Math 9054A Assignment 1

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2: Define

$$\|x\| = \inf\{1/s : s > 0, sx \in B\}$$

Claim: $0 \in B$:
Take some $v \neq 0$, then exists some $t > 0$ such that given $\alpha \in \mathbb{F}$ and $|\alpha| \leq t$ we have $\alpha v \in B$. (Using (*)). Setting $\alpha = 0$, we get $0 \in B$.

To show $\|\cdot\|$ is a norm, we need to check (Definition 2.1 [1]). Let $x, y \in X$ and $\alpha \in \mathbb{F}$.

(a) $\|x\| \geq 0$:
Since the set $\{1/s : s > 0, sx \in B\}$ only contains positive real numbers, its infimum must be greater than or equal to zero.

(b) $\|x\| = 0$ *if and only if* $x = 0$:
We have shown above that $0 \in B$.
Let $x = 0$, then $0 = s \in B$ for any $s > 0$. So norm $\|0\| = \inf\{s > 0\} = 0$. Hence, $\|x\| = 0$.

Let $x \neq 0$ then there exists $t > 0$ such that for $\alpha \in \mathbb{F}$, $|\alpha| \leq t$ if and only if $\alpha x \in B$. Which gives $\inf\{1/s : s > 0, sx \in B\} \geq 1/t$. Hence $\|x\| > 0$.

(c) $\|\alpha x\| = |\alpha|\|x\|$:
Suppose $\alpha(\neq 0) \in \mathbb{F}$ and $x \neq 0$.

$$\begin{aligned}
\|\alpha x\| &= \inf\{1/s : s > 0, s\alpha x \in B\} \\
&= \inf\{1/s : s > 0, s|\alpha|\frac{\alpha}{|\alpha|}x \in B\} \\
&= \inf\{|\alpha|/r : r > 0, r\frac{\alpha}{|\alpha|}x \in B\} \\
&= |\alpha|\inf\{1/r : r > 0, r\frac{\alpha}{|\alpha|}x \in B\}
\end{aligned}$$

By property (*) there is $t > 0$ such that for $\beta \in \mathbb{F}$, $|\beta| \leq t$ if and only if $\beta x \in B$. So, if $r\frac{\alpha}{|\alpha|}x \in B$, then $rx \in B$.

$$\begin{aligned}\|\alpha x\| &= |\alpha|\inf\{1/r : r > 0, rx \in B\} \\ &= |\alpha|\|x\|\end{aligned}$$

If either $\alpha = 0$ in the field or $x = 0$ in X then $\alpha x = 0$ and $|\alpha|\|x\| = \|0\| = \|\alpha x\|$.

(d) $\|x + y\| \leq \|x\| + \|y\|$:
Let $x \neq 0$, $y \neq 0$. Since $\|x\|$ is the infimum of the set $\{1/s : s > 0, sx \in B\}$. If we take $\|x\| + \delta_1$ for any $\delta_1 > 0$, we get $\frac{x}{\|x\|+\delta_1} \in B$. Similarly, $\frac{y}{\|y\|+\delta_2} \in B$ for any $\delta_2 > 0$. Now using the convexity of B, for $t \in [0, 1]$.

$$t\left(\frac{x}{\|x\| + \delta_1}\right) + (1 - t)\left(\frac{y}{\|y\| + \delta_2}\right) \in B$$

Set $t = (\|x\|+\delta_1)/(\|x\|+\|y\|+\delta_1+\delta_2)$, then $1-t = (\|y\|+\delta_2)/(\|x\|+\|y\| + \delta_1 + \delta_2)$

$$\frac{1}{(\|x\| + \|y\| + \delta_1 + \delta_2)}\left((\|x\| + \delta_1)\left(\frac{x}{\|x\| + \delta_1}\right) + (\|y\| + \delta_2)\left(\frac{y}{\|y\| + \delta_2}\right)\right) \in B$$

$$\frac{1}{(\|x\| + \|y\| + \delta_1 + \delta_2)}(x + y) \in B$$

So,

$$(\|x\| + \|y\| + \delta_1 + \delta_2) \in \{1/s : s > 0, s(x + y) \in B\}$$

Then $\|x + y\| = \inf\{1/s : s > 0, s(x+y) \in B\} \leq \|x\| + \|y\| + \delta_1 + \delta_2$. This works for all $\delta_1 > 0$ and $\delta_2 > 0$. Hence, $\|x + y\| \leq \|x\| + \|y\|$.

For the second part, we must show $B = \{x \in X : \|x\| \leq 1\}$.

Let $x \in B$, then $1 \cdot x \in B$. Which means $1 \in \{1/s : s > 0, sx \in B\}$ and $\|x\| \leq 1$.

Suppose $x \in X$ such that $\|x\| \leq 1$. For $\delta > 0$, $\frac{x}{\|x\|+\delta} = \frac{x}{1+\delta} \in B$. Condition (*) tells us there exists $t > 0$ such that for $\alpha \in \mathbb{F}$, $|\alpha| \leq t$ if and only if $\alpha x \in B$. Then $\frac{1}{1+\delta} \leq t$. Taking δ arbitarily small $1 \leq t$. Finally set $\alpha = 1$, then $1 \cdot x = x \in B$.

3: Let $0 < p < 1$ and let S be the vector space of all sequences in $\mathbb{F}$. For $x = (x_1, x_2, ...)$ is in S, set

$$\|x\| = \left(\sum_{k=1}^{\infty}|x_k|^p\right)^{1/p}$$

Let $X = \{x \in S : \|x\| < \infty\}$.

Let us show that $d(x, y) = \|x - y\|^p$ defines a metric on X. We need to check the conditions on Def 1.17 of [1].

Let $x, y, z \in X$,

(a)

$$d(x,y) = \|x-y\|^p = \left(\sum_{k=1}^{\infty} |x_k - y_k|^p\right)$$

Since we are summing over non negative terms, $d(x,y) \geq 0$. Also, since $x \in X$ the sum is finite.

(b) Look at the equivalence

$$\left(\sum_{k=1}^{\infty} |x_k - y_k|^p\right) = 0 \Leftrightarrow x_k = y_k \quad \forall k = 1, 2, ...$$

If for any k, $x_k \neq y_k$ then we will have a postive term. Since each term is non negative, the sum will be positive.

This shows that $d(x,y) = 0$ if and only if $x = y$ (that is $x_k = y_k$ for all $k = 0, 1, ...$).

(c) Since $|x_k - y_k| = |y_k - x_k|$ we have

$$d(x,y) = \left(\sum_{k=1}^{\infty} |x_k - y_k|^p\right) = \left(\sum_{k=1}^{\infty} |y_k - x_k|^p\right) = d(y,x)$$

(d) First we see the triangle inequality the field for $a, b \in \mathbb{F}$.

$$|a+b|^p = \frac{|a+b|}{|a+b|^{1-p}} \leq \frac{|a| + |b|}{|a+b|^{1-p}} \leq \frac{|a|}{|a|^{1-p}} + \frac{|b|}{|b|^{1-p}} \leq |a|^p + |b|^p$$

Now let us look at the triangle inequality the vector space

$$\begin{aligned} d(x,z) &= \sum_{k=1}^{\infty} |x_j - z_j|^p \\ &= \sum_{k=1}^{\infty} |(x_j - y_j) - (z_j - y_j)|^p \\ &\leq \sum_{k=1}^{\infty} |(x_j - y_j)|^p + |(z_j - y_j)|^p \\ &\leq \sum_{k=1}^{\infty} |(x_j - y_j)|^p + \sum_{k=1}^{\infty} |(z_j - y_j)|^p \\ &\leq d(x,y) + d(y,z) \end{aligned}$$

Hence, d is a metric on X.

To show X is a subspace of S. It is enough to show $x - \alpha y \in X$ for

$x, y \in X$ and $\alpha \in \mathbb{F}$.

$$\begin{aligned}
\|x - \alpha y\| &= (\sum_{k=1}^{\infty} |x_j - \alpha y_j|^p)^{1/p} \\
&\leq (\sum_{k=1}^{\infty} |x_j|^p + \sum_{k=1}^{\infty} \alpha |y_j|^p)^{1/p} \\
&\leq 2(\sum_{k=1}^{\infty} |x_j|^p)^{1/p} + 2\alpha(\sum_{k=1}^{\infty} |y_j|^p)^{1/p} \\
&< \infty
\end{aligned}$$

Hence, X is a subspace of S.

Finally, consider $a = (1, 0, 0, ...)$ and $b = (0, 1, 0, 0, ...)$. Then $\|a\| = (1^p + 0^p + 0^p + ...)^{1/p} = 1$ and $\|b\| = (0^p + 1^p + 0^p + ...)^{1/p} = 1$. So $a, b \in X$. Now $\|a + b\| = (1^p + 1^p + 0^p + ...)^{1/p} = 2^{1/p}$. Note that $\|a\| + \|b\| = 2 < 2^{1/p} = \|a + b\|$. This breaks the triangle inequality if this was a norm. Hence, $\|\cdot\|$ can not be a norm.

References

[1] Bryan Rynne and Martin A Youngson. *Linear functional analysis*. Springer Science & Business Media, 2007.