

SOME ASPECTS OF THE CATEGORY OF SUBOBJECTS OF CONSTANT OBJECTS IN A TOPOS

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Let $\mathcal{F}$ be a topos and let $\mathcal{E}$ be a topos defined over $\mathcal{F}$ by a geometric morphism γ . Objects of $\mathcal{E}$ of the form γ^*X for $X \in \mathcal{F}$ are called *constant* objects. In this paper we shall study the full subcategory $\mathcal{E}^*$ of $\mathcal{E}$ consisting of all *subobjects* of constant objects in $\mathcal{E}$. In the case where $\mathcal{F}$ is the category $\mathcal{S}$ of sets we construct, for each complete Heyting algebra H , a simple category $\tilde{H}$ which we show to be equivalent to $\mathcal{E}^*$ when H is the algebra of subobjects of the terminal object in $\mathcal{E}$. This yields a new and especially straightforward proof of the well-known result that a topos defined over $\mathcal{S}$ is equivalent as a category to a Boolean extension of the universe of sets iff it satisfies the axiom of choice. We go on to investigate the properties of $\tilde{H}$ and in Section 2 we extend some of our results to the case in which $\mathcal{F}$ is an arbitrary base topos.

1. Toposes defined over the category of sets

Let $\mathcal{E}$ be a topos defined over the category $\mathcal{S}$ of sets by a geometric morphism γ . In this case we know that $\gamma^*I = \coprod_I 1$ and $\gamma_*X = \mathcal{E}(1, X)$ for $I \in \mathcal{S}$, $X \in \mathcal{E}$. Moreover, the coproduct of any family of subobjects of 1 always exists in $\mathcal{E}$ (cf. the remark on page 120 of [3]), and the objects of $\mathcal{E}^*$ are precisely the objects of $\mathcal{E}$ which are of this form. We first find a particularly simple alternative description of $\mathcal{E}^*$.

Let H be a complete Heyting algebra (frame, locale). We define the category $\tilde{H}$ as follows. The *objects* of $\tilde{H}$ are all functions $I \xrightarrow{a} H$, $a = \langle a_i \rangle_{i \in I}$ for all sets I . If $I \xrightarrow{a} H$, $J \xrightarrow{b} H$ are two objects in $\tilde{H}$, an *arrow* $a \xrightarrow{p} b$ is a function $p: I \times J \rightarrow H$, $p = \langle p_{ij} \rangle_{i \in I, j \in J}$ such that

$$p_{ij} \leq b_j \quad (i \in I, j \in J), \quad (1.1)$$

$$p_{ij} \wedge p_{ij'} = 0 \quad (i \in I, j \neq j' \in J), \quad (1.2)$$

$$\bigvee_{j \in J} p_{ij} = a_i \quad (i \in I). \quad (1.3)$$

(We may think of an object a of $\tilde{H}$ as an ' H -valued set' in which $a(i) \in H$ is the ' H -value' of the statement $i \in a$. An arrow $a \xrightarrow{p} b$ in $\tilde{H}$ may be thought of as an ' H -valued functional relation' between a and b .) If $c = \langle c_k \rangle_{k \in K}$ is an object of $\tilde{H}$ and $q: J \times K \rightarrow H$ is an arrow $b \rightarrow c$ in $\tilde{H}$, the composition $qp = r$ of p and q is defined by

$$r_{ik} = \bigvee_{j \in J} p_{ij} \wedge q_{jk}.$$

It is easy to check that composition is associative and that the identity arrow $\text{id}: a \rightarrow a$ is given by the 'Kronecker delta' function $\delta: I \times I \rightarrow H$ such that

$$\delta_{ii} = 0 \quad (i \neq i'), \quad \delta_{ii} = 1.$$

If $\mathcal{E}$ is an $\mathcal{S}$ -topos, then (cf. the proof of 5.37 of [3]), $\gamma_*\Omega_{\mathcal{E}}$ is a complete Heyting algebra; it is naturally isomorphic to the (partially ordered) set of subobjects of 1 in $\mathcal{E}$. Thus the latter is a complete Heyting algebra.

Now we can prove

1.1. Theorem. *Let $\mathcal{E}$ be an $\mathcal{S}$ -topos, and let H be the complete Heyting algebra of subobjects of 1 in $\mathcal{E}$. Then $\mathcal{E}^* \cong \tilde{H}$. If in addition the axiom of choice holds in $\mathcal{E}$, then (H is a complete Boolean algebra and) $\mathcal{E} \cong \tilde{H}$.*

Proof. We define a functor $F: \tilde{H} \rightarrow \mathcal{E}$ as follows. For each object $a: I \rightarrow H$ in $\tilde{H}$ we put

$$F(a) = \coprod_{i \in I} a_i.$$

If $b: J \rightarrow H$ is an object in $\tilde{H}$ and $p: a \rightarrow b$ an arrow in $\tilde{H}$, we define $F(p): F(a) \rightarrow F(b)$ as follows. From (1.2) and (1.3) we have

$$a_i \cong \coprod_{j \in J} p_{ij} \quad (i \in I)$$

and from the (unique) arrows $p_{ij} \rightarrow b_j$ given by (1.1) we obtain for each $i \in I$ a unique arrow s_i such that the diagram

$$\begin{array}{ccc} p_{ij} & \xrightarrow{\quad} & b_j \\ \downarrow & & \downarrow \\ \coprod_{j \in J} p_{ij} & \xrightarrow{s_i} & \coprod_{j \in J} b_j \end{array}$$

commutes for all $i \in I$, $j \in J$, where the downward arrows are canonical injections. We put p'_i for the composition

$$a_i \xrightarrow{\sim} \coprod_{j \in J} p_{ij} \xrightarrow{s_i} \coprod_{j \in J} b_j.$$

Thus p'_i is the unique arrow making the diagram

$$\begin{array}{ccc} p_{ij} & \xrightarrow{\quad} & b_j \\ \downarrow & & \downarrow \\ a_i & \xrightarrow{p'_i} & \coprod_{j \in J} b_j \end{array} \quad (1.4)$$

commute. We finally define $F(p)$ to be the unique arrow such that the diagram

$$\begin{array}{ccc} a_i & \xrightarrow{\sigma_i} & \coprod_{i \in I} a_i \\ & \searrow p'_i & \downarrow F(p) \\ & & \coprod_{j \in J} b_j \end{array}$$

commutes for each $i \in I$, where σ_i is the canonical injection.

It is not hard to check that F is a functor, and clearly each object in $\mathcal{E}^*$ is (isomorphic to an object) in the range of F . Accordingly, to show that F is an equivalence it suffices to show that F is full and faithful.

To verify the fidelity of F , we first observe that the diagram (1.4) is a pullback for each $i \in I$, $j \in J$. For let

$$\begin{array}{ccc} r_{ij} & \xrightarrow{\quad} & b_j \\ \downarrow & & \downarrow \\ a_i & \xrightarrow{p'_i} & \coprod_{j \in J} b_j \end{array}$$

be a pullback. Then clearly, since (1.4) commutes, we have $p_{ij} \leq r_{ij}$. On the other hand, by the universality of coproducts in $\mathcal{E}$, we have

$$a_i \cong \coprod_{j \in J} r_{ij},$$

so that

$$\bigvee_{j \in J} p_{ij} = a_i = \bigvee_{j \in J} r_{ij}.$$

But it now follows from the disjointness of coproducts in $\mathcal{E}$ that $r_{ij} \wedge r_{ij'} = 0$ when $j \neq j'$. One easily concludes from this that $p_{ij} = r_{ij}$, so (1.4) is indeed a pullback.

Now let $a \xrightarrow{p} b$ and $a \xrightarrow{q} b$ be arrows in $\tilde{H}$ and suppose that $F(p) = F(q)$. Then $p'_i = q'_i$ for all $i \in I$ and so, since (1.4) is a pullback, it follows that $q_{ij} \leq p_{ij}$. Similarly, $p_{ij} \leq q_{ij}$ and so $p = q$. Hence F is faithful as claimed.

Finally, we show that F is full. Suppose that a, b are objects in $\tilde{H}$ and that

$F(a) \xrightarrow{f} F(b)$ is an arrow in $\mathcal{E}$. For each i, j form the pullback

$$\begin{array}{ccc} p_{ij} & \xrightarrow{\quad} & b_j \\ \downarrow & & \downarrow \\ a_i & \xrightarrow{f\sigma_i} & \coprod b_j \end{array} \quad (1.5)$$

By the universality of coproducts in $\mathcal{E}$ we have $\coprod_{j \in J} p_{ij} \cong a_i$, and by the disjointness of coproducts in $\mathcal{E}$ we have $p_{ij} \wedge p_{ij'} = 0$ for $j \neq j'$, whence $\bigvee_{j \in J} p_{ij} = a_i$. Hence $p = \langle p_{ij} \rangle_{i \in I, j \in J}$ is an arrow $a \rightarrow b$ in H . We claim $F(p) = f$. For this to be the case it suffices that $F(p)\sigma_i = f\sigma_i$ for all $i \in I$. But this follows immediately from (1.4), (1.5) and the fact that $p'_i = F(p)\sigma_i$.

Thus F is an equivalence and $\mathcal{E}^* = \tilde{H}$.

Now suppose that $\mathcal{E}$ satisfies the axiom of choice. Then, by 5.3 of [3], the subobjects of 1 form a set of generators in $\mathcal{E}$ and so each object of $\mathcal{E}$ is covered by a family of subobjects of 1. Using the axiom of choice in $\mathcal{E}$, it follows easily from this that each object of $\mathcal{E}$ is isomorphic to a coproduct of subobjects of 1, whence $\mathcal{E} = \mathcal{E}^* = \tilde{H}$. $\square$

We recall that [1] that, for each complete Boolean algebra B , the Boolean extension $V^{(B)}$ of the universe of sets in the sense of Scott–Solovay may be regarded as an $\mathcal{S}$ -topos in a natural way. Since the axiom of choice holds in such a topos (provided it holds in $\mathcal{S}$), Theorem 1.1 yields as an immediate consequence the following well-known result.

1.2. Corollary. *An $\mathcal{S}$ -topos is equivalent to one of the form $V^{(B)}$ for a complete Boolean algebra B if and only if it satisfies the axiom of choice.*

Our next theorem shows that a number of conditions on $\mathcal{E}^*$ and H are equivalent.

1.3. Theorem. *Let $\mathcal{E}$ be an $\mathcal{S}$ -topos and let H be the complete Heyting algebra of subobjects of 1 in $\mathcal{E}$. Consider the conditions:*

- (i) $\mathcal{E}$ satisfies the axiom of choice;
- (ii) $\mathcal{E}^* \hookrightarrow \mathcal{E}$ is an equivalence;
- (iii) $\Omega_{\mathcal{E}}$ is isomorphic to an object in $\mathcal{E}^*$;
- (iv) $\mathcal{E}$ is Boolean;
- (v) $\mathcal{E}^* = \tilde{H}$ is a topos;
- (vi) $\mathcal{E}^* = \tilde{H}$ has a subobject classifier;
- (vii) H is a Boolean algebra;
- (viii) $\tilde{H} = V^{(B)}$ for some complete Boolean algebra B .

Then (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v), and (v) through (viii) are equivalent. If $\mathcal{E}$ is localic over $\mathcal{S}$, then all the conditions are equivalent. Thus conditions (v) through (viii) are equivalent for any complete Heyting algebra H .

Proof. (i) $\Rightarrow$ (ii) follows from Theorem 1.1.

(ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (iv). Recall that an object X of a topos is said to be *decidable* if the diagonal subobject $X \xrightarrow{\Delta} X \times X$ has a complement. It is easy to verify that, since each object X of $\mathcal{S}$ is decidable, so is each object of $\mathcal{E}$ of the form γ^*X and hence so is any subobject of such an object; i.e. any object in $\mathcal{E}^*$ is decidable. Thus condition (iii) implies that $\Omega_{\mathcal{E}}$ is decidable, and this is well known to be equivalent to Booleanness of $\mathcal{E}$.

(iv) $\Rightarrow$ (iii). If $\mathcal{E}$ is Boolean, then $\Omega_{\mathcal{E}} \cong 1 + 1 \cong \gamma^*(1 + 1) \in \mathcal{E}^*$.

(ii) $\Rightarrow$ (i). If (ii) holds, then $\mathcal{E}$ is certainly localic over $\mathcal{S}$; but since (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\mathcal{E}$ is also Boolean. Then by 5.39 of [3] the axiom of choice in $\mathcal{S}$ yields the axiom of choice in $\mathcal{E}$.

(iii) $\Rightarrow$ (vi). Since $\mathcal{E}^*$ is easily seen to be closed under products and subobjects in $\mathcal{E}$, it follows that it is also closed under pullbacks in $\mathcal{E}$. The implication in question now follows easily.

(v) $\Rightarrow$ (vi) is trivial.

(vi) $\Rightarrow$ (vii). Let $\langle b_j \rangle_{j \in J}$ be the subobject classifier in $\tilde{H}$. Then, given $a \in H$, the object $\langle a \rangle$ of $\tilde{H}$ is a subobject of the terminal object $\langle 1 \rangle$ in $\tilde{H}$ and so there are arrows

$$\langle 1 \rangle \xrightarrow{p} \langle b_j \rangle_{j \in J}, \quad \langle 1 \rangle \xrightarrow{q} \langle b_j \rangle_{j \in J}$$

in $\tilde{H}$ such that

$$\begin{array}{ccc} \langle a \rangle & \xrightarrow{\quad} & \langle 1 \rangle \\ \downarrow & & \downarrow q \\ \langle 1 \rangle & \xrightarrow{p} & \langle b_j \rangle_{j \in J} \end{array} \quad (1.6)$$

is a pullback. Since p and q are arrows in $\tilde{H}$, we have

$$\begin{aligned} p_j \wedge p_k &= q_j \wedge q_k = 0 \quad \text{for } j \neq k \in J, \\ 1 &= \bigvee_{j \in J} p_j = \bigvee_{k \in J} q_k, \end{aligned}$$

so that

$$\bigvee_{j \in J} \bigvee_{k \in J} p_j \wedge q_k = 1. \quad (1.7)$$

Since (1.6) commutes, we have

$$a \wedge q_j = a \wedge p_j \quad (j \in J),$$

so that

$$\begin{aligned} a &= a \wedge 1 = \bigvee_{j \in J} a \wedge p_j \\ &= \bigvee_{j \in J} a \wedge p_j \wedge q_j \\ &\leq \bigvee_{j \in J} p_j \wedge q_j. \end{aligned} \quad (1.8)$$

But since (1.6) is a pullback, we must have, for all $c \in H$,

$$\forall j \in J [c \wedge p_j = c \wedge q_j] \Rightarrow c \leq a.$$

In particular, taking $c = \bigvee_{j \in J} p_j \wedge q_j$, we get

$$\bigvee_{j \in J} p_j \wedge q_j \leq a,$$

so that, by (1.8), $a = \bigvee_{j \in J} p_j \wedge q_j$. But then, by (1.7), a has a complement $\bigvee_{j \in J} p_j \wedge q_k$ in H . This gives (vii).

(vii) $\Rightarrow$ (viii). This follows from Theorem 1.1.

(viii) $\Rightarrow$ (v) is trivial.

Finally, if $\mathcal{E}$ is localic over $\mathcal{S}$, then (vii) $\Rightarrow$ (iv) and hence (in this case) (vii) $\Rightarrow$ (i). For suppose that $\mathcal{E}$ is not Boolean; then $\neg : \Omega \rightarrow \Omega$ is not the identity. Hence by the locality of $\mathcal{E}$ there is $U \rightarrow 1$ and $U \xrightarrow{\alpha} \Omega$ such that

$$U \xrightarrow{\alpha} \Omega \neq U \xrightarrow{\alpha} \Omega \xrightarrow{\neg} \Omega.$$

Since Ω is injective there is $1 \xrightarrow{\beta} \Omega$ such that

$$U \xrightarrow{\alpha} \Omega = U \longrightarrow 1 \xrightarrow{\beta} \Omega.$$

Clearly, then

$$1 \xrightarrow{\beta} \Omega \xrightarrow{\neg} \Omega \neq 1 \xrightarrow{\beta} \Omega.$$

But this means that the subobject of 1 classified by β is not equal to its double complement in H , i.e. H is not Boolean. $\square$

Remark. It is well known that the implication (i) $\Rightarrow$ (iv) cannot be reversed; e.g. take $\mathcal{E}$ to be the topos $\mathcal{S}^G$ of G -sets for a group G . A similar counterexample shows the irreversibility of the implication (iv) $\Rightarrow$ (v): take $\mathcal{E}$ to be the topos $\mathcal{S}^M$ of M -sets for a monoid M which is not a group. Then E is not Boolean; on the other hand 1 has only two subobjects 0 and 1 in $\mathcal{E}$, so $\mathcal{E}^* \models \mathcal{S}$ and (v) is satisfied.

2. Toposes defined over an arbitrary base topos

We now suppose that $\mathcal{E}$ is a topos defined over an arbitrary base topos $\mathcal{F}$ by a geometric morphism γ and investigate the extent to which the results and constructions of the previous section carry over to this more general setting. We shall employ freely the internal (Mitchell–Benabou) language of a topos as presented in §5.4 of [3].

To begin with, let us see how to generalize the construction of $\tilde{H}$. Let H be an internally complete Heyting algebra object in $\mathcal{F}$; we define the category $\tilde{H}$ as follows. (It is important to observe that $\tilde{H}$ is an ‘honest-to-goodness’ category, *not* an internal category in $\mathcal{F}$.)

First of all, the objects of $\tilde{H}$ are the objects of $\mathcal{F}/H$, i.e. all arrows $I \xrightarrow{a} H$ in $\mathcal{F}$. Before defining the arrows of $\tilde{H}$ we need some notation. We let

$$\lambda_H : \Omega_{\mathcal{F}} \rightarrow H$$

be the arrow defined by

$$\lambda_H(p) = \bigvee_H \{a \in H : (a = 1_H) \wedge p\},$$

where p is a variable of type $\Omega_{\mathcal{F}}$. For each object J of $\mathcal{F}$, we let

$$\delta_J : J \times J \rightarrow \Omega_{\mathcal{F}}$$

be the classifying arrow of the diagonal subobject of $J \times J$, and we put eq_J for the composition

$$J \times J \xrightarrow{\delta_J} \Omega_{\mathcal{F}} \xrightarrow{\lambda_H} H.$$

Now we can define the arrows of $\tilde{H}$. Given objects $I \xrightarrow{a} H$ and $J \xrightarrow{b} H$ of $\tilde{H}$, an arrow $a \xrightarrow{p} b$ in $\tilde{H}$ is an arrow $I \times J \xrightarrow{p} H$ in $\mathcal{F}$ satisfying the following conditions, where i, j, j', x are variables of types I, J, J, H respectively:

$$F \models p(i, j) \leq_H b(j) \quad (2.1)$$

$$F \models p(i, j) \wedge p(i, j') \leq_H \text{eq}_J(j, j') \quad (2.2)$$

$$F \models \bigvee_H \{x : \exists j [x = p(i, j)]\} = a(i). \quad (2.3)$$

(Notice that these conditions are just the ‘internal’ analogues of the conditions (1.1), (1.2), (1.3).) If $K \xrightarrow{c} H$ is an object of $\tilde{H}$ and $J \times K \xrightarrow{q} H$ is an arrow $b \rightarrow c$ in $\tilde{H}$, the composition $aq = r$ is given by

$$r(i, k) = \bigvee_H \{x : \exists j [x = p(i, j) \wedge q(j, k)]\},$$

where k is a variable of type k . The identity arrow

$$a \xrightarrow{\text{id}_a} a$$

in H is given by

$$\text{id}_a = \wedge_H \langle \text{eq}_1, a\pi_1 \rangle,$$

where π_1 is 'projection onto the first coordinate' and $\wedge_H$ is the meet operation in H . It is readily checked that these data do determine a category.

Now let $\mathcal{E} \xrightarrow{\gamma} \mathcal{F}$ be a geometric morphism. Then ([3], 5.36) $H = \gamma_* \Omega_{\mathcal{E}}$ is an internally complete Heyting algebra object in $\mathcal{F}$, and in this case it is easily verified that the arrow $\lambda = \lambda_H$ has $(\gamma^* \dashv \gamma_*)$ -transpose $\tilde{\lambda}: \gamma^* \Omega_{\mathcal{F}} \rightarrow \Omega_{\mathcal{E}}$ classifying $\gamma^*(\text{true}_{\mathcal{F}})$.

We recall that we have defined $\mathcal{E}^*$ to be the full subcategory of $\mathcal{E}$ whose objects are all subobjects of objects of the form $\gamma^* I$ for $I \in \mathcal{F}$. We shall prove the analogue of 1.1 in this more general context.

2.1. Theorem. $\mathcal{E}^* = (\gamma_* \Omega_{\mathcal{E}})^{\perp}$.

Before giving the proof, we need some more terminology and a lemma. Let $X \xrightarrow{f} Y$ be a partial arrow in $\mathcal{E}$, given by the diagram

$$\begin{array}{ccc} X' & \xrightarrow{f} & Y \\ f' \downarrow & & \downarrow \\ X & & \end{array}$$

We define the *graph* of f , $\text{gph}(f)$, to be the image of the arrow

$$X' \xrightarrow{\langle f', f \rangle} X \times Y,$$

i.e. the extension of the formula

$$\exists x' [(x, y) = \langle f'(x'), f(x') \rangle],$$

where x, x', y are variables of types X, X', Y respectively.

2.2. Lemma. Let $X \times Y \xrightarrow{r} \Omega_{\mathcal{E}}$, let R be the subobject of $X \times Y$ classified by r , and let $[R^1]$ be the corresponding global element of $\Omega^{X \times Y}$. Then the following are equivalent:

- (i) $R = \text{gph}(f)$ for some $X \xrightarrow{f} Y$;
 - (ii) $E = \langle x, y \rangle \in [R^1] \wedge \langle x, z \rangle \in [R^1] \Rightarrow y = z$;
 - (iii) $E = r(x, y) \wedge r(x, z) \leq \Omega_{\mathcal{E}} \delta_Y(y, z)$,
- where x, y, z are variables of type X, Y, Y respectively. Moreover, if these conditions hold, then the subobject X' of X on which f is defined may be taken to be $\{x: \exists y \langle x, y \rangle \in [R^1]\}$, or equivalently $\{x: \exists y [r(x, y) = \text{true}_{\mathcal{E}}]\}$.

Proof. (ii) $\Leftrightarrow$ (iii) holds by definition.

(i) $\Rightarrow$ (ii). We have, introducing variables x', x'' of type X' ,

$$\begin{aligned} \mathcal{E} &= \langle x, y \rangle \in [\text{gph}(f)] \wedge \langle x, z \rangle \in [\text{gph}(f)] \\ &= \exists x' [x = f'(x') \wedge y = f(x')] \wedge \exists x'' [x = f'(x'') \wedge z = f(x'')] \\ &= y = z, \end{aligned}$$

by the monicity of f' (see diagram (2.4)).

(ii) $\Rightarrow$ (i). Form the pullback

$$\begin{array}{ccc} X' & \xrightarrow{f} & Y \\ f' \downarrow & & \downarrow \{ \cdot \} \\ X & \xrightarrow{[\{y: \langle x, y \rangle \in [R^1]\}]} & \Omega^Y \end{array} \quad (2.5)$$

Then we have

$$\begin{aligned} \mathcal{E} = \langle x, y \rangle \in [\text{gph}(f)] &\Leftrightarrow \exists x' [x = f'(x') \wedge y = f(x')] \\ &\Leftrightarrow \{x: \langle x, z \rangle \in [R^1] \} = \{y\} \quad (\text{since (2.5) is a pullback}) \\ &\Leftrightarrow \langle x, y \rangle \in [R^1] \quad (\text{by (iii)}). \end{aligned}$$

Thus $R = \text{gph}(f)$ as required.

To prove the final assertion, we merely observe that, by the above,

$$\mathcal{E} = \exists x' [x = f'(x')] \Leftrightarrow \exists y [\langle x, y \rangle \in [R^1]]. \quad \square$$

Now we can provide the

Proof of Theorem 2.1. We define a functor

$$\beta: (\gamma_* \Omega_{\mathcal{E}})^{\perp} \rightarrow \mathcal{E}^*$$

as follows. Given an arrow $I \xrightarrow{a} \gamma_* \Omega_{\mathcal{E}}$ in $(\gamma_* \Omega_{\mathcal{E}})^{\perp}$, let

$$\gamma^* I \xrightarrow{a} \Omega_{\mathcal{E}}$$

be its transpose across the adjunction $\gamma^* \dashv \gamma_*$, and let $\beta(a)$ be the subobject of $\gamma^* I$ classified by a . Clearly $\beta(a) \in \mathcal{E}^*$ and every object of $\mathcal{E}^*$ is isomorphic to an object of this form.

Next, given an object $J \xrightarrow{b} \gamma_* \Omega_{\mathcal{E}}$ and an arrow $a \xrightarrow{p} b$ in $(\gamma_* \Omega_{\mathcal{E}})^{\perp}$, i.e. an arrow $I \times J \xrightarrow{p} \gamma_* \Omega_{\mathcal{E}}$ in $\mathcal{F}$ satisfying (2.1), (2.2), (2.3) (with $H = \gamma_* \Omega_{\mathcal{E}}$), let

$$\rho: \gamma^* I \times \gamma^* J \cong \gamma^*(I \times J) \longrightarrow \Omega_{\mathcal{E}}$$

be its transpose across $\gamma^* \dashv \gamma_*$. After transposition across $\gamma^* \dashv \gamma_*$, conditions (2.1), (2.2) and (2.3) become the following, where x, y, z are variables of type $\gamma^* I, \gamma^* J, \gamma^* J$,

respectively, and a, b are the transposes of a, b , respectively:

$$\delta \models p(x, y) \leq b(y) \quad (2.1')$$

$$\delta \models p(x, y) \wedge p(x, z) \leq \delta_{\gamma^* J}(y, z) \quad (2.2')$$

$$|\exists y [p(x, y) = \text{true}_\delta]| = a(x). \quad (2.3')$$

From Lemma 2.2 we see that (2.2') implies that there is a partial arrow

$$\gamma^* I \xrightarrow{f} \gamma^* J,$$

unique up to isomorphism, such that $\text{gph}(f)$ is equal to the subobject of $\gamma^* I \times \gamma^* J$ classified by p . Condition (2.3') tells us that f is defined on the subobject $\beta(a)$ of $\gamma^* I$ classified by a , and (2.1') that the image of f is contained in the subobject $\beta(b)$ of $\gamma^* J$ classified by b . Thus we may regard f as an arrow

$$\beta(a) \xrightarrow{f} \beta(b).$$

We put $\beta(p) = f$.

One can now check (tediously!) that β preserves composition and the identity arrows. Thus we have a functor

$$\beta: (\gamma^* \Omega_\delta)^- \longrightarrow \mathcal{E}^*.$$

It remains to show that β is an equivalence of categories. We have already remarked that every object in $\mathcal{E}^*$ is isomorphic to one in the range of β . Also, β is clearly faithful. To show that β is full, let $a, b \in (\gamma^* \Omega_\delta)^-$ and let $\beta(a) \xrightarrow{f} \beta(b)$ be an arrow in $\mathcal{E}^*$. Let p be the characteristic arrow of the subobject of $\gamma^* I \times \gamma^* J$ corresponding to the graph of f . It is then easy to check that (2.1'), (2.2'), (2.3') hold for p , and transposition across $\gamma^* \dashv \gamma_*$ yields (2.1), (2.2), (2.3) for its transpose p . Thus $a \xrightarrow{p} b$ is an arrow in $(\gamma^* \Omega_\delta)^-$, and clearly $\beta(p) = f$. Hence β is full, and therefore an equivalence. $\square$

By taking $\delta = \mathcal{F}$ and γ the identity functor in Theorem 2.1, we immediately obtain

2.3. Corollary. $\bar{\Omega}_\delta \cong \mathcal{F}$. $\square$

Having affirmed that Theorem 1.1 carries over to the case of an arbitrary base topos, we may now ask to what extent the same is true of Theorem 1.3. I have not been able to solve this problem completely, but I shall give a partial solution in Theorem 2.5. First, however, we require another lemma, which gives a canonical representation of objects in $\mathcal{E}^*$.

2.4. Lemma. For each object X of $\mathcal{E}$ the following are equivalent:

- (i) $X \in \mathcal{E}^*$;

(ii) there is a monic $X \rightarrow \gamma^* \gamma_* \bar{X}$, where $\bar{X}$ is the partial map classifier of X . If X is injective, then $\bar{X}$ may be replaced by X .

Proof. (ii) $\Rightarrow$ (i) being trivial, we need only prove (i) $\Rightarrow$ (ii). If $X \in \mathcal{E}^*$, then by definition there is $Y \in \mathcal{F}$ and a monic $X \rightarrow \gamma^* Y$. Hence there is $\gamma^* Y \xrightarrow{\alpha} \bar{X}$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\eta} & \bar{X} \\ \downarrow & \searrow \alpha & \\ \gamma^* Y & & \end{array}$$

commutes. Now if $Y \xrightarrow{\alpha} \gamma_* \bar{X}$ is the transpose of α across $\gamma^* \dashv \gamma_*$, we have the commutative diagram

$$\begin{array}{ccc} \gamma^* Y & \xrightarrow{\alpha} & \bar{X} \\ \downarrow \gamma^*(a) & \searrow \epsilon & \\ \gamma^* \gamma_* \bar{X} & & \end{array}$$

where ϵ is the counit arrow. Hence the composition

$$X \longrightarrow \gamma^* Y \xrightarrow{\gamma^*(a)} \gamma^* \gamma_* \bar{X}$$

is monic.

Clearly, if X is injective, we may replace $\bar{X}$ by X and η by the identity arrow. $\square$

Now we can prove:

2.5. Theorem. Let $\mathcal{F}$ be a topos, let $\mathcal{E}$ be a topos defined over $\mathcal{F}$ by a geometric morphism γ , and let $H = \gamma_* \Omega_\delta$. Consider the conditions:

- (i) Ω_δ is isomorphic to an object in $\mathcal{E}^*$;
- (ii) the counit arrow $\gamma^* \gamma_* \Omega_\delta \xrightarrow{\epsilon} \Omega_\delta$ has a section;
- (iii) $\mathcal{E}$ is Boolean;
- (iv) $\mathcal{E}^* \cong \bar{H}$ has a subobject classifier;
- (v) H is an internal Boolean algebra.

Then (i) $\Leftrightarrow$ (ii). If $\mathcal{F}$ is Boolean, then (i) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v). Finally, if $\mathcal{F}$ is Boolean and $\mathcal{E}$ is localic over $\mathcal{F}$, then all the conditions are equivalent.

Proof. (ii) $\Rightarrow$ (i) is trivial.

(i) $\Rightarrow$ (ii). By Lemma 2.4, if (i) holds, then since Ω_δ is injective we have a monic $\Omega_\delta \rightarrow \gamma^* \gamma_* \Omega_\delta$. Again using the injectivity of Ω_δ , there is an arrow $\gamma^* \gamma_* \Omega_\delta \xrightarrow{\beta} \Omega_\delta$.

such that $\beta \cdot \alpha = \text{id}$. Let $\gamma_* \Omega \xrightarrow{\beta} \gamma_* \Omega$ be the transpose of β across $\gamma^* \dashv \gamma_*$. Then $\beta = \varepsilon \cdot \gamma^*(\beta)$, so

$$\text{id} = \beta \cdot \alpha = \varepsilon \cdot \gamma^*(\beta) \cdot \alpha.$$

From now on we suppose that $\mathcal{F}$ is Boolean.

(i) $\Leftrightarrow$ (iii) is proved in exactly the same manner as (iii) $\Leftrightarrow$ (iv) in Theorem 1.3.

(i) $\Rightarrow$ (iv) is proved in just the same way as (iii) $\Rightarrow$ (vi) in Theorem 1.3.

(iv) $\Rightarrow$ (v). Suppose that $\mathcal{E}^*$ has a subobject classifier

$$1 \xrightarrow{\text{true}'} \Omega'.$$

Since Ω' is a subobject of an object of the form γ^*X for some $X \in \mathcal{F}$ and since each object in the Boolean topos $\mathcal{F}$ is decidable, Ω' itself must be decidable. By standard arguments, the global element

$$1 \xrightarrow{\text{true}'} \Omega'$$

must then have a complement

$$1 \xrightarrow{\text{false}'} \Omega'.$$

It follows that

$$\begin{pmatrix} \text{true}' \\ \text{false}' \end{pmatrix}$$

is an isomorphism between $1 + 1$ and Ω' . Thus $1 + 1$ is the subobject classifier in $\mathcal{E}^*$.

Now, by §1 of [2], we may without loss of generality replace $\mathcal{E}$ by the *localic* topos $\mathcal{F}[H]$ of internal sheaves on H (since $\mathcal{E}^*$ is unaffected by the change). Since $\mathcal{F}[H]$ is localic over $\mathcal{F}$, we have a diagram of the form

$$\begin{array}{ccc} Y & \xrightarrow{\quad} & \gamma^*X \\ \downarrow & & \\ \Omega & & \end{array}$$

where $X \in \mathcal{F}$ and Ω is the subobject classifier in $\mathcal{F}[H]$. Since Ω is injective, we have an epic $\gamma^*X \twoheadrightarrow \Omega$. Form the pullback

$$\begin{array}{ccc} Z & \xrightarrow{\quad} & 1 \\ \downarrow & & \downarrow \text{true} \\ \gamma^*X & \xrightarrow{\alpha} & \Omega. \end{array}$$

Since Z and γ^*X are both in $\mathcal{E}^*$ and $1 + 1$ is the subobject classifier in $\mathcal{E}^*$, we have a

pullback (in $\mathcal{E}^*$, and hence also in $\mathcal{F}[H]$)

$$\begin{array}{ccc} Z & \xrightarrow{\quad} & 1 \\ \downarrow & & \downarrow \sigma_1 \\ \gamma^*X & \xrightarrow{\beta} & 1 + 1 \end{array}$$

where σ_1 is a canonical injection. Combining this with the obvious pullback diagram

$$\begin{array}{ccc} 1 & \xrightarrow{\quad} & 1 \\ \downarrow \sigma_1 & & \downarrow \text{true} \\ 1 + 1 & \xrightarrow{\quad} & \Omega \end{array} \quad \begin{pmatrix} \text{true} \\ \text{false} \end{pmatrix}$$

yields a pullback

$$\begin{array}{ccc} Z & \xrightarrow{\quad} & 1 \\ \downarrow & & \downarrow \text{true} \\ \gamma^*X & \xrightarrow{\quad} & \Omega \end{array} \quad \begin{pmatrix} \text{true} \\ \text{false} \end{pmatrix} \cdot \beta$$

But then α and $\beta \cdot \begin{pmatrix} \text{true} \\ \text{false} \end{pmatrix}$ both classify the subobject Z of γ^*X , so they are equal; and since α is epic, so is $\begin{pmatrix} \text{true} \\ \text{false} \end{pmatrix}$. But $\begin{pmatrix} \text{true} \\ \text{false} \end{pmatrix}$ is obviously monic, and so it is an isomorphism. Therefore $\mathcal{F}[H]$ is Boolean, which implies, by 2.2 of [2], that H is an internal Boolean algebra.

(v) $\Rightarrow$ (iv). Again we may without loss of generality replace $\mathcal{E}$ by $\mathcal{F}[H]$. If H is an internal Boolean algebra, then $\mathcal{F}[H]$ is Boolean by 2.2 of [2]. So the subobject classifier $1 + 1$ of $\mathcal{F}[H]$ is in $\mathcal{E}^*$, and is clearly a subobject classifier there as well.

Finally, if $\mathcal{E}$ is localic over $\mathcal{F}$, then $\mathcal{E} \simeq \mathcal{F}[H]$ by the relative Giraud theorem, and (v) yields (iii) by 2.2 of [2]. $\square$

3. Final remarks

In the original version of this paper, I posed a number of open problems, one of which has recently been solved by Gordon Monro in [4]. These problems were:

- Does $\mathcal{E}^* \simeq \tilde{H}$ always have exponentials?
- If $\mathcal{E}$ is a Boolean topos, defined and localic over a Boolean topos $\mathcal{F}$, must the inclusion $\mathcal{E}^* \hookrightarrow \mathcal{E}$ be an equivalence? (By Theorem 1.3, this is true when $\mathcal{F} = \mathcal{S}$ and the axiom of choice holds. It can also be shown to hold when $\mathcal{F}$ is any topos satisfying the axiom of choice.)
- If the answer to (b) is, in general, no, find a characterization of those toposes defined over a (Boolean) topos $\mathcal{F}$ for which $\mathcal{E} = \mathcal{E}^*$.

Let me sketch a proof that a positive answer to (a) (even just for *Boolean* $\mathcal{E}$) yields the same for (b). To begin with, it is easy to show that $\mathcal{E}^*$ inherits (finite) products from $\mathcal{E}$. So if $\mathcal{E}^*$ has exponentials, where $\mathcal{E}$ is Boolean and defined over a Boolean topos $\mathcal{F}$ by a geometric morphism γ , then $\mathcal{E}^*$ is a Boolean topos with subobject classifier $2 = 1 + 1$ inherited from $\mathcal{E}$. Thus $\mathcal{E}$ and $\mathcal{E}^*$ are both localic $\mathcal{F}$ -toposes and, by the relative Giraud theorem, they are both equivalent over $\mathcal{F}$ to the topos $\mathcal{F}[B]$ of internal sheaves on the internally complete Boolean algebra $B = \gamma_* 2$ in $\mathcal{F}$. Accordingly we have a commutative diagram

$$\begin{array}{ccc} \mathcal{E}^* & \xrightarrow{\alpha} & \mathcal{E} \\ \gamma^* \downarrow & & \downarrow \gamma_* \\ \mathcal{F} & & \mathcal{F} \end{array}$$

where α is an equivalence. If $X \in \mathcal{E}^*$ there is a diagram of the form $X \rightarrow \gamma^* A$ in $\mathcal{E}$ and hence in $\mathcal{E}^*$. Applying α gives $\alpha X \rightarrow \alpha(\gamma^* A) = \gamma^* A$. Therefore $\alpha X \in \mathcal{E}^*$. Since α is an equivalence, every object in $\mathcal{E}$ is isomorphic to one of the form αX , and hence to one in $\mathcal{E}^*$. Therefore $\mathcal{E}^* \hookrightarrow \mathcal{E}$ is an equivalence.

Now Monro has shown that the answer to (b) is, in general, no. In fact he shows that $\mathcal{E}^* \hookrightarrow \mathcal{E}$ can fail to be an equivalence even when the base topos $\mathcal{F}$ is $\mathcal{S}$, provided the axiom of choice fails in a certain (relatively consistent) way in $\mathcal{S}$. He starts with the well-known Halpern–Levy model N of set theory in which the axiom of choice fails but in which every set is totally orderable. Then he constructs a certain complete Boolean algebra B in N such that, in the corresponding Boolean extension $N^{(B)}$, with probability 1 the power set $P(R)$ of the set of real numbers is not totally orderable. Thinking of N as our base topos $\mathcal{S}$ of sets and $N^{(B)}$ as the (Boolean) topos $\mathcal{E}$ defined over $\mathcal{S}$, it follows that the object $P(R)$ of $\mathcal{E}$ cannot be (isomorphic to an object) in $\mathcal{E}^*$, for it is not hard to see that any object in $\mathcal{E}^*$ must be totally orderable. So in this case the inclusion $\mathcal{E}^* \hookrightarrow \mathcal{E}$ cannot be an equivalence. We also see that (a) fails as well: B does not have exponentials.

Problem (c) is, however, still open.

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