Math 9054A Assignment 2

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2: Let $k \geq 3$ and $\omega \in \mathbb{C}$ be a primitve kth root of unity. Then $1 - \omega^k = 0$.

$$1^k - x^k = (1-x)(1 + x + x^2 + ... + x^{k-1})$$

As $k \geq 3$, we have $\omega \neq 1$ so $0 = 1 + \omega + \omega^2 + ... + \omega^{k-1}$. Since $k \geq 3$, $\omega^2 \neq 1$ but $(\omega^2)^k = (\omega^k)^2 = 1$. Which means $0 = 1 + (\omega^2) + (\omega^2)^2 + ... + (\omega^2)^{k-1}$. Also note, $|\omega| = 1$.

We know the polarization identity

$$4(x, y) = \|x + y\| - \|x - y\| + i\,\|x + iy\| - i\,\|x - iy\|$$

Note that $(i)^4 = 1$ and this can be seen as

$$4(x, y) = \|x + y\| + i\,\|x + iy\| + i^2\,\left\|x + (i)^2 y\right\| + i^3\,\left\|x + i^3 y\right\|$$

A natural extension would be to check if

$$k(x, y) = \sum_{j=0}^{k-1} \omega^j \left\|x + \omega^j y\right\|$$

Lets expand one of the terms

$$\begin{aligned}
\left\|x + \omega^j y\right\| &= (x + \omega^j y, x + \omega^j y) \\
&= (x, x) + (\omega^j y, \omega^j y) + (x, \omega^j y) + (\omega^j y, x) \\
&= (x, x) + \omega^j \bar{\omega}^j (y, y) + \bar{\omega}^j (x, y) + \omega^j (y, x) \\
&= (x, x) + |\omega^j|^2 (y, y) + \bar{\omega}^j (x, y) + \omega^j (y, x) \\
&= \|x\| + |\omega|^{2j} \|y\| + \bar{\omega}^j (x, y) + \omega^j (y, x) \\
&= \|x\| + \|y\| + \bar{\omega}^j (x, y) + \omega^j (y, x)
\end{aligned}$$

Now we sum

$$\begin{aligned}\sum_{j=0}^{k-1}\omega^j \left\|x+\omega^j y\right\| &= \sum_{j=0}^{k-1}\omega^j \left(\|x\|+\|y\|+\bar{\omega}^j(x,y)+\omega^j(y,x)\right)\\ &= \sum_{j=0}^{k-1}\omega^j(\|x\|+\|y\|)+\sum_{j=0}^{k-1}|\omega|^{2j}(x,y)+\sum_{j=0}^{k-1}\omega^{2j}(y,x)\\ &= (\|x\|+\|y\|)\sum_{j=0}^{k-1}\omega^j+\sum_{j=0}^{k-1}(x,y)+(y,x)\sum_{j=0}^{k-1}\omega^{2j}\\ &= k(x,y)\end{aligned}$$

Hence, the polarization identity in k terms is as follows

$$k(x,y)=\sum_{j=0}^{k-1}\omega^j \left\|x+\omega^j y\right\|$$

3: Let the range of T be given by M.

M a subspace of ℓ^2: Given $(y_n) \in M$, we have $(x_n) \in \ell^2$, such that $T((x_n)) = (y_n)$. So $y_n = 2^{-n}x_n$,

$$\|(y_n)\|^2 = \left\|(2^{-n}x_n)\right\|^2 = \sum_{n=1}^{\infty} 2^{-2n}x_n^2 \le \sum_{n=1}^{\infty} x_n^2 < \infty$$

as $(x_n) \in \ell^2$. Hence, $(y_n) \in \ell^2$.

Let $a = (2^{-1}, 2^{-2}, ...)$. Clearly $a \in \ell^2$ since it is a geometric sum. Suppose there exists $x = (x_1, x_2, ...) \in \ell^2$ such that $T(x) = a$, then we are forced to have $2^{-j}x_j = 2^{-j}$ for all $j \in \mathbb{N}$. This means $x_j = 1$ for all j. But we have $\|x\|^2 = \sum_{j=1}^{\infty} 1^2 = \infty$ hence $(1, 1, ...)$ is not in ℓ^2. This gives us a contradiction, hence $a \notin M$. Hence M is a proper subset of ℓ^2.

Let $y = (y_1, y_2, ...) \in \ell^2$. Then consider the sequence $\{a_n\} \subset \ell^2$

$$a_n = (2^1y_1, 2^2y_2, 2^3y_3, ..., 2^ny_n, 0, 0, ...)$$

Since each sequence has only finitely many non zero terms, we get $a_n \in \ell^2$. Which gives $T(a_n) \in M$.

$$T(a_n) = (y_1, y_2, ..., y_n, 0, 0, ...)$$

Note $\{T(a_n)\} \subset M$. Finally we will show $T(a_n) \to y$

$$\begin{aligned}\|y - T(a_n)\|^2 &= \|(y_1, ..., y_n, y_{n+1}, ...) - (y_1, ..., y_n, 0, 0, ...)\|^2\\ &= \|(0, ..., 0, y_{n+1}, y_{n+2}, ...)\|^2\\ &= \sum_{k=n+1}^{\infty} |y_k|^2\end{aligned}$$

Since $\sum_{k=0}^{\infty}|y_k|^2 < \infty$, we must have that the tail of the sequence goes to zero, ie, $\sum_{k=N+1}^{\infty}|y_k|^2 \to 0$ as $N \to \infty$. So $\|y - T(a_n)\|^2 \to 0$ as $n \to \infty$. Hence $\{T(a_n)\}$ is a sequence in M that converges in ℓ^2 to $y \in \ell^2$. Therefore, M is dense in ℓ^2.