Meeting Notes

Harshith Sairaj Alagandala

October 4, 2023

Contents

1 2023-09-20: Geometric approach for polynomial convexity

1.1 Abstract

Let K be a compact subset of $\mathbb{C}^n$. If the algebras $P(K) = C(K)$, then $K = \widehat{K}$. The proof of this is found in Theorem 1.2.10 Stout [2]. The aim is to find a geometric proof for the statement without invoking any arguments of uniform algebras and characters as seen in Stout [2].

Oka characterization of polynomial convexity gives a geometric picture. It uses hypersurfaces which escape to infinity.

For simplicity, we can restrict K to be a maximal totally real submanifold. A manifold where the tangent space at a point is a totally real vector space of real dimension n. Note $C(K)$ and $O(K)$ mayn't be equal even if $K^o = \phi$. But for totally real manifolds they are equal.

Analytic discs can aid us with checking polynomial convexity of a set. Let K be a compact set. If it is possible to attach the boundary of a analytic disc to K, ie, an analytic function $f : \mathbb{D} \to \mathbb{C}^n$ such that the boundary $f(b\mathbb{D})$ is inside a compact set K, then $f(\mathbb{D}) \subset K$ if K is polynomially convex (using maximal principle on $P \circ f$).

Given a totally real compact manifold with $C(K) = P(K)$, prove that it does not have a non trivial disc attached to it.

In parallel, look at the same statement for rational convexity. Prove that $C(K) = R(K)$ implies $K = \widehat{K_R}$. Rationally convex hull has a better geometric structure than polynomial hull.

1.2 Continuation: 2023-09-22

Looked at plane and Lipschitz graph. We can not attach a nontrival disc to it. But all of this is local.

Look at $S^1 \times S^1$. We can attach boundary of analytic disc to it. Now look at $P(K) = C(K)$.

1.3 Proof

Let $K \subset \mathbb{C}^n$ be compact with $P(K) = C(K)$. Suppose $f : U \to \mathbb{C}^n$ be an injective holomorphic map where U is a neighbourhood of $\overline{\mathbb{D}}$.

Suppose $f(b\mathbb{D}) \subset K$ and $f(0) \notin K$. Define continuous function g on $f(b\mathbb{D})$ such that $g \circ f = 1/z$ on $b\mathbb{D}$ (by taking $g(z) = 1/f^{-1}(z)$ on $f(b\mathbb{D})$). Extend g to a continuous function on K by Tietze's Extension Theorem (20.4 Rudin [1]).

Since $P(K) = C(K)$, we have $\{p_n\}$ such that $p_n \to g$ uniformly on K. Then $p_n \circ f \to g \circ f$ on $b\mathbb{D}$. Since $p_n \circ f$ is uniformly Cauchy on $b\mathbb{D}$, it converges to a holomorphic function on $\mathbb{D}$ that matches with $g \circ f = 1/z$ on $b\mathbb{D}$. This gives us a contradiction.

2 2023-09-27

Looking at the case with non injective analytic discs: We can prove this by taking a real valued continuous function g on K. Then we get $\{p_n\}$ polynomials that converge to g uniformly on K. Take g (which is real valued) varying on the boundary of the analytic disc. Then $p_n \circ \psi \to g \circ \psi$ uniformly on $b\mathbb{D}$. Get a contradiction for a holomorphic function which is real on the boundary of the circle. It should be constant.

Some definitions, analytic structure, Kessentail hull.

Now for the general theorem, on showing polynomial convexity. Look at **Rossi maximal theorem** in Stout [2].

Side track: Germs of holomorphic functions which are zero at a point is a maximal ideal of all germs at that point. The quotient field is $\mathbb{C}$.

non math: Mocha machine for coffee (at Winners). Get a good valve (made in italy).

Theorem 2.1 (Rossi Theorem)**.** *If $X \subset \mathbb{C}^n$ is compact, if E is a compact subset of $\widehat{X}$, if U is an open subset of $\mathbb{C}^n$ that contains E, and if $f \subset O(U)$, then $\|f\|_E = \|f\|_{(E\cap X)\cup bE}$.*

2.1 Proof

Let K be a compact set of $\mathbb{C}^n$ such that $P(X) = C(X)$. Let $f : \overline{\mathbb{D}} \to \mathbb{C}^n$ be a continuous function such that it is holomorphic on $\mathbb{D}$ and $f(b\mathbb{D}) \subset K$.

Suppose $f(b\mathbb{D})$ is not a singleton set. We can take a real valued continuous function g on K that separates points on $f(b\mathbb{D})$. Then $g \circ f : b\mathbb{D} \to \mathbb{R}$ is a continuous which is non-constant. Since g is continuous on K, we can find a sequence of polynomials $\{p_n\}$ such that $p_n \to g$ uniformly on K. Then the function $p_n \circ f \to g \circ f$ uniformly on $b\mathbb{D}$. Since the sequence of functions is uniformly Cauchy on the boundary, we get that $p_n \circ f$ converges to a holomorphic function continuous upto the boundary of the unit disc. Say the limit function is $\tilde{f}$. Then $\tilde{f}$ is equal to $g \circ f$ on $b\mathbb{D}$. The function $\tilde{f}$ being real valued on the boundary of the unit disc implies that $\tilde{f}$ is constant: the imaginary part of $\tilde{f}$ would be a harmonic function that is zero on the boundary of the unit disc, which has a unique extention as a constant zero function; Hence $\tilde{f}$ is a real valued holomorphic function on the disc, making it constant. But had $g \circ f$ a non constant function on $b\mathbb{D}$, which contradicts the fact that $g \circ f|_{b\mathbb{D}} = \tilde{f}|_{b\mathbb{D}}$ is constant on the boundary.

Hence $f(b\mathbb{D})$ must be a singleton set. So, we can not attach any non trivial analytic disc to K.

3 2023-10-04

The problem with using Rossi maximal theorem would be that $P(K) = C(K)$ does not give us control over continuous functions on $\widehat{K}$.

Set up: Let $K \subset \mathbb{C}^n$ be a compact set such that $P(K) = C(K)$. We wish to show $K = \widehat{K}$.

We can extend continuous function on $g \in C(K)$ to a continuous function in $C(\widehat{K})$: Suppose $\{p_n\}$ is a sequence of polynomials that converges uniformly to g on K. Then we get that $\{p_n\}$ is uniformly Cauchy in $\widehat{K}$; this will converge to a continuous function $\widehat{g}$ on $\widehat{K}$. Further, this extension is unique: Suppose $\{q_n\}$ is another sequence that converges uniformly on K to g. Then $\{p_n - q_n\}$ will converge uniformly to zero on $\widehat{K}$. This shows us that any such extension must be same. So we can define a map $\phi : C(K) \to C(\widehat{K})$ given by the above.

If we wish to use Rossi's maximal theorem to get a contradiction, we would like to find g such that $\widehat{g}$ has the property: $\|\widehat{g}\|_{b_{\widehat{K}}E \cup (E\cap K)} < \|g\|_{bE}$.

Some observations on the algebras: $\phi(C(K)) \subset C(\widehat{K})$ is a closed subalgebra of $C(\widehat{K})$ (To check). Note $\widehat{K} = K$ is equivalent to $\phi(C(K)) = C(\widehat{K})$, using arguments in the lines of Uryson's lemma. We also have some observation on other algebras

$$P(K) = P(\widehat{K}) \stackrel{?}{=} \phi(C(K)) \subset C(\widehat{K})$$

Check if $P(\widehat{K}) \stackrel{?}{=} \phi(C(K))$.

Note: If we manage to show $P(\widehat{K}) = C(\widehat{K})$. Then we are done as $\phi(C(K)) = C(\widehat{K})$.

Using Oka-Weil: $K = \widehat{K} \implies P(K) = \overline{O}(K)$. Further we have $C(K) = P(K) = \overline{O}(K)$.

$$\overline{O}(\widehat{K}) = P(\widehat{K}) \stackrel{?}{=} \phi(C(K)) \subset C(\widehat{K})$$

We can also look at psh functions, using the following: Say $K \subset \mathbb{C}^n$ is a compact. Then $K = \widehat{K} \Leftrightarrow \exists$ psh $\phi : \mathbb{C}^n \to \mathbb{R}$ such that (i) $\phi^{-1}(0) = K, \phi \geq 0$, (ii) ϕ is spsh on $\mathbb{C}^n \setminus K$. on $\phi : \mathbb{C}^n \to \mathbb{R}$

References

[1] Walter Rudin. “Real and complex analysis. 1987”. In: *Cited on* 156 (1987).

[2] Edgar Lee Stout. *Polynomial convexity.* Vol. 261. Springer Science & Business Media, 2007.