SCV Assignment 1

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1.1: Let Ω be a domain in $\mathbb{C}$ and $K = \widehat{K}_\Omega$. We have seen that $\widehat{K}_\Omega$ is K union with relatively compact compontents of $\Omega \setminus K$. Since $\widehat{K}_\Omega = K$, $\Omega \setminus K$ doesn't contain any component that is relatively compact in Ω.

Let $M \subset K$ be a connected component of K. To show M is holomorphically convex it is enough to show $M = \widehat{M}_\Omega$. Or equivalently, $\Omega \setminus M$ has no relatively compact compontents in Ω.

Suppose U be a connected component of $\Omega \setminus M$ that is relatively compact in Ω. We have $\Omega \setminus K \subset \Omega \setminus M$ since $K \supset M$. As M is a connected compontent of K, there are open sets that seperate M and $K \setminus M$. Say V_M is a neighbourhood of M in this seperation and W_M is a neighbourhood of $K \setminus M$ such that $V_M \cap W_M = \emptyset$. Then $V_M \setminus M \subset \Omega \setminus K$.

Note $U \cap (V_M \setminus M)$ is not empty. Because $\overline{U} \cap M \neq \emptyset$ as Ω is connected and U is a component of $\Omega \setminus M$. Now $U \cap (V_M \setminus M) \subset \Omega \setminus K$. Let $\widetilde{U}$ be the component of $\Omega \setminus K$ that contains $U \cap (V_M \setminus M)$. Now as $\Omega \setminus K \subset \Omega \setminus M$, we get $\widetilde{U} \subset U$. Finally, $\overline{\widetilde{U}} \subset \overline{U} \subset \Omega$. Hence, $\widetilde{U}$ is relatively compact in Ω and is a component of $\Omega \setminus K$ which contradicts K being holomorphically convex. So $\Omega \setminus M$ can not any relatively compact components in Ω. Which tells us that M is holomorphically convex in Ω.

1.2: Let $u \in C^2(\Omega)$ on Ω, where $\Omega \subset \mathbb{C}$ is a domain.

(i) Suppose $\Delta u \geq 0$. Let $G \subset \Omega$ be a relatively compact subdomain, h be a continuous function on $\overline{G}$ such that $u \leq h$ on bG. Consider the function

$$v(z) = u(z) - \epsilon + \delta |z|^2$$

where $\epsilon > 0$ and $\delta \in (0, \epsilon/R)$ with $R = \sup\{|z|^2 \colon z \in bG\}$.

Calculate $\Delta v = \Delta u(z) + \delta \Delta |z|^2 = \Delta u(z) + \delta \Delta x^2 + y^2 = \Delta u(z) + \delta 4 \geq 4\delta > 0$. Hence $\Delta v > 0$. And also $v \leq u$ on bG by the condition on R given.

We will show $v \leq h$ on G. Note $\Delta(v - h) = \Delta v - \Delta h = \Delta v > 0$ and $v - h \leq 0$ on bG. We will show that $v - h$ can not attain maximum

on any point of G. This will show $\sup_G(v-h) \leq \sup_{bG}(v-h) \leq 0$. Hence, we will get $v \leq h$ on G.
Denote $f = v - h$ on $\overline{G}$. We have $\Delta f \geq 4\delta > 0$. Let $z \in G$.
Define average $A(r) := \frac{1}{2\pi r}\int_{bB(z,r)} f dS$ where S is the boundary measure and $r > 0$ such that $B(z,r) \subset G$. As f is a continuous function, limiting $r \to 0$, we get $\lim_{r\to 0} A(r) = f(z)\frac{S(bB(z,r))}{2\pi r} = f(z)$ as the measure of $bB(z,r) = 2\pi r$.
Let us look at the derivate of A. Since f is a continuous function over a compact interval, we can differentiate inside the integral

$$\begin{aligned}
A(r) &= \frac{1}{2\pi r}\int_{bB(z,r)} f(\zeta)dS(\zeta) \\
&= \frac{1}{2\pi r}\int_{[0,2\pi]} f(z + r(\cos t, \sin t))rdt \\
&= \frac{1}{2\pi}\int_{[0,2\pi]} f(z + r(\cos t, \sin t))dt \\
A'(r) &= \frac{1}{2\pi}\int_{[0,2\pi]} \frac{d}{dr} f(z + r(\cos t, \sin t))dt \\
&= \frac{1}{2\pi}\int_{[0,2\pi]} \nabla f(z + r(\cos t, \sin t)) \cdot (\cos t, \sin t)dt \\
&= \frac{1}{2\pi r}\int_{[0,2\pi]} \nabla f(z + r(\cos t, \sin t)) \cdot (\cos t, \sin t)rdt \\
&= \frac{1}{2\pi r}\int_{bB(z,r)} \nabla f(\zeta) \cdot \eta(\zeta)dS(\zeta)
\end{aligned}$$

Here η is the unit normal to the boundary. Using the divergence theorem (integration by parts in 2 variables) we get:

$$\begin{aligned}
A'(r) &= \frac{1}{2\pi r}\int_{B(z,r)} \nabla \cdot \nabla f(\beta)d\mu(\beta) \\
&= \frac{1}{2\pi r}\int_{B(z,r)} f_{xx}(\beta) + f_{yy}(\beta)d\mu(\beta) \\
&= \frac{1}{2\pi r}\int_{B(z,r)} \Delta f d\mu(\beta) \\
&\geq \frac{1}{2\pi r}\int_{B(z,r)} 4\delta d\mu(\beta) \\
&\geq \frac{1}{2\pi r}4\delta\pi r^2 \\
&\geq 4\delta r \\
&> 0
\end{aligned}$$

where μ is the area measure. Since A has derivate strictly postive. We get A is strictly monotone. Which gives $A(r) > \lim_{r\to 0} A(r) = f(z)$.

Suppose f had a maxima at $z \in G$. The choose $r > 0$ such that $B(0,r) \subset G$. Then we get $A(r) > f(z)$. But this is impossible as f attains maxima at z.

$$\begin{aligned} A(r) - f(z) &= \frac{1}{2\pi r}\int_{bB(z,r)} f(\zeta) - f(z) dS(\zeta) \\ &\leq 0 \end{aligned}$$

So the average must be lesser than or equal to $f(z)$. This contradicts that f attains maxima at z. Hence, f can not attain maxima at z and supremum of f is only attained on the boundary (as $\overline{G}$ is compact). But on the boundary $f \leq 0$. This gives $f \leq 0$ on G. Therefore $v \leq h$ on G.

So we have that v is a subharmonic function on G. First take $\delta \to 0$, then the resulting function is decreasing limit of subharmonic fuctions hence is subharmonic. Then adding ϵ to the limit function we get u. Which shows that u is subharmonic.

(ii) Suppose $\Delta u(a) < 0$ for some $a \in \Omega$. Then consider by continuity of Δu we get $\Delta u < 0$ in a small ball say $B = B(a, r_o)$. Then $\Delta(-u) > 0$ on G. Following the calculation as above we get

$$-u(a) < \frac{1}{2\pi r_o}\int_{bB} -u dS$$

And hence

$$u(a) > \frac{1}{2\pi r_o}\int_{bB} u dS$$

Suppose h is harmonic on $B(a, r_o)$ such that $h = u$ on bB. By the MVT of harmonic functions we get

$$h(a) = \frac{1}{2\pi r_o}\int_{bB} h dS$$

Suppose u is subharmonic, we must have $h \geq u$ on B. With the previous equation

$$u(a) - h(a) > \frac{1}{2\pi r_o}\int_{bB} (u-h) dS = 0$$

So $u(a) > h(a)$ this contradicts $h \geq u$ on B. Hence, u can not be subharmonic.

1.3: Let $L_1 = \{z \in \mathbb{C}^2 : (z,a) = b\}$ where $a \in \mathbb{C}^2 \setminus \{0\}$ and $b \in \mathbb{C}$. Suppose $u, v \in L_1$ such that $u \neq v$. Denote $w = u - v = (w_1, w_2) \in \mathbb{C}^2$. As $u \neq v$, WLOG we can take $w_2 \neq 0$. We have $(u,a) = b$ and $(v,a) = b$. So $(u-v, a) = (w,a) = 0$. Which gives $w_1\bar{a}_1 + w_2\bar{a}_2 = 0 \implies a_2 = -\frac{\bar{w}_1}{\bar{w}_2}a_1$. Hence $a = (a_1, -\frac{\bar{w}_1}{\bar{w}_2}a_1)$. Also note that $a_1 \neq 0$ as $a \neq 0$. Now to get a condition on b: $(u,a) = u_1\bar{a}_1 + u_2\bar{a}_2 = b$. Dividing by $\bar{a}_1$, we get $u_1 + (-\frac{w_1}{w_2})u_2 = \frac{b}{\bar{a}_1}$.

Now suppose, we have another line $L_2 = \{z \in \mathbb{C}^2 : (z,\alpha) = \beta\}$ where $\alpha \in \mathbb{C}^2\backslash\{0\}$ and $\beta \in \mathbb{C}$. And suppose $u, v \in L_2$. With the same arguments as above, we get $\alpha = (\alpha_1, -\frac{\bar{w}_1}{\bar{w}_2}\alpha_1)$ with $\alpha_1 \neq 0$ and $u_1 + (-\frac{w_1}{w_2})u_2 = \frac{\beta}{\bar{\alpha}_1}$. Which gives $\frac{\beta}{\bar{\alpha}_1} = \frac{b}{\bar{a}_1}$. Finally look at the equivalence

$$(z,a) = b \Leftrightarrow (z,a)\frac{\bar{\alpha}_1}{\bar{a}_1} = b\frac{\bar{\alpha}_1}{\bar{a}_1} \Leftrightarrow (z, a\frac{\alpha_1}{a_1}) = \beta \Leftrightarrow (z,\alpha) = \beta$$

Therefore, $L_1 = L_2$.

1.4: **Part (i)** First we see that d is a metric. Let $f, g, h \in O(D)$. Non negativity $d(f,g) \geq 0$: this follows as all terms in the sequence are non negative. Also $d(f,f) = 0$: as $|f-f|_K = 0$, so each term in the sequence is zero. If $f \neq g$ then there exists a point $a \in D$ such that $|f(a) - g(a)| > 0$. So $d(f,g) > 0$ as there exists a j such that $a \in K_j$ and the jth term would be non zero in the sequence as $|f-g|_{K_j} > 0$ and hence the sequence is greater than zero.

$$\begin{aligned}
d(f,h) &= \sum_{j=1}^{\infty} 2^{-j} \frac{|f-h|_{K_j}}{1+|f-h|_{K_j}} \\
&= \sum_{j=1}^{\infty} 2^{-j} (1 - \frac{1}{1+|f-h|_{K_j}}) \\
&\leq \sum_{j=1}^{\infty} 2^{-j} (1 - \frac{1}{1+|f-g|_{K_j}+|g-h|_{K_j}}) \\
&\leq \sum_{j=1}^{\infty} 2^{-j} \frac{|f-g|_{K_j}+|g-h|_{K_j}}{1+|f-g|_{K_j}+|g-h|_{K_j}} \\
&\leq \sum_{j=1}^{\infty} 2^{-j} \frac{|f-g|_{K_j}}{1+|f-g|_{K_j}+|g-h|_{K_j}} + 2^{-j} \frac{|g-h|_{K_j}}{1+|f-g|_{K_j}+|g-h|_{K_j}} \\
&\leq \sum_{j=1}^{\infty} 2^{-j} \frac{|f-g|_{K_j}}{1+|f-g|_{K_j}} + \sum_{j=1}^{\infty} 2^{-j} \frac{|g-h|_{K_j}}{1+|g-h|_{K_j}} \\
&\leq d(f,g) + d(g,h)
\end{aligned}$$

Hence d is a metric.

Now let us show that the metric induces the topology of compact convergence.

Say $f_n \to g$ in the topology of compact convergence on $O(D)$ where $f_n, g \in O(D)$. Then let us show that $d(f_n, g) \to 0$. Given $\epsilon > 0$. Find J such that $1/2^J < \epsilon/2$.

$$\begin{aligned} d(f_n, g) &= \sum_{j=1}^{\infty} 2^{-j} \frac{|f_n - h|_{K_j}}{1 + |f_n - h|_{K_j}} \\ &= \sum_{j=1}^{J} 2^{-j} \frac{|f_n - h|_{K_j}}{1 + |f_n - h|_{K_j}} + \sum_{j=J+1}^{\infty} 2^{-j} \frac{|f_n - h|_{K_j}}{1 + |f_n - h|_{K_j}} \\ &\leq \sum_{j=1}^{J} 2^{-j} \frac{|f_n - h|_{K_j}}{1 + |f_n - h|_{K_j}} + \sum_{j=J+1}^{\infty} 2^{-j} \\ &\leq \sum_{j=1}^{J} 2^{-j} \frac{|f_n - h|_{K_j}}{1 + |f_n - h|_{K_j}} + 2^{-J} \\ &\leq \sum_{j=1}^{J} 2^{-j} \frac{|f_n - h|_{K_j}}{1 + |f_n - h|_{K_j}} + \epsilon/2 \end{aligned}$$

Now for each $j \in \{1, ..., J\}$ choose N_j such that for all $n \geq N_j$ we have

$$2^{-j} \frac{|f_n - h|_{K_j}}{1 + |f_n - h|_{K_j}} < \epsilon/(2J)$$

This can be choosen as $|f_n - h|_{K_j} \to 0$ for each j. Now choose $N = \max\{N_1, ..., N_J\}$.

Then for $n \geq N$:

$$d(f_n, g) \leq \sum_{j=1}^{J} 2^{-j} \frac{|f_n - h|_{K_j}}{1 + |f_n - h|_{K_j}} + \epsilon/2 \leq \sum_{j=1}^{J} \epsilon/(2J) + \epsilon/2 \leq \epsilon/2 + \epsilon/2 = \epsilon$$

Hence $d(f_n, g) \to 0$.

Conversely, say $d(f_n, g) \to 0$. Take $K \subset D$ compact. Now given $\epsilon > 0$, we will find N such that $n \geq N$ we have $|f_n - g|_K < \epsilon$. This will show $f_n \to g$ uniformly on K.

As $\{K_j\}$ exhaust D, we can find J such that $K \subset K_J$. Find M such that $n \geq M$ implies $d(f_n, g) < \frac{\epsilon}{1+\epsilon} 2^{-J}$. Then specifically, since the terms are non negative, we have $2^{-J} \frac{|f_n - h|_{K_J}}{1 + |f_n - h|_{K_J}} < \frac{\epsilon}{1+\epsilon} 2^{-J}$. Since $t/(1+t)$ is an increasing function on $t > 0$. We get $|f_n - h|_{K_J} < \epsilon$. Hence $f_n \to g$ uniformly on K.

So d induces the same topology as compact convergence on $O(D)$.

Part (ii) Say Σ is bounded.

Given any K compact in D. Find J such that $K \subset K_J$. Set $r = 2^{-J-1}$. We can find $\lambda > 0$ such that $\Sigma \subset \lambda B(0, r)$. Given $f \in \Sigma$ we have

$d(0,f/\lambda)<r$. Then looking at a single term we get

$$2^{-J}\frac{|f/\lambda|_{K_J}}{|f/\lambda|_{K_J}+1} < r = 2^{-J-1}$$
$$\frac{|f/\lambda|_{K_J}}{|f/\lambda|_{K_J}+1} < 1/2$$
$$|f|_{K_J} < \lambda$$

Given any function $f\in\Sigma$ we have got $|f|_K\leq |f|_{K_J}<\lambda$. Hence $\sup\{|f|_K\colon f\in\Sigma\}\leq\lambda<\infty$.

Suppose $\sup\{|f|_K\colon f\in\Sigma\}<\infty$ for all K compact in D. Let $M_j=\sup\{|f|_{K_j}\colon f\in\Sigma\}$. Take any $r\in(0,1)$. Let J be big enough such that $2^{-J}<r/2$. Choose λ big enough such that

$$2^{-j}\frac{M_j/\lambda}{1+M_j/\lambda} < r/2J$$

for $j=1,...,J$. Then as shown in part (i). We get

$$\begin{aligned}
d(0,f/\lambda) &= \sum_{j=1}^{J}2^{-j}\frac{|f/\lambda|_{K_j}}{1+|f/\lambda|_{K_j}}+\sum_{j=J+1}^{\infty}2^{-j}\frac{|f/\lambda|_{K_j}}{1+|f/\lambda|_{K_j}}\\
&\leq \sum_{j=1}^{J}2^{-j}\frac{|f/\lambda|_{K_j}}{1+|f/\lambda|_{K_j}}+\sum_{j=J+1}^{\infty}2^{-j}\\
&\leq \sum_{j=1}^{J}2^{-j}\frac{M_j/\lambda}{1+M_j/\lambda}+2^{-J}\\
&< \sum_{j=1}^{J}r/2J+r/2\\
&< r
\end{aligned}$$

Hence $f/\lambda\in B(0,r)$. That is $f\in\lambda B(0,r)$. So $\Sigma\subset\lambda B(0,r)$. Therefore Σ is bounded.

1.5: Let $\Omega=\mathbb{D}^2(0,1)\setminus\overline{\mathbb{D}^2(0,r)}$ for $0<r<1$, be a domain in $\mathbb{C}^2$. Suppose $f\in O(\Omega)$.

(i) and (ii) Fix z_1 with $|z_1|<1$, we consider two cases

Case 1 $r<|z_1|<1$ **:** Then for any given $|z_2|<1$, we get $(z_1,z_2)\in\Omega$. Since we have $f\in O(\Omega)$, it is holomorphic in each variable; we have $z_2\mapsto f(z_1,z_2)$ is holomorphic at the set of points

$$\{z_2:(z_1,z_2)\in\Omega\}=\{z_2:|z_2|<1\}$$

Hence, we can write $z_2 \mapsto f(z_1, z_2)$ as a power series on the unit ball. Once we fix z_1, we get a power series in z_2; so the coefficients of the power series are dependent on z_1. We get

$$z_2 \mapsto f(z_1, z_2) = \sum_{n=0}^{\infty} a_n(z_1) z_2^n$$

We can let $a_n(z_1) = 0$ when n is a negative integer. Giving us

$$z_2 \mapsto f(z_1, z_2) = \sum_{n=-\infty}^{\infty} a_n(z_1) z_2^n$$

Now we must show that a_n are continuous functions of z_1. Since we have a power series about zero, we can calculate a_n by taking the following integral:

$$a_n(z_1) = \frac{1}{2\pi i} \int_{|\zeta|=r+\delta_r} \frac{f(z_1, \zeta)}{\zeta^{j+1}} d\zeta$$

where $\delta_r > 0$ such that $r + \delta_r < 1$.
We must show that this function is continuous in z_1. Let $w_n \to z_o$

$$\begin{aligned} |a_n(w_n) - a_n(z_o)| &\leq \frac{1}{2\pi} \int_{|\zeta|=r} \left| \frac{f(w_n, \zeta) - f(z_o, \zeta)}{\zeta^{j+1}} \right| d\zeta \\ &\leq \frac{1}{2\pi r^{j+1}} \int_{|\zeta|=r+\delta_r} |f(w_n, \zeta) - f(z_o, \zeta)| d\zeta \end{aligned}$$

Since f is continuous, it is uniformly continuous on the compact set $\{(z_1, z_2) : z_1 \in \overline{\mathbb{D}(z_o, \epsilon_o)}$ and $|z_2| = r\}$ for some small epsilon that such that $\overline{\mathbb{D}(z_o, \epsilon_o)} \subset \mathbb{D}(0, 1)$. Given $\epsilon > 0$, we find some $\delta > 0$ such that for u, w in this compact set such that $|u - w| < \delta$ gives $|f(u) - f(w)| < \epsilon$.
Choose w_n close enough such that $|w_n - u_o| < \delta$:

$$\begin{aligned} |a_n(w_n) - a_n(z_o)| &\leq \frac{1}{2\pi r^{j+1}} \int_{|\zeta|=r+\delta_r} |f(w_n, \zeta) - f(z_o, \zeta)| d\zeta \\ &\leq \frac{1}{2\pi r^{j+1}} \int_{|\zeta|=r+\delta_r} \epsilon d\zeta \\ &\leq \frac{1}{r^j} \epsilon \end{aligned}$$

So $a_n(w_n) \to a_n(z_o)$ as $w_n \to z_o$. Hence, a_n is continuous.
Case 2 $|z_1| \leq r$ **:** For this case we will get an annulus. Then for $r < |z_2| < 1$ we get $(z_1, z_2) \in \Omega$. We get $z_2 \mapsto f(z_1, z_2)$ is holomorphic in

this annulus. Take the Laurents series expansion about zero. When $r < |z_2| < 1$.

$$z_2 \mapsto f(z_1, z_2) = \sum_{n=-\infty}^{\infty} a_n(z_1) z_2^n$$

Where

$$a_n(z_1) = \frac{1}{2\pi i} \int_{|\zeta|=r+\delta_r} \frac{f(z_1,\zeta)}{\zeta^{j+1}} d\zeta$$

Note $(z_1, \zeta) \in \Omega$ for all $|\zeta| = r + \delta_r$. Again, with the same calculation as before we get a_ns are continuous functions in z_1.

Remark: we can not take

$$a_n(z_1) = \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{f(z_1,\zeta)}{\zeta^{j+1}} d\zeta$$

As this evaluatest f at (z_1, ζ) where $|\zeta| = r$ and $|z_1| \leq r$. In this case $(z_1, \zeta) \notin \Omega$. I fixed the issue by taking $|\zeta| = r + \delta_r$.

(iii) To show a_n is holomorphic in z_1 it is enough to check $\frac{\partial}{\partial z} a_n = 0$.

$$\frac{\partial}{\partial z} a_n(z_1) = \frac{1}{2\pi i} \frac{\partial}{\partial z} \int_{|\zeta|=r+\delta_r} \frac{f(z_1,\zeta)}{\zeta^{j+1}} d\zeta$$

We can take the differential operator inside the integral as the function is continuously differentiable in the compact region of integration and is bounded above.

$$\frac{\partial}{\partial z} a_n(z_1) = \frac{1}{2\pi i} \int_{|\zeta|=r+\delta_r} \frac{\partial}{\partial z} \frac{f(z_1,\zeta)}{\zeta^{j+1}} d\zeta = 0$$

As $z_1 \mapsto f(z_1, \zeta)$ is holomorphic. Also note that, since $|\zeta| = r + \delta_r$ and $(z_1, \zeta) \in \Omega$, a_n can also be defined for all $z_1 \in \mathbb{D}(0, 1) = \mathbb{D}$. So $a_n \in O(\mathbb{D})$.

(iv) For $j < 0$, since $a_j \in O(\mathbb{D})$ and $a_j = 0$ on $|z_1| > r$, which contains a limit point, we get that a_j is identically zero on $\mathbb{D}$.

(v) Take $|z_1| \leq r$. Consider the Laurents series expansion we have given before. When $r < |z_2| < 1$.

$$z_2 \mapsto f(z_1, z_2) = \sum_{n=-\infty}^{\infty} a_n(z_1) z_2^n$$

But since a_j is identically zero for $j < 0$, we will get this map as a power series.

$$z_2 \mapsto f(z_1, z_2) = \sum_{n=0}^{\infty} a_n(z_1) z_2^n$$

The radius of convergence given by the root test would match the one we had for the annulus. Which tells us that this map is analytic on the unit disc.

Hence we can define $f(z_1, z_2)$ for any point $|z_2|< 1$. So we can extend f to all points $|z_1|< 1$ and $|z_2|< 1$. Let us denote it by $\tilde{f} : \mathbb{D}^2 \to \mathbb{C}$.

$$\tilde{f}(z_1, z_2) = \sum_{n=0}^{\infty} a_n(z_1) z_2^n$$

It is clear that it is holomorphic in z_2 for any fixed $|z_1|< 1$. If we fix $|z_2|< 1$, then we have

$$\frac{\partial}{\partial z_1} \tilde{f}(z_1, z_2) = \sum_{n=0}^{\infty} \frac{\partial}{\partial z_1} a_n(z_1) z_2^n = 0$$

As a_n is holomorphic on $\mathbb{D}$. So it partially holomorphic in each variable. This is a holomorphic function on $\mathbb{D}^2(0, 1)$ using Hartogs theorem.

Remark: we don't need to use Hartogs Theorem here, we can show that f is continuous by a sequential argument as a_ns are continuous.