Choice Principles in Intuitionistic Set Theory

John L. Bell

In intuitionistic set theory, the law of excluded middle is known to be derivable from the standard version of the axiom of choice that every family of nonempty sets has a choice function. In this paper it is shown that each of a number of intuitionistically invalid logical principles, including the law of excluded middle, is, in intuitionistic set theory, equivalent *to a suitably weakened version of the axiom of choice. Thus these logical principles may be viewed as choice principles.*

We work in intuitionistic Zermelo-Fraenkel set theory **IST** (for a presentation, see [3], where it is called $\mathbf{ZF'_I}$). Let us begin by fixing some notation. For each set A we write PA for the power set of A, and QX for the set of *inhabited* subsets of A, that is, of subsets X of A for which $\exists x\ (x \in A)$. The set of functions from A to B is denoted by B^A; the class of functions with domain A is denoted by Fun(A). The empty set is denoted by 0, {0} by 1, and {0, 1} by 2.

We tabulate the following *logical schemes*:

SLEM $\alpha \vee \neg\alpha$ (α any sentence)

Lin $(\alpha \to \beta) \vee (\beta \to \alpha)$ (α, β any sentences)

Stone $\neg\alpha \vee \neg\neg\alpha$ (α any sentence)

Ex $\exists x[\exists x\alpha(x) \to \alpha(x)]$ ($\alpha(x)$ any formula with at most x free)

Un $\exists x[\alpha(x) \to \forall x\alpha(x)]$ ($\alpha(x)$ any formula with at most x free)

Dis[1] $\forall x[\alpha \vee \beta(x)] \to \alpha \vee \forall x\beta(x)$ (α any sentence, $\beta(x)$ any formula with at most x free)

Over intuitionistic logic, **Lin, Stone** and **Ex** are consequences of **SLEM**; and **Un** implies **Dis**. All of these schemes follow, of course, from the full law of excluded middle, that is **SLEM** for arbitrary formulas.

We formulate the following *choice principles*—here X is an arbitrary set and $\varphi(x,y)$ an arbitrary formula of the language of **IST** with at most the free variables x, y:

---

[1] **Dis** is equivalent, over intuitionistic predicate logic, to what is called in [4] the *higher dual distributive law*—

HDDL $\forall x[\alpha(x) \vee \beta(x)] \to \exists x\alpha(x) \vee \forall x\beta(x)$.

AC_X $\forall x\in X\,\exists y\,\varphi(x,y) \to \exists f\in \mathrm{Fun}(X)\ \forall x\in X\,\varphi(x,fx)$

AC^*_X $\exists f\in \mathrm{Fun}(X)\ [\forall x\in X\,\exists y\,\varphi(x,y) \to \forall x\in X\,\varphi(x,fx)]$

DAC_X $\forall f\in \mathrm{Fun}(X)\ \exists x\in X\,\varphi(x,fx) \to \exists x\in X\ \forall y\,\varphi(x,y)$

DAC^*_X $\exists f\in \mathrm{Fun}(X)\ [\exists x\in X\,\varphi(x,fx) \to \exists x\in X\ \forall y\,\varphi(x,y)]$

The first two of these are forms of the *axiom of choice* for X; while classically equivalent, in **IST** **AC^*_X** implies **AC_X**, but not conversely. The principles **DAC_X** and **DAC^*_X** are *dual* forms of the axiom of choice for X: classically they are both equivalent to **AC_X** and **AC^*_X**, but in **IST** **DAC^*_X** implies **DAC_X**, and not conversely.

We also formulate what we shall call the *weak extensional selection principle*, in which $\alpha(x)$ and $\beta(x)$ are any formulas with at most the variable x free:

WESP $\exists x\in 2\alpha(x) \wedge \exists x\in 2\beta(x) \to \exists x\in 2\exists y\in 2[\alpha(x) \wedge \beta(y) \wedge [\forall x\in 2[\alpha(x) \leftrightarrow \beta(x)] \to x = y]].$

This principle asserts that, for any pair of instantiated properties of members of 2, instances may be assigned to the properties in a manner that depends just on their extensions. **WESP** is a straightforward consequence of **AC_{Q2}**. For taking $\varphi(u, y)$ to be $y \in u$ in **AC_{Q2}** yields the existence of a function f with domain $Q2$ such that $fu \in u$ for every $u \in Q2$. Given formulas $\alpha(x)$, $\beta(x)$, and assuming the antecedent of **WESP,** the sets $U = \{x\in 2: \alpha(x)\}$ and $V = \{x\in 2: \beta(x)\}$ are members of $Q2$, so that $a = fU \in U$, and $b = fV \in V$, whence $\alpha(a)$ and $\beta(b)$. Also, if $\forall x\in 2[\alpha(x) \leftrightarrow \beta(x)]$, then $U = V$, whence $a = b$; it follows then that the consequent of **WESP** holds.

We are going to show that each of the logical principles tabulated above is equivalent (over **IST**) to a choice principle. Starting at the top of the list, we have first:

- **WESP** and **SLEM** are equivalent over **IST**.

Proof. Assume **WESP.** Let σ be any sentence and define

$$\alpha(x) \equiv x = 0 \vee \sigma \qquad \beta(x) \equiv x = 1 \vee \sigma .$$

With these instances of α and β the antecedent of **WESP** is clearly satisfied, so that there exist members a, b of 2 for which (1) $\alpha(a) \wedge \beta(b)$ and (2) $\forall x\ [[\forall x\in 2[\alpha(x) \leftrightarrow \beta(x)] \to a = b$. It follows from (1) that $\sigma \vee (a = 0 \wedge b = 1)$, whence (3) $\sigma \vee a \neq b$. And since clearly $\sigma \to \forall x\in 2[\alpha(x) \leftrightarrow \beta(x)]$ we deduce from (2) that $\sigma \to a = b$, whence $a \neq b \to \neg\sigma$. Putting this last together with (3) yields $\sigma \vee \neg\sigma$, and **SLEM** follows.

For the converse, we argue informally. Suppose that **SLEM** holds. Assuming the antecedent of **WESP**, choose $a \in 2$ for which $\alpha(a)$. Now (using **SLEM**) define an element $b \in 2$ as follows. If $\forall x{\in}2[\alpha(x) \leftrightarrow \beta(x)]$ holds, let $b = a$; if not, choose b so that $\beta(b)$. It is now easy to see that a and b satisfy $\alpha(a) \wedge \beta(b) \wedge [\forall x{\in}2[\alpha(x) \leftrightarrow \beta(x)] \rightarrow a = b]$. **WESP** follows. ■

Remark. The argument for **WESP** $\rightarrow$ **SLEM** is another "stripped down" version of Diaconescu's theorem that, in a topos, the axiom of choice implies the law of excluded middle. The result may be compared with that of [2] to the effect that the presence of extensional ε-terms renders intuitionistic logic classical.

Next, we observe that, while $\mathbf{AC}_1$ is (trivially) provable in **IST,** by contrast

- $\mathbf{AC}_1^*$ and **Ex** are equivalent over **IST.**

Proof. Assuming $\mathbf{AC}_1^*$, take $\varphi(x,y) \equiv \alpha(y)$ in its antecedent. This yields an $f \in \mathrm{Fun}(1)$ for which $\forall y\alpha(y) \rightarrow \alpha(f0)$, giving $\exists y[\exists y\alpha(y) \rightarrow \alpha(y)]$, i.e., **Ex.**

Conversely, define $\alpha(y) \equiv \varphi(0,y)$. Then, assuming **Ex**, there is b for which $\exists y\alpha(y) \rightarrow \alpha(b)$, whence $\forall x{\in}1\exists y\varphi(x,y) \rightarrow \forall x{\in}1\varphi(x,b)$. Defining $f \in \mathrm{Fun}(1)$ by $f = \{\langle 0,b\rangle\}$ gives $\forall x{\in}1\exists y\varphi(x,y) \rightarrow \forall x{\in}1\varphi(x,fx)$, and $\mathbf{AC}_1^*$ follows. ■

Further, while $\mathbf{DAC}_1$ is easily seen to be provable in **IST**, we have

- $\mathbf{DAC}_1^*$ and **Un** are equivalent over **IST.**

Proof. Given α, Define $\varphi(x,y) \equiv \alpha(y)$. Then, for $f \in \mathrm{Fun}(1)$, $\exists x{\in}1\varphi(x,fx) \leftrightarrow \alpha(f0)$ and $\exists x{\in}1\forall y\varphi(x,y) \leftrightarrow \forall y\alpha(y)$. $\mathbf{DAC}_1^*$ then gives

$$\exists f{\in}\mathrm{Fun}(1)[\alpha(f0) \rightarrow \forall y\alpha(y)],$$

from which **Un** follows easily.

Conversely, given φ, define $\alpha(y) \equiv \varphi(0,y)$. Then from **Un** we infer that there exists b for which $\alpha(b) \rightarrow \forall y\alpha(y)$, i.e. $\varphi(0,b) \rightarrow \forall y\varphi(0,y)$. Defining $f \in \mathrm{Fun}(1)$ by $f = \{\langle 0,b\rangle\}$ then gives $\varphi(0,f0) \rightarrow \exists x{\in}1\forall y\varphi(x,y)$, whence $\exists x{\in}1\varphi(x,fx) \rightarrow \exists x{\in}1\forall y\varphi(x,y)$, and **Un** follows. ■

Next, while $\mathbf{AC}_2$ is easily proved in **IST**, by contrast we have

- **$\mathbf{DAC}_2$** and **Dis** are equivalent over **IST**.

Proof. The antecedent of **$\mathbf{DAC}_2$** is equivalent to the assertion

$$\forall f \in \mathrm{Fun}(2)[\varphi(0, f0) \vee \varphi(1, f1)],$$

which, in view of the natural correlation between members of Fun (2) and ordered pairs, is equivalent to the assertion

$$\forall y \forall y'[\varphi(0, y) \vee \varphi(1, y')].$$

The consequent of **$\mathbf{DAC}_2$** is equivalent to the assertion

$$\forall y \in Y \varphi(0,y) \vee \forall y' \in Y \varphi(1,y')$$

So **$\mathbf{DAC}_2$** itself is equivalent to

$$\forall y \forall y'[\varphi(0,y) \vee \varphi(1,y')] \rightarrow \forall y \varphi(0,y) \vee \forall y' \varphi(1,y').$$

But this is obviously equivalent to the scheme

$$\forall y \forall y'[\alpha(y) \vee \beta(y')] \rightarrow \forall y \alpha(y) \vee \forall y' \beta(y'),$$

where y does not occur free in β, nor y' in α. And this last is easily seen to be equivalent to **Dis**. ■

Now consider $\mathbf{DAC}_2^*$. This is quickly seen to be equivalent to the assertion

$$\exists z \exists z'[\varphi(0,z) \vee \varphi(1,z') \rightarrow \forall y \varphi(0,y) \vee \forall y' \varphi(1,y'),$$

i.e. to the assertion, for arbitrary $\alpha(x)$, $\beta(x)$, that

$$\exists z \exists z'[\alpha(z) \vee \beta(z') \rightarrow \forall y \alpha(y) \vee \forall y' \beta(y')].$$

This is in turn equivalent to the assertion, for any sentence α,

$$\exists y[\alpha \vee \beta(y) \rightarrow \alpha \vee \forall y \beta(y)] \qquad (*)$$

Now (*) obviously entails **Un.** Conversely, given **Un**, there is b for which $\beta(b) \rightarrow \forall y \beta(y)$. Hence $\alpha \vee \beta(b) \rightarrow \alpha \vee \forall y \beta(y)$, whence (*). So we have shown that

- Over **IST**, $\mathbf{DAC}_2^*$ is equivalent to **Un**, and hence also to $\mathbf{DAC}_1^*$.

In order to provide choice schemes equivalent to **Lin** and **Stone** we introduce

$\mathbf{ac}^*_X$ $\exists f\in 2^X\ [\forall x\in X\, \exists y\in 2\ \varphi(x,y) \to \forall x\in X\ \varphi(x,fx)]$

$\mathbf{wac}^*_X$ $\exists f\in 2^X\ [\forall x\in X\, \exists y\in 2\ \varphi(x,y) \to \forall x\in X\ \varphi(x,fx)]$ provided $\vdash_{\mathbf{IST}} \forall x[\varphi(x,0) \to \neg\varphi(x,1)]$

Clearly $\mathbf{ac}^*_X$ is equivalent to

$$\exists f\in 2^X\ [\forall x\in X[\varphi(x,0) \vee \varphi(x,1)] \to \forall x\in X\ \varphi(x,fx)]$$

and similarly for $\mathbf{wac}^*_X$.

Then

- Over **IST**, $\mathbf{ac}^*_1$ and $\mathbf{wac}^*_1$ are equivalent, respectively, to **Lin** and **Stone.**

Proof. Let α and β be sentences, and define $\varphi(x,y) \equiv x = 0 \wedge [(y = 0 \wedge \alpha) \vee (y = 1 \wedge \beta)]$. Then $\alpha \leftrightarrow \varphi(0,0)$ and $\beta \leftrightarrow \varphi(0,1)$, and so $\forall x\in 1[\varphi(x,0) \vee \varphi(x,1)] \leftrightarrow \varphi(0,0) \vee \varphi(0,1) \leftrightarrow \alpha \vee \beta$. Therefore

$$\begin{aligned}\exists f\in 2^1\ [\forall x\in 1[\varphi(x,0) \vee \varphi(x,1)] \to \forall x\in 1\ \varphi(x,fx)] &\leftrightarrow \exists f\in 2^1[\alpha \vee \beta \to \varphi(0,f0)]\\ &\leftrightarrow [\alpha \vee \beta \to \varphi(0,0)] \vee [\alpha \vee \beta \to \varphi(0,1)]\\ &\leftrightarrow [\alpha \vee \beta \to \alpha] \vee [\alpha \vee \beta \to \beta]\\ &\leftrightarrow \beta \to \alpha \ \vee\ \alpha \to \beta.\end{aligned}$$

This yields $\mathbf{ac}^*_1 \to$ **Lin.** For the converse, define $\alpha \equiv \varphi(0,0)$ and $\beta \equiv \varphi(0,1)$ and reverse the argument.

To establish the second stated equivalence, notice that, when $\varphi(x,y)$ is defined as above, but with β replaced by ¬α, it satisfies the provisions imposed in $\mathbf{wac}^*_1$. As above, that principle gives $(\neg\alpha \to \alpha) \vee (\alpha \to \neg\alpha)$, that is, $\neg\alpha \vee \neg\neg\alpha$. So **Stone** follows from $\mathbf{wac}^*_1$. Conversely, suppose that φ meets the condition imposed in $\mathbf{wac}^*_1$ Then from $\varphi(0,0) \to \neg\varphi(0,1)$ we deduce $\neg\neg\varphi(0,0) \to \neg\varphi(0,1)$; now, assuming **Stone**, we have $\neg\varphi(0,0) \vee \neg\neg\varphi(0,0)$, whence $\neg\varphi(0,0) \vee \neg\varphi(0,1)$. Since $\neg\varphi(0,0) \to [\varphi(0,0) \to \varphi(0,1)]$ and $\neg\varphi(0,1) \to [\varphi(0,1) \to \varphi(0,0)]$ we deduce $[\varphi(0,0) \to \varphi(0,1)] \vee [\varphi(0,1) \to \varphi(0,0)]$. From the argument above it now follows that $\exists f\in 2^1\ [\forall x\in 1[\varphi(x,0) \vee \varphi(x,1)] \to \forall x\in 1\ \varphi(x,fx)]$. Accordingly $\mathbf{wac}^*_1$ is a consequence of **Stone**.

In conclusion, we show how certain of the principles we have introduced can be derived in the presence of *term-forming operators.*

The ε- and τ-*operators* are term-forming operators yielding, for formulas $\alpha(x)$, terms $\varepsilon_x\alpha$ and $\tau_x\alpha$ in which the variable x is no longer free; they are introduced in conjunction with the axioms—the ε- and τ-*schemes*:

$$\exists x\alpha(x) \to \alpha(\varepsilon_x\alpha) \qquad \alpha(\tau_x\alpha) \to \forall x\alpha(x).$$

It is an easy matter to derive **Un** from the τ-scheme when τ is merely allowed to act on formulas with at most one free variable. When τ's action is extended to formulas with two free variables, the τ-scheme applied in **IST** yields the full dual axiom of choice $\forall X\ \mathbf{DAC}^*_X$. For under these conditions we have, for any formula $\varphi(x,y)$,

$$\forall x{\in}X[\varphi(x,\tau_y\varphi(x,y)) \to \forall y\varphi(x,y)] \qquad (*)$$

Let $t \in \text{Fun}(X)$ be the map $x \mapsto \tau_y\varphi(x,y)$. Assuming that $\forall f{\in}Y^X\exists x{\in}X\varphi(x, fx)$, let $a \in X$ satisfy $\varphi(a,ta)$. We deduce from (*) that $\forall y{\in}Y\varphi(a,y)$, whence $\exists x{\in}X\forall y{\in}Y\varphi(x, y)$. The dual axiom of choice follows.

In the case of the ε-operator, the number of free variables in the formulas on which the operator is allowed to act is an even more sensitive matter. If ε is allowed to act only on formulas with at most one free variable (so yielding only closed terms), the corresponding ε-scheme applied in **IST** is easily seen to yield both **Ex** and $\mathbf{ac}^*_1$, and so also **Lin**. But it is (in essence) shown in [1] that, if only closed ε-terms are admitted, **SLEM** is not derivable, and so therefore neither is **WESP.** The situation changes dramatically when ε is permitted to operate on formulas with *two* free variables. For then from the corresponding ε-scheme it is easy to derive $\mathbf{AC}_X$ for all sets X, and in particular $\mathbf{AC}_{Q2}$, and hence also **SLEM.**

I have found three ways of strengthening, or modifying, the single-variable ε-scheme so as to enable it to yield **SLEM.** The first, presented originally in [2], is to add to the ε-scheme *Ackermann's Extensionality Principle*, viz.

$$\forall x[\alpha(x) \leftrightarrow \beta(x)] \to \varepsilon_x\alpha = \varepsilon_x\beta\ .$$

From these **WESP** is easily derived, and so, *a fortiori*, **SLEM.**

The second approach is to take the ε-axiom in the (classically equivalent) form

$$(*) \qquad \alpha(\varepsilon_x\alpha) \vee \forall x\neg\alpha(x).$$

From this we can intuitionistically derive **SLEM** as follows:

Given a sentence β, define $\alpha(x)$ to be the formula

$$(x = 0 \wedge \beta) \vee (x = 1 \wedge \neg\beta).$$

Then from (*) we get

$$[(\varepsilon_x\alpha = 0 \wedge \beta) \vee ([(\varepsilon_x\alpha = 1 \wedge \neg\beta)] \vee \forall x\neg[(x = 0 \wedge \beta) \vee (x = 1 \wedge \neg\beta)],$$

which implies

$$[\beta \vee \neg\beta) \vee [\forall x\neg(x = 0 \wedge \beta) \wedge \forall x\neg(x = 1 \wedge \neg\beta)],$$

whence

$$[\beta \vee \neg\beta) \vee [\neg \beta \wedge \neg\neg\beta],$$

winding up with

$$\beta \vee \neg\beta.$$

The third method is to allow ε to act on *pairs* of formulas, each with a *single* free variable. Here, for each pair of formulas $\alpha(x)$, $\beta(x)$ we introduce the "relativized" ε-term $\varepsilon_x\alpha/\beta$ and the "relativized" ε-axioms

$$(1)\ \exists x\, \beta(x) \rightarrow \beta(\varepsilon_x\alpha/\beta) \qquad\qquad (2)\ \exists x\, [\alpha(x) \wedge \beta(x)] \rightarrow \alpha(\varepsilon_x\alpha/\beta).$$

That is, $\varepsilon_x\alpha/\beta$ may be thought of as an individual that satisfies β if anything does, and which in addition satisfies α if anything satisfies both α and β. Notice that the usual ε–term $\varepsilon_x\alpha$ is then $\varepsilon_x\alpha/x = x$. In the classical ε-calculus $\varepsilon_x\alpha/\beta$ may be defined by taking

$$\varepsilon_x\alpha/\beta = \varepsilon_y[[y = \varepsilon_x(\alpha \wedge \beta) \wedge \exists x\, (\alpha \wedge \beta)] \vee [y = \varepsilon_x\beta \wedge \neg\exists x\, (\alpha \wedge \beta)]].$$

But the relativized ε-scheme is not derivable in the intuitionistic ε-calculus since it can be shown to imply **SLEM**. To see this, given a formula γ define

$$\alpha(x) \equiv x = 1 \qquad\qquad \beta(x) \equiv x = 0 \vee \gamma.$$

Write a for $\varepsilon_x\alpha/\beta$. Then we certainly have $\exists x\beta(x)$, so (1) gives $\beta(a)$, i.e.

(3) $$a = 0 \vee \gamma$$

Also $\exists x\, (\alpha \wedge \beta) \leftrightarrow \gamma$, so (2) gives $\gamma \rightarrow \alpha(a)$, i.e.

$$\gamma \rightarrow a = 1,$$

whence

$$a \neq 1 \rightarrow \neg\gamma,$$

so that

$$a = 0 \rightarrow \neg\gamma.$$

And the conjunction of this with (3) gives $\gamma \vee \neg\gamma$, as claimed.

References

[1] Bell, John L. *Hilbert's epsilon operator in intuitionistic type theories*, Math. Logic Quarterly, 39, 1993.

[2] Bell, John L. *Hilbert's epsilon-operator and classical logic*, Journal of Philosophical Logic, 22, 1993.

[3] Grayson, R. J. *Heyting-valued models for intuitionistic set theory.* In Fourman, M. P., Mulvey, C. J., and Scott, D. S. (eds.) (1979) *Applications of Sheaves. Proc. L.M.S. Durham Symposium* 1977. Springer Lecture Notes in Mathematics 753, pp. 402-414.

[4] Lawvere, F.W. and Rosebrugh, R. *Sets for Mathematics.* Cambridge University Press, 2003.