J_W(1) = \partial_s \Gamma_W(0,1) = \left.\frac{d}{ds}\right|_{s=0} \exp_p(V + sW) = (\exp_p)_{*,V} W$$

and

$$J_W(0) = \partial_s \Gamma_W(0,0) = \left.\frac{d}{ds}\right|_{s=0} \exp_p 0 = 0$$

Hence, $(\exp_p)_{*,V}$ fails to be injective precisely when there exists a nonzero vector $W \in T_V(T_pM)$ such that $J_W(1) = 0$. Such a vector does not exist: if $J_W(1) = 0$ then p and q are conjugates along γ as J_W is a nontrivial Jacobi field that vanishes at p and q. Hence $\exp_p$ is a local diffeomorphism near V. □

Lemma 3.3. *If all sectional curvatures of M are nonpositive, then the conjugate locus of any point is empty.*

Proof. Let $\gamma : [a,b] \to M$ be a geodesic segment joining p and q. Suppose J is a nontrivial Jacobi field along γ with $J(0) = 0$.

By Theorem 2.4,

$$D_t^2 J = -R(J, \dot\gamma)\dot\gamma.$$

By the hypothesis of the lemma, the sectional curvature in the direction J and $\dot\gamma$ is non-positive:

$$\left\langle D_t^2 J, J \right\rangle = \langle -R(J,\dot\gamma)\dot\gamma, J\rangle = -\langle R(J,\dot\gamma)\dot\gamma, J\rangle \geq 0. \quad (3)$$

The norm square $\rho : [a,b] \to \mathbb{R}_{\geq 0}$ of J is smooth function along the curve:

$$\rho(t) = \langle J, J\rangle .$$

We have $\rho(0) = 0$. The first derivative is:

$$\frac{d}{dt}\rho = 2\langle D_t J, J\rangle .$$

Taking the second derivative gives:

$$\frac{d^2}{dt^2}\rho = 2\left\langle D_t^2 J, J\right\rangle + 2\left\langle D_t J, D_t J\right\rangle > 0.$$

The inequality follows from equation (3) and postive definiteness of Riemannian metric.

Hence, ρ is concave up. The function has range $[0,\infty)$ and $\rho(0) = 0$ and $\rho'(0) = 0$. Therefore, the function is strictly increasing in the interval $[0,1]$. This $\rho(1) > 0$ □

4 The Cartan-Hadamard Theorem

Lemma 4.1. *Suppose $\widetilde{M}$ and M are connected Riemannian manifolds, with $\widetilde{M}$ complete, and $\pi : \widetilde{M} \to M$ is a local isometry. Then M is complete and π is a covering map.*

Proof. We start by showing the path lifting property of π for geodesics: let $p \in M$ and $p' \in \pi^{-1}(p)$. Consider a geodesic $\gamma : I \to M$ starting at p. To lift γ to $\widetilde{M}$, we construct the unique geodesic $\widetilde{\gamma}$ passing through p' with derivative

$$\dot{\widetilde{\gamma}}(0) = \pi_{*,p'}^{-1}\dot{\gamma}(0)$$

(the derivative π_* has an inverse as π is a local isometry). The geodesic $\widetilde{\gamma}$ can be defined for all time as $\widetilde{M}$ is complete. As π is a local isometry, it takes geodesics to geodesics: $\pi \circ \widetilde{\gamma}$ is a geodesic in M. This curve starts at $\pi \circ \widetilde{\gamma}(0) = p$ in the direction

$$\pi_{*,p'}\dot{\widetilde{\gamma}}(0) = \dot{\gamma}.$$

By the uniqueness of geodesic, $\pi \circ \widetilde{\gamma} = \gamma$ where defined. Further, as $\pi \circ \widetilde{\gamma}$ is defined for all time, γ extends for all time. Therefore, M is complete.

A covering map is surjective. Let us show π is surjective: fix $p' \in \widetilde{M}$, and let $q \in M$ be arbitrary. As M is connected and complete there is a geodesic segment $\gamma : [a,b] \to M$ that joins $\pi(p')$ and q. By the above argument, we have a geodesic lift $\widetilde{\gamma}$ of γ starting at p'. Then $\pi(\widetilde{\gamma}(b)) = \gamma(b) = q$. Hence, π is surjective.

To show π is a covering map, we must show that for every point $p \in M$ there is a neighborhood U that is evenly covered.

Let $p \in M$ and $U = B_\epsilon(p)$ be a geodesic ball at p. Let $\{p_i'\} = \pi^{-1}(p)$ and $\widetilde{U}_i$ denote the metric ball of radius ϵ around p_i'. For $i \neq j$, consider a minimizing geodesic $\widetilde{\gamma}$ from p_i' to p_j'. The projection $\gamma = \pi \circ \widetilde{\gamma}$ will be a geodesic from p to p. Since U is a geodesic ball, γ cannot be contained in U. So γ must leave U the ϵ geodesic ball and enter. Since length of γ and $\widetilde{\gamma}$ are the same (local isometry preserves length of curves), $\widetilde{\gamma}$ must have length greater than 2ϵ. Hence, the distance $d(p_i', p_j') > 2\epsilon$ and $\widetilde{U}_i \cap \widetilde{U}_j = \emptyset$

As π is a local isometry, and a metric ball of radius ϵ maps to a metric ball of radius ϵ. Hence, $\pi(\widetilde{U}_i) \subset U$. To show $\pi^{-1}(U) \subset \bigcup_i \widetilde{U}_i$. Consider $q' \in \pi^{-1}(U)$, then $q' = \pi(q)$ for some $q \in U$. Let $\gamma : [a, b] \to M$ be the minimizing geodesic segment from q to p in the geodesic ball U. Then $d(p, q) \leq \epsilon$. Let $\widetilde{\gamma}$ lift γ starting at q'. Then $\pi(\widetilde{\gamma}(b)) = \gamma(b) = p$, $\widetilde{\gamma}(b) = p_i$ for some i, and $d(q', p_i') \leq b - a < \epsilon$. Hence, $q' \in \widetilde{U}_i$. We have shown that the neighbourhood U is evenly covered: $\pi^{-1}(U) = \bigcup_i \widetilde{U}_i$.

The map $\pi : \widetilde{U}_i \to U$ is bijective as the inverse can be constructed by sending a radial geodesic to its lift (as shown before): let $q \in U$ and $\gamma : [a, b] \to M$ be the geodesic connecting p and q in the geodesic ball U. Consider the unique lift $\widetilde{\gamma}$ of γ starting at p_i, and set $\pi^{-1}(q) = \widetilde{\gamma}(b)$. This is a bijection as $\pi(\widetilde{\gamma}(b)) = \gamma(b) = q$. As $\pi : \widetilde{U}_i \to U$ is a local diffeomorphism (isometry) that is bijective, it is a diffeomorphism. Hence, $\pi : \widetilde{M} \to M$ is a covering map. □

Proof of Theorem 1.1. Let $p \in M$. As M has nonpositive curvature, p has no conjugate points by lemma 3.3. Hence, $\exp_p$ is a local diffeomorphism on all of T_pM by Proposition 3.2.

Let $\tilde{g}$ be the 2-tensor field pullback $\exp_p^* g$ defined on T_pM. Being a local diffeomorphism, $\exp_p^*$ is nonsingular everywhere. Hence,

$$\exp_p : (T_pM, \tilde{g}) \to (M, g)$$

is a local isometry. By lemma 4.1, $\exp_p$ is a covering map. The tangent space $T_pM \cong \mathbb{R}^n$ is simply connected. Hence, $T_pM \cong \mathbb{R}^n$ is the universal covering space of M. □

References

The primary reference for this study is the book: Riemannian Manifolds by John M. Lee (1991). I have also referred to online notes of Zuoqin Wang for the details of Jacobi fields.