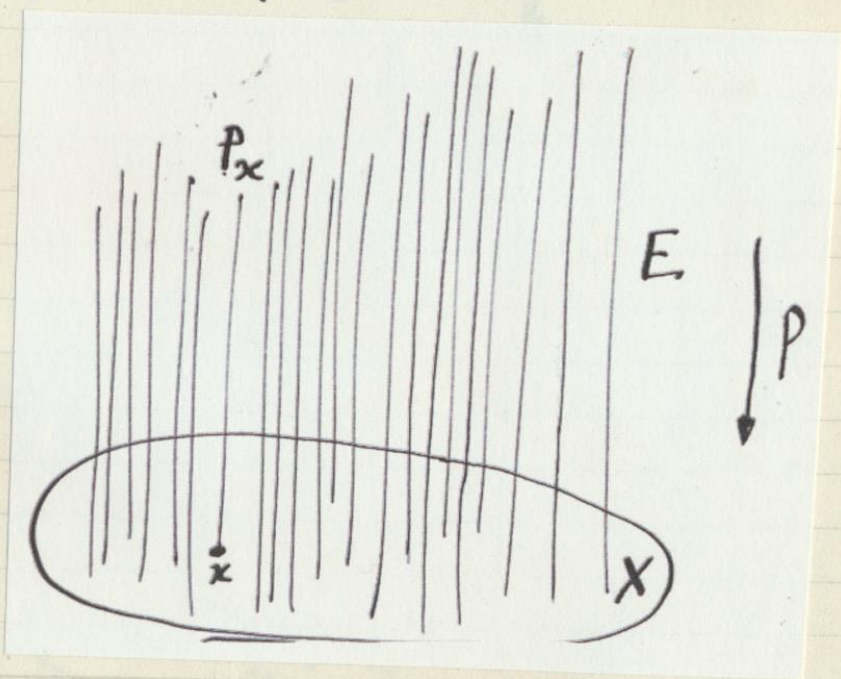


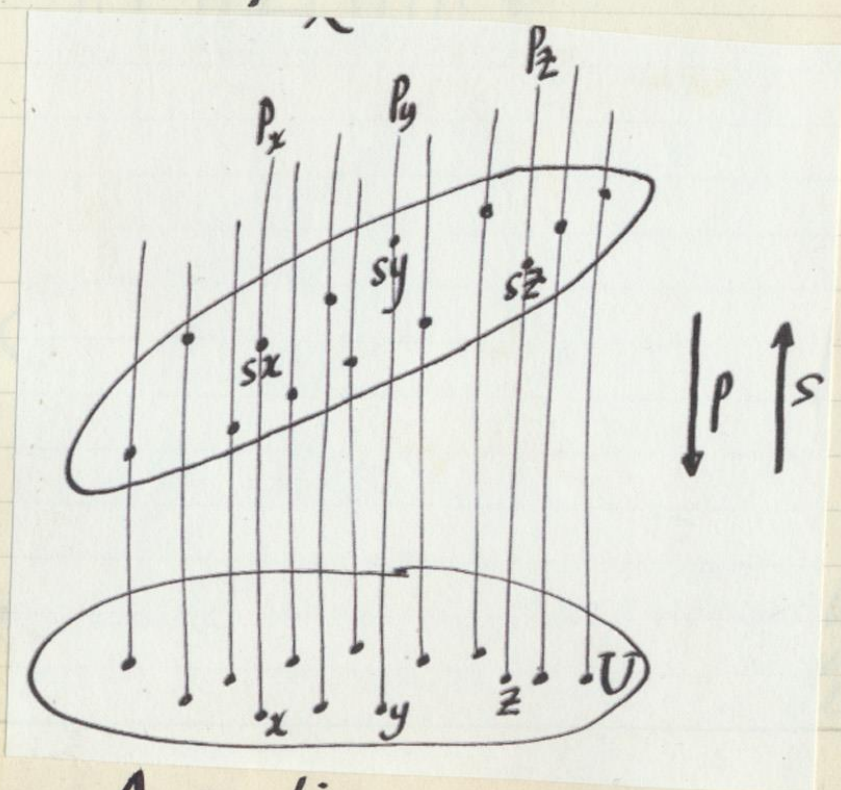
Bundles, Stacks & Sheaves

A bundle over a topological space X is just a continuous map $E \rightarrow X$ from some topological space E . (The domain of) A bundle over X may be viewed as the disjoint union of the fibres $P_x = p^{-1}(x)$ for $x \in X$:



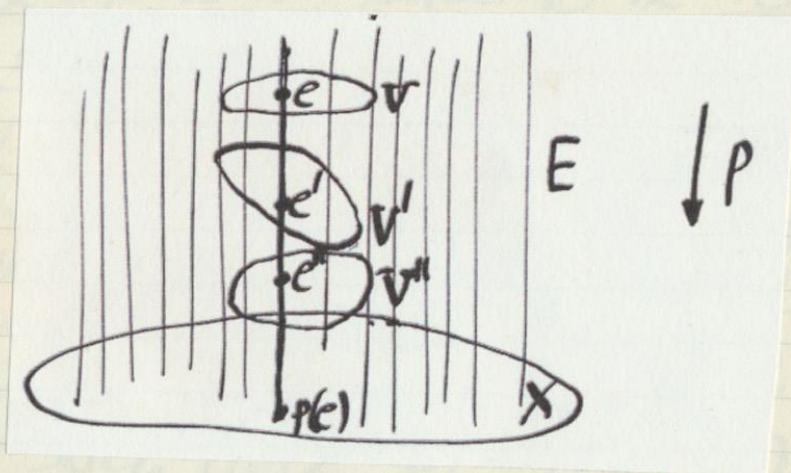
A Bundle

Let U be an open set in X .
A (cross)-section of the bundle
 $E \xrightarrow{p} X$ over U is a map $s: U \rightarrow E$
for which $p \circ s$ is the inclusion map
 $U \hookrightarrow X$. If $x \in X$, a section of
 p around x is a section of p
over some open neighbourhood of
 x .



A section over U

A stack over X is a bundle $E \xrightarrow{p} X$ for which p is a local homeomorphism, i.e. to each $e \in E$ there is an open neighbourhood V of e such that $p[V]$ is open in X and $p|_V$ is a homeomorphism $V \rightarrow p[V]$.



A Stack

If we call such a neighbourhood V of $e \in E$ a stratum at e , then

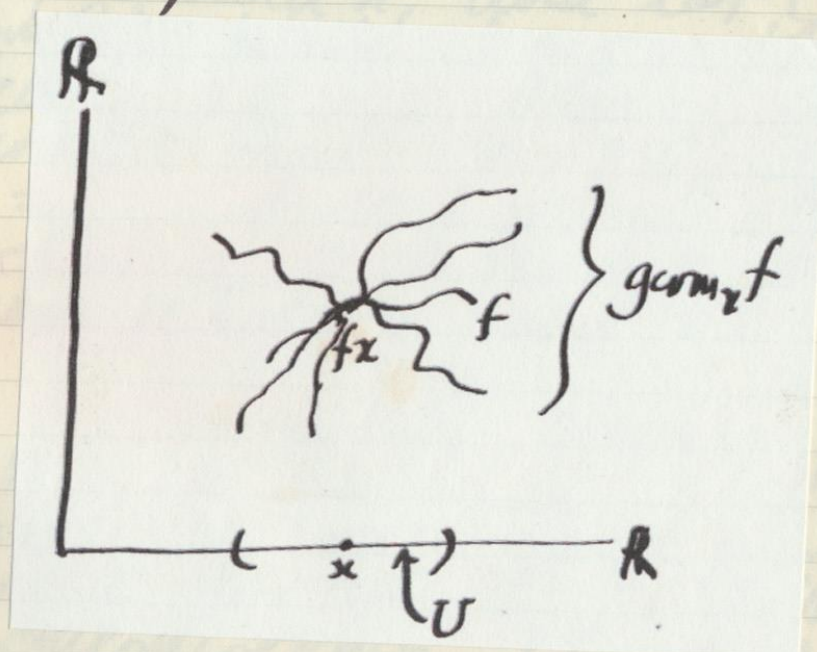
the vicinity of each fibre in a stack can be pictured as a bunch of strata "stacked up".

Now consider topological spaces X and Y and a point $x \in X$. A continuous map $h: Y$ with domain an open neighbourhood of x is said to be locally defined at x .

Let f and g be locally defined around x . We say that f and g are locally identical around x , written $f \sim_x g$, if there is some open neighbourhood W of x such that $W \subseteq \text{dom}(f) \cap \text{dom}(g)$ and $f|_W = g|_W$. It is easy to see that $\sim$ is an equivalence relation; the equivalence class under this relation of a given locally

defined f is called the germ
of f at x and is written
 $\text{germ}_x f$.

When $X = Y = \mathbb{R}$ the germ
of a map looks like this:



The germ of a map

Clearly, $\text{germ}_x f = \text{germ}_x g$
 $\Rightarrow f_x = g_x$, but not conversely.

Now let $E \xrightarrow{p} X$ be a stack over X and $x \in X$. Then points on the fibre P_x correspond bijectively to germs of sections of p around x . For suppose s is a section of p around x , and let $U = \text{dom}(s)$. Write $s_x = s_x$; then clearly $s_x \in P_x$. Also if $t \in \text{germs}_x s$ then $s_x = t_x$, whence $s = t$. So each germ of a section of p around x determines a unique point of P_x .

Reciprocally, given a point $c \in P_x$, i.e. for which $p(c) = x$, there is an open neighborhood U of c for which $p|_U$ is a homeomorphism $U \rightarrow p[U]$, and the latter is an open neighborhood of x . The map $s = p^{-1}|_{p[U]}$ is then a section

of p around x giving rise to its corresponding germ. Note that this germ is independent of the choice of U .

These correspondences are readily shown to be mutually inverse.

A presheaf F on a topological space X is an assignment, for each open set U in X , of a set $F(U)$ and for each pair U, V of open sets for which $U \subseteq V$, of a map $F_{UV}: F(V) \rightarrow F(U)$ satisfying

- $F_{UU} = 1_U$

- $F_{UW} = F_{UV} \circ F_{VW}$

for $U \subseteq V \subseteq W$

If $v \in FV$ and $U \subseteq V$ we write $v|_U$ for $F_U(v)$ and call it the restriction of v to U .

A presheaf F on X is a sheaf if it satisfies the following condition:

- Given any open covering $\{U_i : i \in I\}$ of an open set U in X and any family $\{u_i : i \in I\}$ with $u_i \in F U_i$ such that for any pair (i, j) with $u_i|_{U_i \cap U_j} = u_j|_{U_i \cap U_j}$, there exists a unique $u \in F U$ for which $u_i = u|_{U_i}$ for all $i \in I$.

Each bundle $E \xrightarrow{p} X$ over X determines a sheaf $\tilde{p}$ on X called the sheaf of sections of p . Here, if U is open in X ,

$\tilde{p}U$ is the set of sections of p over U and for $U \subseteq V$,

$\tilde{p}_{UV}: \tilde{p}V \rightarrow \tilde{p}U: s \mapsto s|_U$.
It is easy to check that $\tilde{p}$ is a sheaf.

Conversely, each sheaf on X can be regarded as the sheaf of sections of a suitable

bundle — in fact a stack —
over X .

The bijective correspondence
— noted above — between
points of (the domain of) a
stack and germs of sections
provides the "germ" of the
construction of the stack whose
sheaf of sections coincides with
a given sheaf. The idea is
to extend the concept of germ
to an arbitrary presheaf.

Suppose then that F is
a presheaf on X , $x \in X$, and
 U, V are open neighborhoods

of x . If $s \in F\bar{U}$, $t \in F\bar{V}$,
we say that s and t are
concordant around x , and
write $s \sim_x t$, provided there
exists an open neighborhood
 W of x such that $W \subseteq U \cap V$
and $s|_W = t|_W$.

It is not hard to show that
concordance around x is an
equivalence relation; for $s \in F\bar{U}$,
the equivalence class containing
 s is called the germ of s at
 x and denoted by $\text{germ}_x s$.

Now define

$$P_x = \{ \text{germ}_x s : s \in F U, U \text{ open nbd of } x \}^*$$

to be the set of all germs at x . We amalgamate the various sets P_x of germs in the disjoint union (over $x \in X$), to obtaining the set

$$\Sigma_p = \coprod_{x \in X} P_x,$$

and define $p = p_F : \Sigma_p \rightarrow X$ as the map sending each $\text{germ}_x s$ to the point $x \in X$ where it is taken.

* P_x is called the stalk of F at x .

To turn p into a stack over X we have to topologize Σ_p . This is done as follows: Each $s \in FU$ determines a map $s^*: U \rightarrow \Sigma_p$ given by $s^*x = \text{germ}_x s$ for $x \in U$.

Clearly s^* is a section of p . (Notice that in this way each element s of the given presheaf is replaced by an actual function s^* to the set Σ_p of germs.) Now we topologize Σ_p by taking as a basis of open sets all the

image sets $s^*[U] \subseteq \Sigma_p$;
 thus an open set in Σ_p is
 a union of images of the
 regions S^* . This topology
 makes p and every s^* con-
 tinuous, and it is not hard
 to show that $s^*: U \rightarrow s[U]$
 is a homeomorphism.

So $p: \Sigma_p \rightarrow X$ is a bundle,
 it is in fact a stack. For each
 point germ s in Σ_p has the
 open neighborhood $s^*[U]$; and
 $p|_{s^*[U]}$ has $s^*: U \rightarrow s^*[U]$
 as a two-sided inverse, making
 it a homeomorphism.

It can then be shown

that, if F is a sheaf, then F is isomorphic in the appropriate sense to the sheaf of sections of the stack Σ_F . In other words, each sheaf is the sheaf of sections of a stack.

Finally, let $Sh(X)$ be the category of sheaves on X with natural transformations as arrows, and let $St(X)$ be the category of stacks over X in which an arrow between stacks

$E \xrightarrow{p} X, E' \xrightarrow{p'} X$ is
 a continuous map $f: E \rightarrow E'$
 such that the diagram

$$\begin{array}{ccc} E & \xrightarrow{f} & E' \\ p \searrow & & \swarrow p' \\ & X & \end{array}$$

commutes. It can then be
 shown that the correspondence
 $F \mapsto p_F$ between sheaves
 and stacks extends to an
 equivalence between $\text{Sh}(X)$ and
 $\text{St}(X)$. This is the precise
 sense in which sheaves and

stacks are "equivalent".