SCV Assignment 3

Harshith Sairaj Alagandala
Student number: 251388575

October 27, 2023

3.1: Let f be holomorphic in the complement of the ball B_r and let it be bounded in that region by a constant $M > 0$.

Let us consider the domain $D = \mathbb{C}^n$ and $K = \overline{B_r}$. Since the ball is bounded, we have K is comapct and is compactly contained in D. The set $D \setminus K$ is a connected set as it is open and path connected. By the Hartogs Kugelsatz theorem, any function $f \in \mathscr{O}(D \setminus K)$ extends holomorphically to D. So the function f extends to $\hat{f} \in \mathscr{O}(\mathbb{C}^n)$. Which tells that $\hat{f}$ is an entire function.

Specifically, $\hat{f}$ is continuous on K. So $\hat{f}$ is bounded by a constant $M' > 0$ on K. Since $f = \hat{f}$ on $D \setminus K$, we get $\hat{f}$ is bounded above on D by $M_o = \max\{M, M'\}$. So, $\hat{f}$ is an entire function that is bounded. By considering any point $z \in \mathbb{C}^n$ and letting the coefficient j vary, we get the map $w_j \mapsto \hat{f}(z_1, ..., z_{j-1}, w_j, z_{j+1}, ..., z_n)$. This is an entire map on $\mathbb{C}$ that is bounded. Hence, it must be constant. Doing this process for any z and $j = 1, ..., n$, we get that $\hat{f}$ is constant. In turn, we get f is constant on its domain $D \setminus K$.

3.2: (i) Let $w \in bD$. We must show that the tangential Cauchy-Riemann operator $\bar{\partial}_T$ acting on f gives zero at w. The operator $\bar{\partial}_T$ is the $\bar{\partial} - \bar{\partial}_N$, where $\bar{\partial}_N$ is the normal direction to the boundary at w. Suppose $\bar{\partial}_T|_w = \sum a_j \frac{\partial}{\partial \bar{z}_j} d\bar{z}_j$. Keeping the coefficients $\{a_j\}$ fixed, we can define this form on any point of D as $\omega = \sum a_j \frac{\partial}{\partial \bar{z}_j} d\bar{z}_j$. Then we see that ωf is the same as $\bar{\partial}_T f$ at the point w. On any point $z \in D$, we have $\omega f = 0$: since $\frac{\partial f}{\partial \bar{z}_j}|_z = 0$ as f is holomorphic. Since f is C^1, we must have ωf to be a form with continuous coefficients on $D \cup \{w\}$. Since w is a boundary point, we have a sequence of points $\{a_j\}$ in D such that $a_j \to w$. And hence $\omega f = 0$ at w. Which means, the tangential Cauchy-Riemann operator acting on f gives zero at any boundary point of D.

(ii) Let $M = \{z \in \mathbb{C}^n : x_n = 0\}$, consider the function

$$f(x_1, y_1, ..., x_{n-1}, y_{n-1}, y_n) = \begin{cases} 0 & y_n < 0 \\ e^{-1/y_n} & y_n \geq 0 \end{cases}$$

This function is C^1 on M as the directional derivatives with respect to each of $x_1, y_1, ..., x_{n-1}, y_{n-1}, y_n$ is continuous. Moreover $\frac{\partial}{\partial \bar{z}_j} f = 0$ for $j = 1, ..., n-1$. The normal directoin is x_n, so the complex normal would be complex span of x_n which is real span of x_n and y_n. Which means the complex tangential direction is the real span of $x_1, y_1, ..., x_{n-1}, y_{n-1}$. So $\bar{\partial}_T f = \sum_{j=1}^{n-1} \frac{\partial}{\partial \bar{z}_j} f = 0$. The function is CR since $\bar{\partial}_T f = 0$.

This function cannot be extended holomorphically in a neighbourhood of M. If it could have been, then there is a power series representation of the extension of f about zero . This power series when restricted to M should match f. This is not possible as the function cannot have a power series. As the power series would have been the zero function, but the function is non zero at all points $y_n > 0$.

3.3: First, we will show that f is partially holomorphic even at the boundary of the polydisc. Choose $z^{(0)} = (z_1, ..., z_n) \in b\mathbb{D}^n$ (Toplogical boundary) such that $|z_k| < 1$ and $|z_j| = 1$ for some $k, j \in \{1, ..., n\}$. WLOG, suppose $|z_1| = 1$ and suppose $|z_2| < 1$. Claim: $g(w) := f(z_1, w, z_3, ..., z_n)$ is holomorphic near z_2. Find a sequence of points $(a^{(j)})$ in $\mathbb{D}^n$ that converges to $z^{(0)}$ such that $a_2^{(k)} = w$. Let $g_k(w) = f(a_1^{(k)}, w, a_3^{(k)}, ..., a_n^{(k)})$. As g_k is holomorphic:

$$g_k(w) = \int_{B_r(w)} \frac{f(a_1^{(k)}, \eta, a_3^{(k)}, ..., a_n^{(k)})}{\eta - w} d\eta$$

For $r > 0$ small enough to contain $B_r(w)$ in the unit ball.
We get $g_k(w) \to g(w)$ as $k \to \infty$ by the continuity of f.

$$g(w) = \lim_{k\to\infty} \int_{B_r(w)} \frac{f(a_1^{(k)}, \eta, a_3^{(k)}, ..., a_n^{(k)})}{\eta - w} d\eta$$

Since f is uniformly continuous on $\mathbb{D}^n$, we can take the limit into the integral:

$$g(w) = \int_{B_r(w)} \lim_{k\to\infty} \frac{f(a_1^{(k)}, \eta, a_3^{(k)}, ..., a_n^{(k)})}{\eta - w} d\eta$$

Since each component converges, we get $a_j^{(k)} \to a_j^{(0)}$ as $k \to \infty$ for $j = 1, 3, 4, ..., n$.

$$g(w) = \int_{B_r(w)} \frac{f(a_1^{(0)}, \eta, a_3^{(0)}, ..., a_n^{(0)})}{\eta - w} d\eta$$

Now it is clear that g is holomorphic. As $\frac{d}{d\bar{w}} g(w) = 0$.

Suppose $|f|$ attains a maxima at $a \in \overline{\mathbb{D}^n}$. If $a \notin b_o\mathbb{D}^n$, we have $|a_j| < 1$ for some $j = 1, ..., n$. Consider the function $g(w) := f(a_1, ..., a_{j-1}, w, a_{j+1}, ..., a_n)$. This function is holomorphic in $\mathbb{D}$: if $a \in \mathbb{D}^n$ then by the partial holomorphicity of holomorphic functions else we use the argument stated above.

By the maximal modulus principle, we obtain that there is a point $b_j \in b\mathbb{D}$ (ie, $|b_j|= 1$) such that $|g(b_j)|\geq |g(a_j)|$. If $(a_1, ..., a_{j-1}, b_j, a_{j+1}, ..., a_n) \notin b_o\mathbb{D}$, repeat the same process for $(a_1, ..., a_{j-1}, b_j, a_{j+1}, ..., a_n)$. Until we obtain a point z such that all the components have modulus 1. We get $|f(z)|\geq |f(a)|$, but since $|f|$ attains maxima at a on $\overline{\mathbb{D}^n}$, we have $|f(z)|= |f(a)|$. So, $|f|$ attains a maxima at the point z of the distinguished boundary.

3.4: (i) Consider a point $(w_1, w_2) \in A$ such that $(w_1, w_2) \neq (0, 0)$. Then clearly $w_1 \neq 0$ and $w_2 \neq 0$. We will show that A is regular at (w_1, w_2). Consider the map $F(z_1, z_2) = (z_1, z_1^2 - z_2^3)$. Let us calculate the complex Jacobian of this map

$$|DF(z_1, z_2)|= \det \begin{pmatrix} 1 & 0 \\ 2z_1 & 3z_2^2 \end{pmatrix} = 3z_2^2$$

In particular, at (w_1, w_2), we get the Jacobian is not zero as $w_2 \neq 0$. We can apply the inverse function theorem for complex holomorphic functions. We get a local biholomorphism $G = (g_1, g_2)$ between a neighbourhood U of (w_1, w_2) and V of $(w_1, w_1^2 - w_2^3)$. And the set $A \cap U = [g_2 = 0]$. Hence locally A is a submanifold. So the point (w_1, w_2) is a regular point.

Suppose $(0, 0)$ is regular, and we have a submanifold at $(0, 0)$ that corresponds to A locally. At $(0, 0)$, if we restrict z_1 to its real part x_1 and z_2 to its real part x_2. We get that $\{(t^3, t^2) : t \in \mathbb{R}\}$ is a submanifold of $\mathbb{R}^2$. Which is not true.

(ii) Consider the map $\phi : \mathbb{C} \to A$ given by $\phi(z) = (z^3, z^2)$. It is a continuous map as each component is a polynomial. It is clear the range is in A as $z^6 = z^6$. Define the map $\psi : A \to \mathbb{C}$

$$\psi(z_1, z_2) = \begin{cases} z_1/z_2 & z_2 \neq 0 \\ 0 & z_2 = 0 \end{cases}$$

. This is a continuous map: it is continuous at all points where $z_2 \neq 0$ (rational function). At zero: Let $((a_n, b_n)) \subset A$ limit to $(0, 0)$ such that $b_n \neq 0$. Then $a_n^2 = b_n^3$, so $|\psi(a_n, b_n)|= |\frac{a_n}{b_n}|= \frac{|a_n|}{|b_n|} = (\frac{|a_n^3|}{|b_n^3|})^{1/3} = (|\frac{a_n^3}{b_n^3}|)^{1/3} = (|a_n|)^{1/3}$. Since $(a_n, b_n) \to (0, 0)$, we have $|\psi(a_n, b_n) - \psi(0, 0)|= |\psi(a_n, b_n) - 0|= |\psi(a_n, b_n)|= |a_n|^{1/3}\to 0$. Hence, this map is continuous.

When $z \in A$ and $z_2 \neq 0$, we have $\phi\circ\psi(z_1, z_2) = \phi(z_1/z_2) = (\frac{z_1^3}{z_2^3}, \frac{z_1^2}{z_2^2}) = (z_1\frac{z_1^2}{z_2^3}, z_2\frac{z_1^2}{z_2^3}) = (z_1, z_2)$. When $z_2 = 0$, $z = (0, 0)$, then we get the composition to be $(0, 0)$. Hence that the map is a bijection between $\mathbb{C}$ and A. And both the forward and backward maps are continuous. So we have a homeomorphism.