Math 9054A Assignment 4

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1: Let $\{e_n\}$ be an orthonormal basis of Hilbert space H and $T : H \to H$ be a bounded linear operator.

Since $\{e_n\}$ is an orthonormal basis we have $x = \sum_{k=1}^{\infty}(x, e_k)e_k$. Using linearity for T, we get

$$Tx = T(\sum_{k=1}^{\infty}(x, e_k)e_k) = \sum_{k=1}^{N} T((x, e_k)e_k) + T(\sum_{k=N}^{\infty}(x, e_k)e_k)$$

for any $N \in \mathbb{N}$.
If we let $N \to \infty$, we get $\|\sum_{k=N}^{\infty}(x, e_k)e_k\| \to 0$. So, by the continuity of T we get $\|T(\sum_{k=N}^{\infty}(x, e_k)e_k)\| \to 0$. Hence limiting $N \to \infty$.

$$Tx = \lim_{n\to\infty} \sum_{k=1}^{N} T((x, e_k)e_k) + \lim_{n\to\infty} T(\sum_{k=N}^{\infty}(x, e_k)e_k) = \sum_{k=1}^{\infty} T((x, e_n)e_n) + 0$$

Finally, we look at the represetation of $\sum_{k=1}^{\infty} T((x, e_n)e_n)$ in terms of the orthonormal sequence.

$$\sum_{k=1}^{\infty} T((x, e_k)e_k) = \sum_{n=1}^{\infty}(\sum_{k=1}^{\infty} T((x, e_k)e_k), e_n)e_n$$

Now we use $\lim(a_n, b) = (\lim a_n, b)$

$$\sum_{n=1}^{\infty}(\sum_{k=1}^{\infty} T((x, e_k)e_k), e_n)e_n = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(T((x, e_k)e_k), e_n)e_n$$

Using linearity of T over the field

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(T((x, e_k)e_k), e_n)e_n = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}((x, e_k)T(e_k), e_n)e_n$$

Finally, using the linearity of the inner product.

$$Tx = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}((x, e_k)T(e_k), e_n)e_n = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(x, e_k)(T(e_k), e_n)e_n$$

3: The set ℓ^p is an ideal of ℓ^∞: Let $(a_n) \in \ell^p$ and $(b_n) \in \ell^\infty$. Let M be a bound on the terms of (b_n). We have $(a_n)(b_n) = (a_n b_n)$. Let us look at $\|(a_n b_n)\|_p^p = \sum |a_n b_n|^p \leq M^p \sum |a_n|^p = M^p \|(a_n)\|_p^p < \infty$. So, $(a_n)(b_n) \in \ell^p$. Hence ℓ^p is an ideal of ℓ^∞.

The space c_o is an ideal of ℓ^∞: Let $(a_n) \in c_o$ and $(b_n) \in \ell^\infty$. Let M be a bound on the terms of (b_n). We have $(a_n)(b_n) = (a_n b_n)$. Given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|a_n| < \epsilon M^{-1}$. So, $|a_n b_n| < \epsilon M^{-1} M = \epsilon$. Hence, $(a_n b_n) = (a_n)(b_n) \in c_o$ converges to zero. Therefore, c_o is an ideal of ℓ^∞.

The space c_o is closed in ℓ^∞: Let $((a_n)_j)$ be a sequence in c_o that convergers to (b_n) in ℓ^∞. Given $\epsilon > 0$, choose N_0 such that for $j \geq N_0$, $\|(a_n)_j - (b_n)\|_{\ell^\infty} < \epsilon/2$. Now choose N_1 such that for $n \geq N_1$, we have $|a_n^{(N_0)}| < \epsilon/2$. So, $|b_n| \leq |b_n - a_n^{(N_0)}| + |a_n^{(N_0)}| < \epsilon/2 + \epsilon/2$, ie, $|b_n| < \epsilon$ for $n \geq N_1$. Hence (b_n) converges to zero, that is, $(b_n) \in c_0$. Therefore, c_o is closed in ℓ^∞.

The closure of ℓ^p in ℓ^∞ is c_o: Clearly $\ell^p \subset c_o$. Since c_o is closed we have $\overline{\ell^p} \subset c_o$ (Closure will be with respect to ℓ^∞). Let $(b_n) \in c_o$, then consider the trucated sequences $A_1 = (b_1, 0, 0, ...)$, $A_2 = (b_1, b_2, 0, 0, ...)$ and $A_j = (b_1, ..., b_j, 0, 0, ...)$. Then (A_j) is a sequences in ℓ^p as each of the sequence has only finitetly many nonzero elements. Now look at $\|A_j - (b_n)\|_{\ell^\infty} = sup_{n \geq j}\{|b_n|\}$. Since (b_n) converges to zero, as $j \to \infty$, $\sup_{n \geq j}\{|b_n|\} \to 0$. Hence ℓ^p is dense in c_o. So, $\overline{\ell^p} = c_o$.