Abstract and Variable Sets in Category Theory[1]

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In 1895 Cantor gave a definitive formulation of the concept of set (*menge*), to wit,

> *A collection to a whole of definite, well-differentiated objects of our intuition or thought.*

Let us call this notion a *concrete set.* More than a decade earlier Cantor had introduced the notion of cardinal number (*kardinalzahl*) by appeal to a process of abstraction:

> *Let* M *be a given set, thought of as a thing itself, and consisting of definite well-differentiated concrete things or abstract concepts which are called the elements of the set. If we abstract not only from the nature of the elements, but also from the order in which they are given, then there arises in us a definite general concept…which I call the power or the* cardinal number *belonging to M.*

As this quotation shows, one would be justified in calling *abstract sets* what Cantor called termed cardinal numbers[2]. An abstract set may be considered as what

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[1] This paper has its origins in a review [2] of Lawvere and Rosebrugh's book [5].

[2] This usage of the term "abstract set" is due to F. W. Lawvere: see [4] and [5]. Lawvere's usage contrasts strikingly with that of Fraenkel, for example, who on p. 12 of [3] remarks:

arises from a concrete set when each element has been purged of all intrinsic qualities *aside from the quality which distinguishes that element from the rest.* An abstract set is then an image of pure discreteness, an embodiment of raw plurality; in short, it is an assemblage of featureless *but nevertheless distinct* "dots" or "motes"[3]. The sole intrinsic attribute of an abstract set is the number of its elements.

Concrete sets are typically obtained as *extensions of attributes*. Thus to be a member of a concrete set C is precisely to possess a certain attribute A, in short, to *be* an A. (It is for this reason that Peano used $\in$, the first letter of Greek εστι, "is", to denote membership.) The identity of the set C is completely determined by the attribute A. As an embodiment of the relation between object and attribute, membership naturally plays a central role in concrete set theory; indeed the usual axiom systems for set theory such as Zermelo-Fraenkel and Gödel-Bernays take membership as their sole primitive relation. Concrete set theory may be seen as a theory of extensions of attributes.

By contrast, an abstract set cannot be regarded as the extension of an attribute, since the sole "attribute" possessed by the featureless dots—to which we shall still refer as *elements*—making up an abstract set is that of bare distinguishability from its fellows. Whatever abstract set theory is, it cannot be a theory of extensions of attributes. Indeed the object/attribute relation, and so *a fortiori* the membership relation between objects and sets cannot act as a primitive within the theory of abstract sets.

The key property of an abstract set being *discreteness*, we are led to derive the principles governing abstract sets from that fact. Now it is characteristic of discrete collections, and so also of abstract sets, that relations between them are reducible to relations between their constituting elements[4]. Construed in this way, relations between abstract sets provide a natural first basis on which to build a theory thereof[5]. And here categorical ideas can first be glimpsed, for relations can be *composed* in the evident way, so that abstract sets and relations between them form a *category*, the category **Rel.**

In fact **Rel** does not play a central role in the categorical approach to set theory, because relations have too much specific "structure" (they can, for example, be intersected and inverted). To obtain the definitive category associated with abstract sets, we replace arbitrary relations with *maps* between sets. Here a map from an abstract set X to an abstract set Y is a relation f between X and Y which correlates each element of X with a *unique* element of Y. In this situation we

Whenever one does not care about what the nature of the members of the set may be one speaks of an abstract set.

Fraenkel's "abstraction" is better described as "indifference".

[3] Perhaps also as "marks" or "strokes" in Hilbert's sense.

[4] This is to be contrasted with relations between *continua*. In the case of straight line segments, for example, the relation of being double the length is clearly not reducible to any relation between points or "elements". In the case of continua, and geometric objects generally, the relevant relations take the form of *mappings*.

[5] We conceive a relation R between two abstract sets X and Y is as correlating (some of) the elements of X with (some of) the elements of Y.

write $f: X \to Y$, and call X and Y the *domain*, and *codomain*, respectively, of f. Since the composite of two maps is clearly a map[6], abstract sets and mappings between them form a category **Set** known simply as the *category of abstract sets.*

While definitions in concrete set theory are presented in terms of membership and extensions of predicates, in the category of abstract sets definitions are necessarily formulated in terms of maps, and correlations of maps. This is the case in particular for the concept of membership itself. Thus in **Set** an element of a set X is defined to be a mapping $1 \to X$, where 1 is any set "consisting of a single dot", that is, satisfying the condition, for any set Y, that there is a unique mapping $Y \to 1$. In categorical terms, 1 is a *terminal element* of **Set.** In **Set** 1 has the important property of being a *separator* for maps in the sense that, for any maps f, g with common domain and codomain, if the composites of f and g with any element of their common codomain agree, then f and g are identical.

The "empty" set $\varnothing$ may be characterized as an *initial* object of **Set**, i.e., such that, for any set Y, there is a unique map $\varnothing \to Y$.

In **Set** the concept of set inclusion is replaced by that of monic (or one-to one) map, where a map $m: X \to Y$ is monic if, for any $f, g: A \to X$, $m f = m g \Rightarrow f = g$. A monic map to a set Y is also known as a *subobject* of Y.

Any two-element set 2 (characterized categorically as the sum of a pair of 1s; to be specific, we may choose 2 to be the set $\{\varnothing, 1\}$) plays the role of a *subobject classifier* or *truth-value object* in **Set.** This means that, for any set X, maps $X \to 2$ correspond naturally to subobjects of X. Maps $X \to 2$ correspond to *attributes* on X, with the members of 2 playing the role of *truth values*: $\varnothing$ "false" and 1 "true".

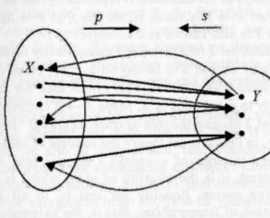
Along with $\in$, in concrete set theory the concept of identity or equality of *sets* —essentially defined in terms of $\in$—plays a seminal role. In abstract set theory, i.e. in **Set,** by contrast, it is the equality of maps which is crucial; it is, in fact, taken as a primitive notion. Equality for sets is, to all intents and purposes, replaced by the notion of *isomorphism,* that is, the existence of an invertible map between assemblages of dots. An abstract set is then defined "up to isomorphism"—the precise identity of the "dots" composing the set in question being irrelevant, the sole identifying feature is the "form" of the set.

In abstract or categorical set theory sets are identified not as extensions of predicates but through the use of the omnipresent categorical concept of *adjunction*. Consider, for instance, the definition of exponentials. In concrete set theory the exponential B^A of two sets A, B is defined to be the set whose elements are all functions from A to B. In categorical set theory B^A is introduced in terms of an adjunction, that is, the postulation of an appropriately defined natural bijective correspondence, for each set X, between maps $X \to B^A$ and mappings $X \times A \to B$. (Here $X \times A$ is the Cartesian product of X and A, itself defined by means of a suitable adjunction expressing the fact that maps from an arbitrary set Y to $A \times X$ are in natural bijective correspondence with pairs of maps from Y to X and A. We note in passing that relations between X and A can be identified with subobjects of

[6] If $f: X \to Y$ and $g: Y \to Z$, we write $g f: X \to Z$ for the composite of g and f.

$X \times A$.) Thus defined, B^A is then determined uniquely up to isomorphism, that is, as an assemblage of dots. We note that the exponential 2^X then corresponds to the *power set* of X.

The *axiom of choice* is a key principle in the theory of abstract sets. Stated in terms of maps, it takes the following form. Call a map $p: X \to Y$ *epic* if $f, g: Y \to A$, $fp = gp \Rightarrow f = g$: p is then epic if it is "onto" Y in the sense that each element of Y is the image under p of an element of X. A map $s: Y \to X$ is a *section* of p if the composite ps is the identity map on Y. Now the axiom of choice for abstract sets is the assertion that any epic map in **Set** has a section (and, indeed usually many). This principle is taken to be correct for abstract sets because of the totally arbitrary nature of the maps between them. Thus in the figure below the choice of a section s of the epic map p can be made on purely combinatorial grounds since no constraint whatsoever is placed on s (aside, of course, from the fact that it must be a section of p).



An abstract set X is said to be *infinite* if there exists an isomorphism between X and the set $X + 1$ obtained by adding one additional "dot" to X. It was the discovery of Dedekind in the 19th century that the existence of an infinite set in this sense is equivalent to that of the system of natural numbers. The *axiom of infinity*, which is also assumed to hold in abstract set theory, is the assertion that an infinite set exists.

The category **Set** is thus supposed to satisfy the following axioms:

1. There is a 'terminal' object 1 such that, for any object X, there is a unique arrow $X \to 1$
2. Any pair of objects A, B has a Cartesian product $A \times B$.
3. For any pair of objects A, B one can form the 'exponential' object B^A of all maps $A \to B$.

4. There is an object of truth values Ω such that for each object X there is a natural correspondence between subobjects (subsets) of X and arrows $X \to \Omega$. (In Set, as we have observed, one may take Ω to be the set $2 = \{\varnothing, 1\}$.)
5. 1 is not isomorphic to $\varnothing$.
6. The axiom of infinity.
7. The axiom of choice.
8. "Well-pointedness" axiom: 1 is a separator.

A category satisfying axioms **1. – 6.** (suitably formulated in the first-order language of categories) is called an elementary nondegenerate topos with an infinite object, or simply a *topos*[7]. The category of abstract sets is thus a topos satisfying the special additional conditions **7.** and **8.**

The objects of the category of abstract sets have been conceived as pluralities which, in addition to being discrete, are also *static* or *constant* in the sense that their elements undergo no change. There are a number of natural category-theoretic approaches to bringing *variation* into the picture. For example, we can introduce a simple form of *discrete* variation by considering as objects *bivariant sets*, that is, maps $F: X_0 \longrightarrow X_1$ between abstract sets. Here we think of X_0 as the "state" of the bivariant set F at stage 0, or "then", and X_1 as its "state" at stage 1, or "now". The bivariant set may be thought of having undergone, via the "transition" F, a change from what it was then (X_0) to what it is now (X_1). Any element x of X_0, that is, of F "then" becomes the element Fx of X_0 "now". Pursuing this metaphor, two elements "then" may become one "now" (if F is not monic), or a new element may arise "now", but because F is a map, no element "then" can split into two or more "now" or vanish altogether[8].

The appropriate maps between bivariant sets are pairs of maps between their respective states which are compatible with transitions. Thus a map from $F: X_0 \longrightarrow X_1$ to $G: Y_0 \longrightarrow Y_1$ is a pair of maps $h_0: X_0 \longrightarrow Y_0$, $h_2: X_1 \longrightarrow Y_1$ for which G E $h_1 = h_2$ E F. Bivariant sets and maps between them defined in this way form the category **Biv** of bivariant sets.

Now, like **Set, Biv** is a topos but the introduction of variation causes several new features to emerge. To begin with, the subobject classifier Ω in **Biv** is no longer a two -element constant set but the bivariant set $i: \Omega_0 \to \Omega_1 = 2$, where Ω_0 is the *three-element* set $\{\varnothing, \Phi, 1\}$, that is, 2 together with a new element Φ, and i sends $\varnothing$ to $\varnothing$ and both Φ and 1 to 1.

[7] See, e.g., [6] or [7].
[8] Note that had we employed relations rather than maps the latter two possibilities would have to be allowed for, complicating the situation considerably.

And while axioms **5** and **6** continue to hold in **Biv**, axioms **7** and **8** fail[9]. In short, the axiom of choice and well-pointedness are incompatible with even the most rudimentary form of discrete variation.

Abstract sets can also be subjected to *continuous* variation. This can be done in the first instance by considering, in place of abstract sets, *bundles over topological spaces*. Here a *bundle* over a topological space X is a continuous map p from some topological space Y to X. If we think of the space Y as the union of all the "fibres" $A_x = p^{-1}(x)$ for $x \in X$, and A_x as the "value" at x of the abstract set A, then the bundle p itself may be conceived as the *abstract set A varying continuously over X*. A *map* $f: p \to p'$ between two bundles $p: Y \to X$ and $p': Y' \to X$ over X is a continuous map $f: Y \to Y'$ respecting the variation over X, that is, satisfying $p' E f = p$. Bundles over X and maps between them form a category **Bun**(X), the *category of bundles* over X.

While categories of bundles do represent the idea of continuous variation in a weak sense, they fail to satisfy the topos axioms **3.** and **4.** and so fall short of being suitable generalizations of the category of abstract sets to allow for such variation. To obtain these, we confine attention to special sorts of bundles known as *sheaves*. A bundle $p : Y \to X$ over X is called a *sheaf* over X when p is a *local homeomorphism* in the following sense: to each $a \in Y$ there is an open neighbourhood U of a such that pU is open in X and the restriction of p to U is a homeomorphism $U \to pU$. The domain space of a sheaf over X "locally resembles" X in the same sense as a differentiable manifold locally resembles Euclidean space. It can then be shown that the category **Shv**(X) of sheaves over X and maps between them (as bundles) is a topos the elements of whose truth-value object correspond bijectively with the open subsets of X[10]. *Categories of sheaves are appropriate generalizations of the category of abstract sets to allow for continuous variation*, and the term *continuously varying set* is taken to be synonymous with the term *sheaf*. In general, both the axiom of choice and the axiom of well-pointedness fail in sheaf categories[11], showing that both principles are incompatible with continuous variation.

If we take X to be a space consisting of a single point, a sheaf over X is a discrete space, so that the category of sheaves over X is essentially the category of abstract sets. In other words, an abstract set varying continuously over a one-point

[9] The axiom of choice fails in **Biv** since it is easy to construct an epic from the identity map on $\{0, 1\}$ to the map $\{0, 1\} \to \{0\}$ with no section. That 1 is not a separator follows from the fact that $\varnothing \to 1$ has many different maps from it but no maps from $1 \to 1$ to it.

[10] See, e.g., [6].

[11] This is most easily seen by taking X to be the unit circle S^1 in the Euclidean plane, and considering the "double-covering" D of S^1 in 3-space. The obvious projection map $D \to S^1$ is a sheaf over S^1 with no elements in **Shv**(S^1), so the latter cannot be well-pointed. The same fact shows that the natural epic map in **Shv**(S^1) from $D \to S^1$ to the identity map $S^1 \to S^1$ (the terminal object of **Shv**(S^1) has no section, so that the axiom of choice fails in **Shv**(S^1).

space is just a (constant) abstract set. In this way arresting continuous variation leads back to constant discreteness[12].

There is an alternative description of sheaf categories which brings forth their character as categories of variable sets even more strikingly. For **Shv**(X) can also be described as the category of sets *varying* (in a suitable sense) *over open sets in X*[13]. This type of variation can be further generalized to produce categories of sets *varying over a given* (small) *category*. Each such category is again a topos[14]. Further refinements of this procedure yield so-called *smooth toposes*, categories of variable sets in which the form of variation is honed from mere continuity into smoothness[15]. Regarded as universes of discourse, in a smooth topos all maps between spaces are smooth, that is, arbitrarily many times differentiable. Even more remarkably, the objects in a smooth topos can be seen as being generated by the motions of an infinitesimal object—a "generic tangent vector"—as envisioned by the founders of differential geometry.

So, starting with the category of abstract sets, and subjecting its objects to increasingly strong forms of variation leads from discreteness to continuity to smoothness. The resulting unification of the continuous and the discrete is one of the most startling and far-reaching achievements of the categorical approach to mathematics.

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[12] We note that had we chosen categories of bundles to represent continuous variation, then the corresponding arresting of variation would lead, not to the category of abstract sets—constant discreteness—but to the category of topological spaces—constant continuity. This is another reason for not choosing bundle categories as the correct generalization of the category of abstract sets to incorporate continuous variation.

[13] See, e.g. [6].

[14] See, e.g. [6].

[15] See [1], [7] and [8].

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