SCV Assignment 2

Harshith Sairaj Alagandala
Student number: 251388575

October 13, 2023

2.1: The domain of convergence is the set of all points where the power series converges in some neighbourhood of that point.

Consider the power series $p(x) = \sum_{k=0}^{\infty} z_1^k z_2^k$. By Abel's lemma: If $(a, b) \in \mathbb{C}^n$ such that $|a^k b^k| < M < \infty$ for any term in the power series p, then p converges in the polydisc $\mathbb{D}(0, (|a|, |b|))$.

Take $(a, b) \in \mathbb{C}^n$ such that $|ab| < 1$. We can choose $\epsilon > 0$ small enough that $|(1+\epsilon)a(1+\epsilon)b| < 1$. Set $p = (1+\epsilon)a$ and $q = (1+\epsilon)b$. . Clearly, $|p^k q^k| = (pq)^k < 1 < \infty$. So, the power series converges in the polydisc $\mathbb{D}(0, (p, q))$. And $(a, b) \in \mathbb{D}(0, (p, q))$. Since the polydisc is an open set, there is some neighbourhood of (a, b) in it. Hence, the power series converges in a neighbourhood of the point.

Suppose $(a, b) \in \mathbb{C}^n$ such that $|ab| \geq 1$. Given any neighbourhood U of (a, b) we can choose $\epsilon > 0$ small enough such that $((1+\epsilon)a, (1+\epsilon)b)$ lies in U. At this point, the terms of the power series would have modulus $|(1+\epsilon)^{2k}(ab)^k| = |(1+\epsilon)^{2k}||ab|^k \geq |(1+\epsilon)^{2k}| > 1$. So the power series can not converge absolutely at this point. Hence, (a, b) will not have any neighbourhood that the power series converges in.

By the above, the domain of convergence of p is precisely given by

$$\Omega = \{(a, b) \in \mathbb{C}^n : |ab| < 1\}$$

This can not be a polydisc as: It is not convex: $(2, 1/3), (1/3, 2) \in \Omega$ but

$$\left(\frac{2 + 1/3}{2}, \frac{2 + 1/3}{2}\right) = \left(\frac{7}{6}, \frac{7}{6}\right) \notin \Omega$$

Let us take a point $(x, y) \in A := \lambda(\Omega \setminus \{z_1.z_2...z_n = 0\})$. By the definition of A we have $(z, w) \in \Omega$ such that $\lambda(z, w) = (x, y)$. We know $|zw| < 1$, so $\ln(|zw|) < 0$ which gives $\ln(|z|) + \ln(|w|) = x + y < 0$.

Now suppose we are given any points $(a, b) \in \mathbb{R}^2$ with $a + b < 0$, then as a point in $\mathbb{C}^2$, $(\exp a, \exp b) \in \Omega$ as $\exp(a + b) < 1$.

Therefore $A = \{(x, y) \in \mathbb{R}^n : x + y < 0\}$. This is a half-plane and is convex.

2.2: (i) We have $A \subset \mathbb{C}^n$:

$$A = \bigcup_{j=1}^{\infty} \{z_n = 1/j\}$$

and f is a holomorphic function on $\mathbb{C}^n$ such that $f|_A = 0$.
Fix any $z' = (z_1, z_2, ..., z_{n-1})$. Then the map

$$z_n \to f(z', z_n)$$

is holomorphic in one variable on $\mathbb{C}$. Since f is zero on A. This one variable function is zero at $1/j$ for all $j \in \mathbb{N}$. We know the set of zeros of a non-constant holomorphic function in one variable can not have a limit point. But $z_n = 0$ is a limit point of the zeros of $z_n \to f(z', z_n)$. Hence, the function is constant, so it must be the zero function. Which tells us that $f(z', z_n) = 0$ for all $z_n \in \mathbb{C}$. This works for any choice of z'. Hence $f \equiv 0$.

(ii) Consider the points

$$A = \{(1/j_1, 1/j_2, ..., 1/j_n) \in \mathbb{C}^n : j_1, j_2 ..., j_n \in \mathbb{N}\}$$

We see that 0 is a limit point for this set. Suppose f is holomorphic in $\mathbb{C}^n$ such that $f|_A = 0$. Fix $j_1, ..., j_{n-1}$, then the function

$$z_n \to f(1/j_1, ..., 1/j_{n-1}, z_n)$$

must be a zero function as it is a holomorphic function in one variable whose zero set $\{1/j : j \in \mathbb{N}\}$ has a limit point 0. This can be done for any choice of the first $n-1$ coordinates in A. So f is zero on the set

$$A_1 = \{(1/j_1, ..., 1/j_{n-1}, z_n) \in \mathbb{C}^n : j_1, j_2 ..., j_{n-1} \in \mathbb{N}, z \in \mathbb{C}\}$$

Now fix the first $n-2$ term as per A and fix any $z_n' \in \mathbb{C}$, then the function

$$z_{n-1} \to f(1/j_1, ..., 1/j_{n-2}, z_{n-1}, z_n')$$

must be a zero function as the function is zero on $z_{n-1} = 1/j$ for $j \in \mathbb{N}$, as f is zero on A_1. So f is zero on the set

$$A_2 = \{(1/j_1, ..., 1/j_{n-2}, z_{n-1}, z_n) \in \mathbb{C}^n : j_1, j_2 ..., j_{n-2} \in \mathbb{N}, z_{n-1}, z_n \in \mathbb{C}\}$$

Continuing this process we get f is zero on $\mathbb{C}^n$.

2.3: Let M_1 and M_2 be closed connected complex submanifolds of $\mathbb{C}^n$. Let us define a set $A \subset \mathbb{C}^n$ as the set of points in $M_1 \cap M_2$ such that for any point $p \in A$ there exists a neighbourhood U in $\mathbb{C}^n$ about p such that $M_1 \cap U = M_2 \cap U$.

Note on dimensions: Since M_1 and M_2 match in some neighbourhood U in $\mathbb{C}^n$. They would locally be represented as the zero set of the same set of holomorphic functions. Hence the dimension of M_1 and M_2 are the same. Let us suppose it is $0 < k < n$.

We are given that A is not empty. For $j = 1, 2$, we will show that A is closed and open in M_j. Since $A \subset M_j$ and M_j is connected, we must have $A = M_j$. Therefore, $M_1 = M_2$.

The set A is open in M_1: given $p \in A$, we have a neighbourhood U in $\mathbb{C}^n$. Then consider $U_p = M_1 \cap U$. This is a neighbourhood of p in M_1. For any point in $q \in U_p$, the same open set U works, that is, $q \in U$ and $M_1 \cap U = M_2 \cap U$. Hence, $U_p \subset A$. So, A is open in M_1.

The set A is closed in M_1: Suppose $p \in M_1$ is a limit point of A. By the definition of submanifold, we have a neighbourhood U_1 about p in $\mathbb{C}^n$ such that $M_1 \cap U_1$ is the zero set of $f_1, ..., f_k$ where $F = (f_1, f_2, ..., f_k, f_{k+1}, ..., f_n)$ is biholomorphism from U_1 to an open set of $\mathbb{C}^n$. Similarly, we have a neighbourhood U_2 about p in $\mathbb{C}^n$ such that $M_2 \cap U_2$ is the zero set of $g_1, ..., g_k$ where $G = (g_1, g_2, ..., g_k, g_{k+1}, ..., g_n)$ is biholomorphism from U_2 to an open set of $\mathbb{C}^n$.

Set $U = U_1 \cap U_2$. For convenience, we restrict F and G to U. Then look at the biholomorphism $F \circ G^{-1}$.

As $U \cap M_1$ is open in M_1 and p is a limit point of A, we can find $q \in (U \cap M_1) \cap A$. So we have a neighbourhood U_q in $\mathbb{C}^n$ such that $M_1 \cap U_q = M_2 \cap U_q$.

On the set $V := G(U_q \cap M_1)$ the first k coordinates of $F \circ G^{-1}$ is zero on $\{z_1 = z_2 = ... = z_k = 0\} \cap V$. More precisely, set $H := F \circ G^{-1}$. Then $H = (h_1, ..., h_n)$, with $h_1 = h_2 = ... = h_k = 0$ on $\{z_1 = z_2 = ... = z_k = 0\} \cap V$. Note that h_j are holomorphic on $G(U)$. Fixing all first $n-1$ variables in $\{z_1 = z_2 = ... = z_k = 0\} \cap V$ we get the map $z_n \to h_1(0, 0, ..., 0, z'_{k+1}, ..., z'_{n-1}, z_n)$ is a holomorphic map. And since the zero set would be a non empty open set for this, we have this map to be zero. Now repeat the same process for z_{n-1} but this time we have the liberty to choose any point in the domain for z_n. This process will end up telling us that: $h_1 = 0$ on $\{z_1 = z_2 = ... = z_k = 0\} \cap G(U)$. Similarly $h_2, ..., h_k = 0$ on $\{z_1 = z_2 = ... = z_k = 0\} \cap G(U)$. Which tells us that the zero set of $f_1, ..., f_k$ on U is the same as the zero set of $g_1, ..., g_k$ on U. Hence $U \cap M_1 = U \cap M_2$.

Finally, with the connected argument given before, we have $M_1 = M_2$.

2.4: Let us show that $\phi : \mathbb{B}^n \to \mathbb{C}^n$

$$\phi(z', z_n) = \left(\frac{z'}{1 + z_n}, i \frac{1 - z_n}{1 + z_n} \right)$$

is a biholomorphism onto $\mathbb{H}$.

This is a holomorphic map from $\mathbb{B}^n \to \mathbb{C}^n$ as each of the coordinates is a rational function defined on $\mathbb{B}^n$.

Let us see that the range of ϕ is in $\mathbb{H}$. Let $w = \phi(z)$, for $z = (z_1, ..., z_n) \in \mathbb{B}^n$. Let us calculate $\Im(w_n)$. By the formula for ϕ, we get $w_n = i\frac{1-z_n}{1+z_n}$.

$$\begin{aligned}
\Im(w_n) &= \frac{1}{2i}(w_n - \overline{w_n}) \\
&= \frac{1}{2i}\left(i\frac{1-z_n}{1+z_n} - \overline{i\frac{1-z_n}{1+z_n}}\right) \\
&= \frac{i}{2i}\left(\frac{1-z_n}{1+z_n} + \frac{1-\bar{z}_n}{1+\bar{z}_n}\right) \\
&= \frac{1-|z_n|^2}{|1+z_n|^2}
\end{aligned}$$

Let us calculate $|w'|^2$

$$|w'|^2 = \left|\frac{z'}{1+z_n}\right|^2 = \frac{|z'|^2}{|1+z_n|^2} = \frac{\sum_{j=1}^{n-1}|z_j|^2}{|1+z_n|^2}$$

Since we are in the unit ball $|z|^2 = \sum_{j=1}^{n}|z_n|^2 < 1$. Which gives $1 - |z_n|^2 > \sum_{j=0}^{n-1}|z_j|^2$. And hence we have $\Im(w_n) > |w'|^2$.

Remark: In the assignment question, $\mathbb{H}$ was defined as

$$\mathbb{H} = \{z \in \mathbb{C}^n : y_n < |z'|^2, z' = (z_1, ..., z_{n-1})\}$$

But as seen above, the map takes $\Im(w_n) > |w'|^2$. So I will take $\mathbb{H}$ to be

$$\mathbb{H} = \{z \in \mathbb{C}^n : y_n > |z'|^2, z' = (z_1, ..., z_{n-1})\}$$

Now let us show ϕ is injective. Say $\phi(z) = \phi(u)$. Then

$$\left(\frac{z'}{1+z_n}, i\frac{1-z_n}{1+z_n}\right) = \left(\frac{u'}{1+u_n}, i\frac{1-u_n}{1+u_n}\right)$$

Comparing the second coordinates gives us $z_n = u_n$. Then we compare the first one to get $z' = u'$. Hence $z = u$.

Now we show surjectivity. Suppose $w \in \mathbb{H}$. Then $w_n = i\frac{1-z_n}{1+z_n}$. After a bit of algebra we get $z_n = \frac{i-w_n}{i+w_n}$. And $w' = \frac{z'}{1+z_n}$; solving for z' by substituting w_n gives $z' = w'\left(1 + \frac{i-w_n}{i+w_n}\right)$. These values are well defined when $w \in \mathbb{H}$ as $w_n \neq -i$ and solving for the norm of the inverse gives us that $|z|^2 < 1$ if we take into consideration $\Im(w_n) > |w'|^2$.

Again, the inverse map is holomorphic as it is again a rational function in each variable that is defined on the domain. Hence ϕ is a biholomorphism.