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LOGICAL REFLECTIONS ON THE KOCHEN-SPECKER THEOREM

IN THEIR WELL-KNOWN PAPER, Kochen and Specker (1967) introduce the concept of *partial Boolean algebra* (pBa) and show that certain (finitely generated) partial Boolean algebras arising in quantum theory fail to possess morphisms to any Boolean algebra (we call such pBa's *intractable* in the sequel). In this note we begin by discussing partial Boolean algebras within a *category-theoretic* framework¹; our analysis will result in what appear to be some new formulations of intractability in purely *logical* terms, and an open problem.

Partial Boolean algebras arise naturally in connection with *ortholattices* (a concept we take to be familiar: see, e.g. Birkhoff (1984)). Let $L = (L, \vee, \perp, 1)$ be an ortholattice and let \approx be the binary relation ("compatibility") on L defined by: $x \approx y \Leftrightarrow \{x, y\}$ generates a Boolean subalgebra (= distributive subortholattice) of L . Clearly \approx is reflexive and symmetric. Now consider the partial algebra L^\approx obtained from L by restricting the domain of \vee to those pairs (x, y) of L for which $x \approx y$. The concept of partial Boolean algebra employed here is obtained by abstracting from the properties of L^\approx .

Thus a *partial Boolean algebra* (pBa) is a structure $B = (B, \vee, \perp, 1, \approx)$ in which B is a nonempty set, $1 \in B$, \approx is a reflexive, symmetric relation (the *compatibility* relation) on B , and \perp, \vee are maps to B from B and $\{(x, y) : x \approx y\}$ respectively, satisfying the conditions:

$$\begin{array}{l} 1^\perp \neq 1 ; \\ \text{for all } x, y \in B: 1 \approx x ; \\ x \approx y \Rightarrow x \approx y^\perp \ \& \ (x \vee y) \approx x ; \\ \text{for any } x \approx y \text{ in } B: \text{ the substructure of } B \text{ generated by } \{x, y\} \\ \text{is a Boolean algebra.} \end{array}$$

We note that the partial Boolean algebras of Kochen and Specker satisfy the additional condition that $\{x_1, \dots, x_n\}$ generates a Boolean subalgebra whenever $x_i \approx x_j$ for any i, j : this condition is satisfied by pBa's induced by *orthomodular* ortholattices. For our purposes here, there is no need to impose this additional constraint.

We can now define the category of partial Boolean algebras. We de-

¹An earlier investigation of the category of partial Boolean algebras, in which results different from those included here are obtained, appears in Kamber (1964).

fine a *morphism* of pBa's B, B' to be a map $h: B \rightarrow B'$ such that:

$$\begin{aligned} \text{for any } x \approx y \text{ in } B: & \quad h(x) \approx h(y) ; \\ & \quad h(x \vee y) = h(x) \vee h(y) ; \\ & \quad h(x^\perp) = h(x)^\perp . \end{aligned}$$

The category \mathcal{PBA} of partial Boolean algebras then has as objects all partial Boolean algebras and as arrows all morphisms between them. Clearly the category \mathcal{Bool} of Boolean algebras and (Boolean) homomorphisms is a subcategory of \mathcal{PBA} which is *full* in the sense that if B, B' are objects of \mathcal{Bool} , then any morphism of B to B' in \mathcal{PBA} is also in \mathcal{Bool} (i.e. is a Boolean homomorphism in the usual sense).

Although it is obvious that not every pBa is a Boolean algebra, it is natural to ask whether, nonetheless, any pBa B has a best "Boolean approximation" in the following sense: there is a pair (\tilde{B}, i) consisting of a Boolean algebra \tilde{B} and a morphism $i: B \rightarrow \tilde{B}$ such that, for any morphism $f: B \rightarrow C$ to a Boolean algebra C , there is a (unique) morphism $g: \tilde{B} \rightarrow C$ such that the diagram

$$\begin{array}{ccc} B & \xrightarrow{i} & \tilde{B} \\ & \searrow f & \downarrow g \\ & & C \end{array}$$

commutes: we call this the *universal condition* on (\tilde{B}, i) . We shall call a pBa B *tractable* if a pair (\tilde{B}, i) satisfying the universal condition exists.

Kochen and Specker showed, in effect, that not every pBa is tractable: more on this later. For the moment, consider the subcategory \mathcal{Trac} of \mathcal{PBA} whose objects are all *tractable* pBa's: clearly \mathcal{Bool} is a (full) subcategory of \mathcal{Trac} . Every object B of \mathcal{Trac} has a "Boolean approximation" (\tilde{B}, i) which is easily shown to be unique up to isomorphism in the evident sense. In that case, the map $B \rightarrow \tilde{B}: \mathcal{Trac} \rightarrow \mathcal{Bool}$ defines what category-theorists call a *reflection*: for any object B of \mathcal{Trac} , \tilde{B} is the "reflection" of B in the subcategory \mathcal{Bool} .

We are going to characterize the objects in \mathcal{Trac} , and provide an explicit description of the reflection \tilde{B} of any tractable pBa.²

²Kamber (1964) constructs the Boolean reflection of a pBa by a method different from that to be formulated here. Moreover, because he allows degenerate (= 1-element) pBa's - a possibility which has been explicitly excluded here - *every* (not necessarily tractable) pBa has a (possibly degenerate) reflection in his sense. However for that reason, the concept of tractability, which occupies centre stage here, plays no role in his discussion.

Given a partial Boolean algebra B , let \mathcal{L}_B be the (classical) propositional language with propositional variables $\{p_b : b \in B\}$, and let Σ_B be the set of all sentences of the form:

$$\begin{aligned} p_a \vee p_b &\leftrightarrow p_{a \vee b}, \text{ for } a \sim b \text{ in } B; \\ \neg p_c &\leftrightarrow p_{c^\perp}, \text{ for } c \in B. \end{aligned}$$

Theorem 1

Let B be a pBa. Then the following conditions are equivalent:

- (i) B is tractable;
- (ii) Σ_B is consistent;
- (iii) there is a morphism from B to the two element Boolean algebra 2 ;
- (iv) there is a morphism from B to some Boolean algebra.

Proof

(i) \Rightarrow (iv) is obvious, as is (iv) \Rightarrow (iii) since every Boolean algebra has a homomorphism to 2 .

(iii) \Rightarrow (ii). Let $h: B \rightarrow 2$ be a morphism, and let $\llbracket \cdot \rrbracket$ be the valuation on \mathcal{L}_B induced by h (i.e. such that $\llbracket p_b \rrbracket = h(b)$ for $b \in B$). It is easily verified that $\llbracket \phi \rrbracket = 1$ for every $\phi \in \Sigma_B$, so that Σ_B is consistent.

(ii) \Rightarrow (i). Suppose that Σ_B is consistent. Then the Lindenbaum-Tarski algebra \tilde{B} of Σ_B (obtained by identifying formulas of \mathcal{L}_B when their equivalence follows from Σ_B) is a nondegenerate Boolean algebra, and the canonical map $b \mapsto [p_b]$ (where $[\phi]$ is the image - i.e., equivalence class modulo provable equivalence from Σ_B - of a formula ϕ in \tilde{B}) is a morphism $i: B \rightarrow \tilde{B}$. We have to show that (\tilde{B}, i) satisfies the universal condition.

To this end, suppose that f is a morphism of B to a Boolean algebra C . Let Form be the set of formulas of \mathcal{L}_B and define the map $\underline{f}: \text{Form} \rightarrow C$ recursively by:

$$\begin{aligned} \underline{f}(p_b) &= f(b) \text{ for } b \in B, \\ \underline{f}(\phi \vee \psi) &= \underline{f}(\phi) \vee \underline{f}(\psi), \\ \underline{f}(\neg \phi) &= \underline{f}(\phi)^\perp. \end{aligned}$$

Now define $g: \tilde{B} \rightarrow C$ by $g([\phi]) = \underline{f}(\phi)$ for $\phi \in \text{Form}$. Clearly g will be a homomorphism provided we can show that it is well-defined. To do this, it suffices to show that, if $\phi, \psi \in \text{Form}$, then $[\phi] = [\psi] \Rightarrow \underline{f}(\phi) = \underline{f}(\psi)$. To this end, let h be any homomorphism $C \rightarrow 2$ and let $\llbracket \cdot \rrbracket$ be the truth valuation on Form induced by taking $\llbracket p_b \rrbracket = h(\underline{f}(p_b))$ for $b \in B$. It is easily shown by induction on formulas that $\llbracket \phi \rrbracket = h(\underline{f}(\phi))$ for any $\phi \in \text{Form}$. Clearly also $\llbracket \sigma \rrbracket = 1$ for any $\sigma \in \Sigma_B$. Thus if $[\phi] = [\psi]$, then $\Sigma_B \vdash \phi \leftrightarrow \psi$, so $\llbracket \phi \leftrightarrow \psi \rrbracket = 1$, whence $h(\underline{f}(\phi)) = \llbracket \phi \rrbracket = \llbracket \psi \rrbracket = h(\underline{f}(\psi))$. Since this equality

holds for arbitrary $h:C \rightarrow 2$, and homomorphisms to 2 distinguish the points of any Boolean algebra, it follows that $f(\phi)=f(\psi)$ as required.

Therefore g is a homomorphism; clearly $g \circ i = f$, and it is easy to see that it is the unique such homomorphism. So (\tilde{B}, i) satisfies the universal condition. ■

As a consequence, we obtain the

Corollary

A pBa is tractable if and only if all of its finitely generated sub-pBa's are tractable.

Proof

If the pBa B is tractable, there is a morphism $h:B \rightarrow 2$ whose restriction to any sub-pBa B' of B is a morphism $B' \rightarrow 2$. Accordingly, B' is tractable.

Conversely, suppose B is intractable. Then Σ_B is inconsistent and so there are finite subsets $\{a_1, \dots, a_n\}$, $\{b_1, \dots, b_n\}$, $\{c_1, \dots, c_m\}$ of B for which $a_i \approx b_i$ ($i=1, \dots, n$) and the set of sentences

$$\{p_{a_i} \vee p_{b_i} \leftrightarrow p_{a_i} \vee b_i : i=1, \dots, n\} \cup \{\neg p_{c_j} \leftrightarrow p_{c_j} \perp : j=1, \dots, m\}$$

is inconsistent. It follows that, if A is the sub-pBa of B generated by

$$\{a_1, \dots, a_n\} \cup \{b_1, \dots, b_n\} \cup \{c_1, \dots, c_m\}$$

then Σ_A is inconsistent, so that A is intractable. ■

It is natural to single out those tractable pBa's B for which the canonical morphism $i:B \rightarrow \tilde{B}$ is *injective*. In this connection - following Kochen and Specker - we call a morphism $h:B \rightarrow B'$ in \mathcal{PBA} an *embedding* (resp. *weak embedding*) if $a \neq b \Rightarrow h(a) \neq h(b)$ (resp. $a \neq b$ & $a \approx b \Rightarrow h(a) \neq h(b)$) for $a, b \in B$. It is straightforward to adapt the proofs of Theorem 1 to establish:

Theorem 2

The following conditions on a pBa B are equivalent:

- (i) B is tractable and $i:B \rightarrow \tilde{B}$ is an embedding (resp. weak embedding);
- (ii) $\Sigma_B \not\vdash p_a \leftrightarrow p_b$ for any $a \neq b$ (resp. $a \neq b$ & $a \approx b$) in B ;
- (iii) for any $a \neq b$ (resp. $a \neq b$ & $a \approx b$) in B there is a morphism $h:B \rightarrow 2$ such that $h(a) \neq h(b)$;
- (iv) there is an embedding (resp. weak embedding) of B into some Boolean algebra. ■

Simple examples of pBa's which, while not Boolean algebras themselves, are embeddable therein, arise in the following way.

Let (A, f) be a structure consisting of a nonempty set A and a map $f: A \rightarrow A$ such that $f(x) \neq x$ and $f(f(x)) = x$ for every $x \in A$. Let $0, 1$ be two distinct objects not in A , and on the set $\bar{A} = A \cup \{0, 1\}$ define the relation \approx by:

$$\approx = \{(x, x) : x \in \bar{A}\} \cup \{(x, fx) : x \in A\} \cup \{0, 1\} \times \bar{A} \cup \bar{A} \times \{0, 1\} .$$

Clearly \approx is reflexive and symmetric. Define $\perp: \bar{A} \rightarrow \bar{A}$ by:

$$\begin{aligned} 0^\perp &= 1 ; \\ 1^\perp &= 0 ; \\ \text{for } x \in A: \quad x^\perp &= fx , \end{aligned}$$

and $\vee: \{(x, y) : x \approx y\} \rightarrow \bar{A}$ by:

$$\begin{aligned} 0 \vee 0 &= 0 ; \\ 0 \vee 1 &= 1 \vee 0 = 1 \vee 1 = 1 ; \\ \text{for every } x \approx y, x \neq y \text{ in } A: \quad x \vee y &= 1 . \end{aligned}$$

Then $(\bar{A}, \vee, \perp, 1, \approx)$ is a pBa which is evidently not a Boolean algebra if A has more than two elements.

To show that \bar{A} is embeddable in a Boolean algebra, we argue as follows. Suppose given $a \neq b$ in A . If $b \neq fa$, let k be a function (whose existence is ensured by the axiom of choice for pairs) which selects an element from each pair $\{x, fx\}$ for $x \in A - \{a, b\}$. Now define $h: \bar{A} \rightarrow 2$ by:

$$\begin{aligned} h(a) &= h(1) = 1 ; \\ h(b) &= h(0) = 0 ; \\ \text{for } x \in A - \{a, b\}: \quad h(x) &= 1, \text{ if } x \in \text{range}(k) ; \\ &= 0, \text{ otherwise.} \end{aligned}$$

It is easy to see that h is a morphism and $h(a) \neq h(b)$. When $b = fa$, or one or both of a or b is 0 or 1 , the construction of h is similar. It follows from Theorem 2 that \bar{A} is embeddable in a Boolean algebra.³

We now turn our attention to *intractable* pBa's. If B is intractable,

³The standard case arises of course when A is the set of rays in the plane E_2 and f the map assigning to each ray its orthogonal complement in E_2 . In this case, we do not need to invoke the axiom of choice to prove the existence of two-valued morphisms, since a simple geometric argument suffices. However, in the general case we do need to use the axiom of choice, since it is not hard to show that the assertion that *every* partial Boolean algebra of the form \bar{A} is embeddable in a Boolean algebra is *equivalent* to the axiom of choice for pairs.

then, as we have seen, Σ_B is inconsistent, which means that there exist finite subsets $\{a_1, \dots, a_n\}$, $\{b_1, \dots, b_n\}$, $\{c_1, \dots, c_m\}$ of B such that $a_i \approx b_i$ ($i=1, \dots, n$) and the sentence

$$\bigwedge_{i=1}^n p_{a_i} \vee p_{b_i} \leftrightarrow p_{a_i \vee b_i} \wedge \bigwedge_{j=1}^m \neg p_{c_j} \leftrightarrow p_{c_j^\perp} \quad (*)$$

is a classical contradiction. Now in (*) we may delete any conjunct in which a_i , b_i , or c_j is 0 or 1 and still ensure that the remaining conjunction - call it Φ - is a contradiction, since if Φ is satisfied by some truth valuation $\llbracket \cdot \rrbracket$, then (*) is satisfied by the valuation $\llbracket \cdot \rrbracket'$ which agrees with $\llbracket \cdot \rrbracket$ except that $\llbracket p_0 \rrbracket' = 0$, $\llbracket p_1 \rrbracket' = 1$. Next, in Φ delete each conjunct of the form $\neg p_{c_j} \leftrightarrow p_{c_j^\perp}$ and each conjunct $p_{a_i} \vee p_{b_i} \leftrightarrow p_{a_i \vee b_i}$ for which $a_i \vee b_i \neq c_j^\perp$ for any $j=1, \dots, m$. In each remaining conjunct, replace the term $p_{a_i \vee b_i}$ by $\neg p_{c_j}$, where $a_i \vee b_i = c_j^\perp$. The result is a formula of the form

$$\bigwedge_{\langle i, j, k \rangle \in R} q_i \vee q_j \leftrightarrow \neg q_k \quad (**)$$

in which each q_i , $i=1, \dots, N$ is a propositional variable and R is a set of triples of integers $1, \dots, N$ such that $\langle i, j, k \rangle \in R \Rightarrow i \neq j$. Since (*) was a contradiction, it is not hard to see that (**) is also. Moreover, for each $\langle i, j, k \rangle \in R$, there exist (unique) $a, b, c \in B$ for which q_i is p_a , q_j is p_b , and q_k is p_c , where $a \approx b$ and $a \vee b = c^\perp$. It follows that, if we assign the values a, b, c in B to q_i, q_j, q_k respectively, the formula $q_i \vee q_j \leftrightarrow \neg q_k$ receives value 1. Doing this for each conjunct in (**) thus assigns value 1 to (**).

Now for each natural number $N \geq 1$ write \underline{N} for the set $\{1, \dots, N\}$. Let us define an N -skeleton to be a structure \mathcal{S} of the form (\underline{N}, R) with $R \subseteq N^3$ satisfying $\langle i, j, k \rangle \in R \Rightarrow i \neq j$. Given an N -skeleton \mathcal{S} , let q_1, \dots, q_N be propositional variables and denote the resulting formula (**) by $\phi_{\mathcal{S}}$: we shall call this the formula associated with \mathcal{S} . A representation of \mathcal{S} in a $\mathcal{P}Ba$ B is an injective map $b: \underline{N} \rightarrow B - \{0, 1\}$ such that, writing b_i for $b(i)$,

$$\forall i, j, k \in \underline{N} [\langle i, j, k \rangle \in R \Rightarrow b_i \approx b_j \ \& \ b_i \vee b_j = b_k^\perp] .$$

Clearly any representation b of \mathcal{S} in B assigns the (well-defined) value $b(\phi_{\mathcal{S}}) = 1$ to $\phi_{\mathcal{S}}$ in B . If, in addition, $\phi_{\mathcal{S}}$ is a classical contradiction, then B must be intractable. For if h were any morphism $B \rightarrow 2$, then $h(b(\phi_{\mathcal{S}}))$ is the value in 2 of $\phi_{\mathcal{S}}$ under the valuation obtained by assigning the value $h(b_i)$ to each q_i ; this value must be 0 since $\phi_{\mathcal{S}}$ is a classical contradiction. On the other hand, since $b(\phi_{\mathcal{S}}) = 1$, we must also have $h(b(\phi_{\mathcal{S}})) = 1$. This inconsistency shows that B has no two-valued morphisms, and so is

intractable.

We may sum up our findings so far by the assertion:

A pBa B is intractable if and only if there is a skeleton which is representable in B and whose associated formula is a classical contradiction.

If S is representable in B and ϕ_S is a classical contradiction, it may be regarded as a *canonical* example of a formula which is true in B and at the same time a classical contradiction.

For any Hilbert space H , write $B(H)$ for the pBa of closed subspaces of H , and E_n for n -dimensional Euclidean space. Kochen and Specker showed, in effect, that some finitely generated subalgebra of the pBa $B(E_3)$ is intractable. (We note that, by the Corollary above, it suffices for this purpose to show that $B(E_3)$ itself is intractable, which of course follows from Gleason's (1957) theorem.) For intractable subalgebras of $B(E_3)$, we can always find a skeletal representation of a particularly simple form which gives rise to a canonical contradiction. This is because each element of $B(E_3)$ is of the form $0, 1, a, a^\perp$ where a is an *atom* (= 1-dimensional subspace of E_3). Accordingly, corresponding to each conjunct $q_i \vee q_j \leftrightarrow \neg q_k$ in a canonical contradiction for an intractable subalgebra of $B(E_3)$ is an equality $b_i \vee b_j = b_k^\perp$ in which b_i, b_j, b_k are compatible, $b_i \neq b_j$, and each is either an atom or a complement of one. It is easy to see that there are then only three possibilities:

- (i) b_i, b_j, b_k are mutually orthogonal atoms;
- (ii) b_i, b_j^\perp are atoms, $b_k = b_j^\perp$, and $b_i \leq b_j$;
- (iii) b_i^\perp, b_j are atoms, $b_k = b_i^\perp$, and $b_j \leq b_i$.

Now for each i define $e_i \in B(E_3)$ by $e_i = b_i$ if b_i is an atom, $e_i = b_i^\perp$ if b_i^\perp is an atom. Then corresponding to each conjunct $q_i \vee q_j \leftrightarrow \neg q_k$ in a canonical contradiction is a triple (e_i, e_j, e_k) of atoms for which either (a) e_i, e_j, e_k are mutually orthogonal, or (b) e_i is orthogonal to e_j , and e_k is either e_i or e_j . (Similar, but more complicated, representations can be formulated for $B(E_n)$ with $n \geq 4$.)

It is interesting to note that the negation of the formula constructed by Kochen and Specker ((1967), Cor. to Thm. 4) to establish the intractability of a finitely generated subalgebra of $B(E_3)$ is canonical in the sense of this note (and in fact only involves conjuncts arising under clause (a) above). For the negation of their formula is a conjunction of formulas of the form

$$q_i + q_j + q_k + q_i \wedge q_j \wedge q_k \quad (***)$$

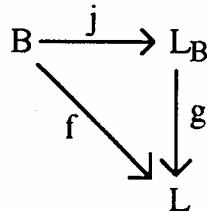
(where "+" denotes exclusive disjunction) over a finite set of orthogonal triples of atoms (e_i, e_j, e_k) in $B(E_3)$. And it is easy to see that (***) is equivalent to the formula

$$(q_i \vee q_j \leftrightarrow \neg q_k) \wedge (q_k \vee q_i \leftrightarrow \neg q_j) \wedge (q_j \vee q_k \leftrightarrow \neg q_i)$$

since both assert that exactly one of q_i, q_j, q_k is true. Therefore, any conjunction of formulas of the form (***) is canonical.

I conclude with two problems. The first of these is, to my knowledge, unresolved. The second was resolved some years ago, but this fact does not seem to be well-known.

In producing examples of intractable pBa's, Kochen and Specker in effect showed that there is no reflection from \mathcal{PBA} into its subcategory \mathcal{Bool} . Suppose, however, that we replace \mathcal{Bool} by the category \mathcal{Orth} of ortholattices and orthohomomorphisms. \mathcal{Orth} may be identified as a (non-full) subcategory of \mathcal{PBA} by identifying each ortholattice L with the corresponding pBa $L^{\#}$ defined at the beginning of the paper. We may now ask whether there is a reflection $\mathcal{PBA} \rightarrow \mathcal{Orth}$, that is, corresponding to each pBa B there is a pair (L_B, j) consisting of an ortholattice L_B and a morphism $j: B \rightarrow L_B$ such that, for any morphism f of B to an ortholattice L , there is a unique orthohomomorphism $g: L_B \rightarrow L$ such that the diagram



commutes. L_B , if it exists, would be the best "ortholattice approximation" to B . It is not hard to see that L_B exists if and only if there is a morphism of B to some ortholattice: since all the examples of intractable pBa's known to me arise as partial subalgebras of ortholattices, L_B certainly exists in these cases. I do not know whether L_B exists for arbitrary (intractable) B .

Finally, we note that the intractability of $B(E_3)$ implies the intractability of any $B(E_n)$ with $n > 3$, despite the fact that $B(E_3)$ is not a partial subalgebra of $B(E_n)$ (if it were, the implication would be automatic). For if h were any two-valued morphism on $B(E_n)$ with $n > 3$, then h must send some atom a to 1, so that the restriction of h to the partial

Boolean algebra of subspaces of any E_3 containing a is a two-valued homomorphism. Now consider any *infinite* dimensional Hilbert space H . Is $B(H)$ intractable? Since no $B(E_n)$ is actually a partial subalgebra of $B(H)$, and a two-valued morphism on $B(H)$ need not send *any* finite-dimensional subspace of H to 1 (since it need not define a *countably* additive measure in the sense of Gleason (1957)), the intractability of the latter does *not* immediately follow.⁴

However, a straightforward argument, (essentially) due to Jost (1976)⁵, shows that each $B(E_n)$ is *embeddable* in $B(H)$, from which the latter's intractability follows easily. To show that $B(E_n)$ is embeddable in $B(H)$, let $\{e_0, e_1, \dots\}$, $\{a_0, \dots, a_{n-1}\}$ be orthonormal bases for H and E_n respectively, and let H_n be the subspace of H generated by $\{e_{kn} : k=0, 1, \dots\}$. Then the map

$$e_{kn} \otimes a_i \mapsto e_{kn+i} \quad i=0, \dots, n-1$$

is a bijection between the bases of $H_n \otimes E_n$ and H , which induces in a natural way an isomorphism between $H_n \otimes E_n$ and H , and hence an isomorphism j between $B(H_n \otimes E_n)$ and $B(H)$. It follows that the map which sends each subspace S of E_n to $j(H_n \otimes S)$ is an embedding of $B(E_n)$ into $B(H)$.

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⁴This observation seems first to have been made in Cook (1968). I thank William Demopoulos for bringing this review to my attention.

⁵I am grateful to Michael Kernaghan and William Demopoulos for bringing this paper to my attention.