Math 9054A Assignment 3

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2: We have

$$Y = \{(a_n) \in X : \sum_{n=1}^{\infty} a_n = 0\}$$

Suppose $(b_n) \in Y^{\perp}$, then for any $(a_n) \in Y$ we get $((a_n),(b_n)) = 0$. Since $(b_n) \in X$, we have $m \in \mathbb{N}$ such that $b_j = 0$ for $j > m$. Consider $(\tilde{b}_n) \in Y$ defined as $\tilde{b}_j = b_j$ for $1 \leq j \leq m$, $\tilde{b}_j = -b_{j-m}$ for $m+1 \leq j \leq 2m$ and $\tilde{b}_j = 0$ for $j \geq 2m+1$. Indeed $(\tilde{b}_j) \in Y$: Taking the sum of terms

$$\sum_{j=1}^{\infty} \tilde{b}_j = \sum_{j=1}^{m} \tilde{b}_j + \sum_{j=m+1}^{2m} \tilde{b}_j + \sum_{j=2m+1}^{\infty} 0 = \sum_{j=1}^{m} b_j + \sum_{j=1}^{m} -b_j = 0$$

Hence $(\tilde{b}_j) \in Y$. Let us look at $((a_n),(b_n))$:

$$((\tilde{b}_n),(b_n)) = \sum_{j=1}^{\infty} \tilde{b}_j \bar{b}_j = \sum_{j=1}^{m} \tilde{b}_j \bar{b}_j + \sum_{j=m+1}^{\infty} \tilde{b}_j \bar{0} = \sum_{j=1}^{m} b_j \bar{b}_j = \sum_{j=1}^{m} |b_j|^2$$

Since $(\tilde{b}_n) \in Y$ we get the inner product to be zero. Hence $\sum_{j=1}^{m} |b_j|^2 = 0$ which implies $|b_j| = 0$ for $j = 1, ..., m$. Which forces $(b_j) = 0$ as all the terms are zero. So any element in $Y^{\perp}$ is zero. Hence $Y^{\perp} = \{0\}$.

Note $\{0\}^{\perp}$ is the whole of X; given any element in $(x_n) \in X$, $(0,(x_n)) = 0$. Hence $(Y^{\perp})^{\perp} = X$.

Finally to see that $X \neq Y$. Note that $(1,1,1,0,0,0,...) \in X$, but not in Y, since the sequence sums to 3.

Hence $(Y^{\perp})^{\perp} = X \neq Y$.

3: Suppose M and N are closed subspaces of a Hilbert space H.

Let $v \in M \cap N$. Take any element $u \in M^{\perp} + N^{\perp}$, then $u = m + n$ then $m \in M^{\perp}$ and $n \in N^{\perp}$. We get $(v,u) = (v, m+n) = (v,m) + (v,n)$. Since $v \in M$ and $m \in M^{\perp}$: $(v,m) = 0$. Similary $(v,n) = 0$. Hence $(v,u) = 0 + 0 = 0$ for all $u \in M^{\perp} + N^{\perp}$ which gives $v \in (M^{\perp} + N^{\perp})^{\perp}$. So $M \cap N \subset (M^{\perp} + N^{\perp})^{\perp}$.

Now take an element $v \in (M^\perp + N^\perp)^\perp$. Then let us show that $v \in (M^\perp)^\perp$. Given any element in $m \in M^\perp$, we have $m = m + 0 \in M^\perp + N^\perp$; so $(v, m) = 0$. Since this holds for all $m \in M^\perp$, we have $v \in (M^\perp)^\perp$. Now we use a result proved in class regarding closed subspaces of Hilbert space: Since M is a closed subspace we get $M = (M^\perp)^\perp$. Hence $v \in M$. Using the same arguments with M replaced by N we get $v \in N$. Hence $v \in M \cap N$. So $(M^\perp + N^\perp)^\perp \subset M \cap N$.

Therefore $(M^\perp + N^\perp)^\perp = M \cap N$.

To show that $(M \cap N)^\perp = M^\perp + N^\perp$ may fail, we consider the following argument subspaces of ℓ^2. Let U and W be as follows:

$$U = \overline{\text{span}}\{e_{2m} : m \in \mathbb{N}\}$$

It is clear that U is closed as it is a closure of a set.
Set $v_n = \cos(\frac{1}{n^2})e_{2n} + \sin(\frac{1}{n^2})e_{2n+1}$, and set $V = \overline{\text{span}}\{e_1, v_1, v_2, ...\}$. Finally, set $W = \{z\}^\perp \cap V$. Again W is closed since it is intersection of two closed sets (V is a closed set as it is a closure and perp of any subspace is closed).

In class we have shown that U and W are closed subspaces such that $U + W$ is not closed as $e_1 \notin U + W$ but $e_1 \in \overline{U + W}$.

Now we will consider $M = U^\perp$ and $N = W^\perp$. Then $(M \cap N)^\perp$ is closed as perp of a subspace is always closed. Now $M^\perp + N^\perp = (U^\perp)^\perp + (W^\perp)^\perp$. Since U is a closed subspace of the Hilbert ℓ^2, we have $U = (U^\perp)^\perp$, and similary $V = (V^\perp)^\perp$. So $M^\perp + N^\perp = (U^\perp)^\perp + (W^\perp)^\perp = U + V$. But $U + V$ is not closed. Hence $M^\perp + N^\perp$ is not closed in this case.

So we can not have $(M \cap N)^\perp = M^\perp + N^\perp$ as one subspace is closed and the other is not. Therefore this maynot be true.