

APPENDIX: FREGE'S THEOREM
AND THE ZERMELO-BOURBAKI LEMMA⁴⁰

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This Appendix establishes the existence of an infinite well-ordering as a (hitherto unremarked) consequence of a general version of Zermelo's Well-ordering theorem. We also indicate how this fact can be derived along 'Fregean' lines within a certain system \mathbb{F} of many-sorted first-order logic whose sorts correspond to Frege's domains of objects, relations, and first and second level concepts. We show that the system of axioms we formulate within \mathbb{F} constitutes a consistent fragment of Frege's original (inconsistent) system sufficient for the development of arithmetic.

We begin by specifying the basic constituents of the system \mathbb{F} .

Sorts (or domains)

- \mathcal{O} — objects
- \mathcal{B} — basic (first level) concepts
- \mathcal{R} — relations
- \mathcal{S}_b — second level concepts
- \mathcal{S}_r — second level relational concepts

Variables and Constants

<i>Sort</i>	<i>Variable</i>	<i>Constant</i>
\mathcal{O}	x, y, z, \dots	a, b, c, \dots
\mathcal{B}	X, Y, Z, \dots	A, B, C, \dots
\mathcal{R}	$\underline{X}, \underline{Y}, \underline{Z}, \dots$	$\underline{A}, \underline{B}, \underline{C}, \dots$
\mathcal{S}_b	$\underline{\sim}X, \underline{\sim}Y, \underline{\sim}Z, \dots$	$\underline{\sim}A, \underline{\sim}B, \underline{\sim}C, \dots$
\mathcal{S}_r	$\underline{\underline{X}}, \underline{\underline{Y}}, \underline{\underline{Z}}, \dots$	$\underline{\underline{A}}, \underline{\underline{B}}, \underline{\underline{C}}, \dots$

A *term* is a variable or a constant or one of the concept or relation or extension terms to be introduced shortly. A variable of sort \mathcal{B} or \mathcal{S}_b will be called a *concept variable* for brevity.

We assume the presence of an *identity* sign $=$ yielding atomic statements of the form $s = t$ where s and t are terms of the *same*

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sort. On all domains except \mathcal{O} , $=$ is to be thought of as *intensional* equality.

We also assume the presence of a *predication* sign η yielding atomic statements of the form $s \eta t$, $(s't') \eta u$ where s is of sort \mathcal{O} , \mathcal{B} , \mathcal{R} and t is of sort \mathcal{B} , \mathcal{S}_b , \mathcal{S}_r respectively; and s' , t' are both of sort \mathcal{O} and u is of sort \mathcal{R} . We read ' $s \eta t$ ' as '*s falls under t.*'

We shall assume the following comprehension scheme for concepts:

Corresponding to any formula $\Phi(x)$, $\Phi(x, y)$, $\Phi(X)$ or $\Phi(\underline{X})$ we are given a term s of sort \mathcal{B} , \mathcal{R} , \mathcal{S}_b , \mathcal{S}_r , respectively, for which we adopt as an axiom the formula

$$\forall \left\{ \begin{array}{c} x \\ xy \\ X \\ \underline{X} \end{array} \right\} \left[\left[\left\{ \begin{array}{c} x \\ (xy) \\ X \\ \underline{X} \end{array} \right\} \eta s \leftrightarrow \Phi \left\{ \begin{array}{c} x \\ xy \\ X \\ \underline{X} \end{array} \right\} \right] \right].$$

We write $\hat{x}\Phi$, $(xy)\hat{}\Phi$, $\hat{X}\Phi$, $\hat{\underline{X}}\Phi$ for s , as the case may be. A term of the first, third and fourth types is called the *concept* (term) determined by Φ , and a term of the second type the *relation* (term) determined by Φ .

We define the relation \equiv of *extensional equality* on the domains \mathcal{B} , \mathcal{R} , \mathcal{S}_b , \mathcal{S}_r by

$$\begin{aligned} X \equiv Y &\iff_{df} \forall x(x \eta X \leftrightarrow x \eta Y) \\ \underline{X} \equiv \underline{Y} &\iff_{df} \forall x \forall y[(xy) \eta \underline{X} \leftrightarrow (xy) \eta \underline{Y}] \\ \underline{\underline{X}} \equiv \underline{\underline{Y}} &\iff_{df} \forall X[X \eta \underline{\underline{X}} \leftrightarrow X \eta \underline{\underline{Y}}] \\ \underline{\underline{X}} \equiv \underline{\underline{Y}} &\iff_{df} \forall \underline{X}[\underline{X} \eta \underline{\underline{X}} \leftrightarrow \underline{X} \eta \underline{\underline{Y}}]. \end{aligned}$$

Clearly concepts are determined uniquely by formulas up to extensional equality. We assume that \mathbb{F} contains

- a term e such that $e(\mathfrak{X})$ is well-formed and of sort \mathcal{O} for any concept variable \mathfrak{X} ;
- a predicate symbol E such that $E(\mathfrak{X})$ is well-formed for any concept variable \mathfrak{X} .

We finally assume the axioms

- (1) $\forall \mathfrak{X} \forall \mathfrak{Y}[E(\mathfrak{X}) \wedge E(\mathfrak{Y}) \rightarrow [e(\mathfrak{X}) = e(\mathfrak{Y}) \leftrightarrow \mathfrak{X} \equiv \mathfrak{Y}]]$
- (2) $\forall \mathfrak{X} \forall \mathfrak{Y}[E(\mathfrak{X}) \wedge \mathfrak{X} \equiv \mathfrak{Y} \rightarrow E(\mathfrak{Y})]$

where in both (1) and (2) \mathfrak{X} and \mathfrak{Y} are concept variables of the same sort.

If we think of $e(\mathfrak{X})$ as an object representing \mathfrak{X} , Axiom 1 above expresses the idea that *extensional equality* of any concepts satisfying E is equivalent to *identity* of their representing objects. That is, for any concept \mathfrak{X} satisfying E , $e(\mathfrak{X})$ may be regarded as the *extension* of \mathfrak{X} . And the predicate E itself represents the property of *possessing an extension*. For these reasons Axiom 1 will be called the *Axiom of Extensions*. As for Axiom 2, it states the reasonable requirement that any concept extensionally equivalent to a concept possessing an extension *itself* possesses one (that is, \equiv is a *congruence* relation with respect to E).

A straightforward Russell type argument in \mathbb{F} enables us to infer $\neg\forall\mathfrak{X}E(\mathfrak{X})$,⁴¹ that is, *not every concept possesses an extension*. This being the case, what concepts do we need to (consistently) assume possess extensions in order to enable an infinite well-ordering to be constructed? It was Frege's remarkable *discovery* that for this it suffices to assume just that extensions be possessed by the members of a certain class of simple and natural *second-order* concepts—those that, following Boolos,⁴² we shall term *numerical*.

Numerical concepts are defined as follows. First, we formulate the relation \approx of equinumerosity or equipollence on \mathcal{B} as usual:

$$\begin{aligned} X \approx Y \iff_{df} & \exists Z[\forall x\forall y[(xy) \eta Z \rightarrow x \eta X \wedge y \eta Y] \\ & \wedge \forall x\forall y\forall z[(xy) \eta Z \wedge (xz) \eta Z \rightarrow y = z] \\ & \wedge \forall x[x \eta X \rightarrow \exists y (xy) \eta Z] \\ & \wedge \forall y[y \eta Y \rightarrow \exists x (xy) \eta Z] \end{aligned}$$

With any basic concept X we associate the second level concept

$$\|X\| =_{df} \widehat{Y} [X \approx Y].$$

Concepts of the form $\|X\|$ are called *numerical*.

⁴¹To be explicit, define $A =_{df} \hat{x}[\forall X[e(X) = x \wedge E(X) \rightarrow \neg x \eta X]]$. Then $\neg E(A)$ is inferrable in \mathbb{F} .

⁴²"The standard of equality of numbers," Chapter 8, below.

If we assume that every numerical concept possesses an extension (i.e., $\forall X E(\|X\|)$), then the extension

$$|X| =_{df} e(\|X\|)$$

is called the (cardinal) *number* of X . Objects of the form $|X|$ are called (cardinal) *numbers*. Under these assumptions it is easy to derive *Hume's principle*, viz.

$$\forall X \forall Y [X \approx Y \leftrightarrow |X| = |Y|].$$

We shall call a concept X (Dedekind) *infinite* if $\exists Y [Y \subsetneq X \wedge X \approx Y]$, where $Y \subsetneq X$ of course stands for $\forall x(x \eta Y \rightarrow x \eta X) \wedge Y \neq X$. Objects of the form $|X|$ with X infinite are called *infinite numbers*.

We are going to show how, in \mathbb{F} , the existence of an infinite well-ordering (i.e. an infinite well-ordered concept) may be derived as a special case of a general set-theoretic result—formulable and provable in \mathbb{F} —which is normally used to derive Zermelo's Well-ordering theorem. In its original form this result is what we shall call the

Zermelo-Bourbaki lemma. ⁴³ *Let E be a set, \mathcal{F} a family of subsets of E and $p : \mathcal{F} \rightarrow E$ a map such that $p(X) \notin X$ for all $X \in \mathcal{F}$. Then there is a subset M of E and a well-ordering \leq of M such that, writing S_x for $\{y : y < x\}$,*

- (i) $\forall x \in M [S_x \in \mathcal{F} \wedge p(S_x) = x]$
- (ii) $M \notin \mathcal{F}$.

Bourbaki employs this result to construct an elegant derivation of Zermelo's Well-ordering theorem from the Axiom of Choice. In the present context, however, it will be used to produce an equally elegant proof of what we shall call, following a suggestion of Boolos,

⁴³Lemma 3, § 2, Ch. 3 of N. Bourbaki, *Théorie des ensembles*, 2nd ed. Paris: Hermann, 1963. Bourbaki's proof is a generalization of Zermelo's argument for his Well-ordering theorem in his "Proof that every set can be well-ordered" (1904) in Jean van Heijenoort, ed., *From Frege to Gödel: a sourcebook in mathematical logic, 1879–1931*, Cambridge: Harvard University Press, 1967, Stefan Bauer-Mengelberg, tr.

Frege's theorem. Suppose given a set E and a map $n : PE \rightarrow E$ such that

$$(*) \quad \forall X \subseteq E \forall Y \subseteq E [n(X) = n(Y) \leftrightarrow X \approx Y].$$

Then E has an infinite well-ordered subset.

Proof. We apply the Zermelo-Bourbaki lemma with \mathcal{F} the family of all subsets X of E for which $n(X) \notin X$ and p the map n . We obtain $M \subseteq E$ and a well-ordering \leq of M such that (i) $n(S_x) = x$ for all $x \in M$, (ii) $n(M) \in M$. Writing m for $n(M)$ we have $m \in M$ by (ii), whence $n(S_m) = m = n(M)$ by (i). Condition (*) now yields $S_m \approx M$. Since $m \notin S_m$, S_m is a proper subset of M and it follows that the latter is infinite. \square

Now both of these results can be translated into and proved within \mathbb{F} . Carrying this out for the Zermelo-Bourbaki lemma yields the

Zermelo-Bourbaki lemma in \mathbb{F} . Let \mathcal{S} be any second level concept with respect to which \equiv is a congruence relation and t a term such that $t(X)$ is an object for all basic concepts X and satisfies

$$\forall X \forall Y [X \equiv Y \wedge X \eta \mathcal{S} \rightarrow t(X) = t(Y)]$$

$$\forall X [X \eta \mathcal{S} \rightarrow \neg t(X) \eta X].$$

Then there is a relation R such that R is a well-ordering and, writing M for its field, and R_x for $\widehat{y}[(yx) \eta R \wedge y \neq x]$,

- (i) $\forall x [x \eta M \rightarrow R_x \eta \mathcal{S} \wedge t(R_x) = x]$
- (ii) $\neg M \eta \mathcal{S}$.

In the case of Frege's theorem, the same process yields

Frege's theorem in \mathbb{F} . Suppose that every numerical concept has an extension. Then there exists an infinite well-ordered concept and hence an infinite number.

Since, as is well known, Frege's original system in the *Grundgesetze* was inconsistent, we should assure ourselves that the axioms of

\mathbb{F} , together with the hypothesis of Frege's theorem—that every numerical concept has an extension—are consistent. The easiest way to see this is by noting that the following set-theoretic interpretations yield a model of the axioms of \mathbb{F} in which the hypothesis of Frege's theorem holds. To wit, interpret \mathcal{O} as $\omega+1$, \mathcal{B} as $P(\omega+1)$, \mathcal{R} as $P((\omega+1) \times (\omega+1))$, \mathcal{S}_b as $PP(\omega+1)$, \mathcal{S}_r as $PP((\omega+1) \times (\omega+1))$, E as the subset of $PP(\omega+1)$ consisting of all elements of the form $\|X\| =_{df} \{Y \in P(\omega+1) : X \approx Y\}$ with $X \in P(\omega+1)$, and e as the map $PP(\omega+1) \rightarrow \omega+1$ which sends each $\|X\|$ with $X \in P(\omega+1)$ to its cardinality $|X|$ ($\in \omega+1$) and everything else to 0. Thus the axioms of \mathbb{F} — together with the hypothesis of Frege's theorem — may be regarded as a consistent fragment of Frege's original system.