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QuasiBoolean Algebras and Simultaneously Definite Properties in Quantum Mechanics

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We define and characterize a new abstract notion of "quasiBoolean algebra," intermediate in nature between an (ortho)lattice and a Boolean algebra. It will turn out that such algebras are natural candidates for representing the simultaneously definite properties of a quantum system. We then prove a general theorem about maximal quasiBoolean subalgebras of an ortholattice which we use to derive a number of different proposals in the literature for what properties of a quantum system should be regarded as simultaneously definite.

1. INTRODUCTION

There are a number of different proposals in the literature for what observables of a quantum system should be taken to have simultaneously definite values. "Orthodox" proposals include those which limit the properties one may ascribe to a quantum system to those which can either be predicted with certainty given the state of the system (von Neumann, Dirac), or those properties whose values get strictly correlated to the pointer values of a classically described measurement device (Bohr). "Nonorthodox" proposals include hidden variable interpretations of the theory that regard certain observables of a quantum system, like position, as always having definite values (Bohm), and "modal" interpretations that, though they do not posit (nontrivial) observables with always definite values, still seek to attribute properties so that macroscopic systems in entangled states, like Schrödinger's cat, nevertheless retain sufficient properties to ensure that classical everyday perceptions are not contradicted (van Fraassen, Kochen, Dieks).

It is not our purpose to advocate any one of these proposals here, but rather to point out that they all share a common structural feature: namely,

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that the different property sets these proposals embody all form what may naturally be called "quasiBoolean algebras."

Our aim is first to introduce and characterize the new lattice-theoretic concept of *quasidistributivity*, the weakened form of distributivity that will play a central role in our analysis. To this end, in Section 2 we briefly review the elements of lattice theory necessary for our formulation and analysis of the concept of quasidistributive lattice. Then in Section 3 we will prove a theorem characterizing such lattices in terms of four different, but equivalent, conditions. There is also a natural notion of "strongly quasidistributive lattice" that emerges, and again we prove a theorem characterizing such lattices, this time in terms of five equivalent conditions.

In Section 4 we confine attention to ortholattices, and introduce the term (strongly) quasiBoolean *algebra* for a (strongly) quasidistributive ortholattice. It will turn out that all the proposals mentioned above realize *strongly* quasi-Boolean algebras in this sense, though we will show that the requirement that the set of definite properties of a quantum system form merely a quasiBoolean algebra has a more direct physical motivation in the context of Hilbert space quantum mechanics. In order to employ the idea of a quasiBoolean algebra to recover different proposals for the set of definite properties of a quantum system, we shall prove in Section 4 a general theorem about a class of quasiBoolean algebras obtainable as maximal subalgebras of an ortholattice. We then show in Section 5 how such algebras emerge as the natural mathematical objects for representing the set of definite properties of a quantum system.

Finally, in Section 6, we use our general theorem of Section 4 to *derive* all the different proposals mentioned above; for it turns out that they all represent the definite properties of a quantum system in terms of a maximal quasiBoolean subalgebra of the ortholattice of subspaces of the Hilbert space representing the system. This result should go some way toward separating the purely mathematical from the *physical* content of these proposals, helping to clarify their essential differences. In fact, we shall see that the proposals differ physically only in regard to what the atoms of the maximal quasiBoolean subalgebra they select should be taken to be. Once these are fixed, the theorem of Section 4 (which does not depend on Hilbert space) determines the subalgebra selected by each proposal uniquely.

2. FILTERS, IDEALS, AND HOMOMORPHISMS

We begin with our brief review of the elements of lattice theory needed to characterize quasidistributive lattices.

Let L be a lattice with bottom element 0 and top element 1; we shall always assume $0 \neq 1$. A nonempty subset F of L is a *filter* if (1) $x \in F$, $x \leq y \Rightarrow y \in F$, (2) $x, y \in F \Rightarrow x \wedge y \in F$, and (3) $0 \notin F$. A nonempty subset

I of L is an *ideal* if (1) $x \in I$, $y \leq x \Rightarrow y \in I$, (2) $x, y \in I \Rightarrow x \vee y \in I$, and (3) $1 \notin I$. Clearly, for any $x\ (\neq 0) \in L$, $x\uparrow \equiv \{y \in L: x \leq y\}$ is a filter, and for any $x\ (\neq 1) \in L$, $x\downarrow \equiv \{y \in L: y \leq x\}$ is an ideal: the former is the *principal filter* generated by x, the latter the *principal ideal* generated by x.

A filter F (ideal I) is *prime* if $x \vee y \in F \Rightarrow x \in F$ or $y \in F$ ($x \wedge y \in I \Rightarrow x \in I$ or $y \in I$) for all $x, y \in L$. It is easy to check that $F \subseteq L$ is a prime filter iff $L - F$ is a prime ideal (equivalently, that $I \subseteq L$ is a prime ideal iff $L - I$ is a prime filter); this establishes a bijective correspondence between prime filters and prime ideals in a lattice. An *ultrafilter* is a filter maximal under $\subseteq$. [In a distributive lattice, every ultrafilter is prime, and in a Boolean algebra, conversely (Bell and Machover, 1977; Davey and Priestley, 1990).]

There is a bijective correspondence between prime filters in a lattice L and two-valued lattice homomorphisms on L, defined in the following way. To each prime filter P in L, associate the homomorphism $h: L \rightarrow 2$ given by $h(x) = 1 \Leftrightarrow x \in P$; and to each homomorphism $h: L \rightarrow 2$, associate the prime filter $h^{-1}(1)$ in L. Note that this correspondence extends to ortholattices and ortholattice homomorphisms, since any lattice homomorphism from an ortholattice to a Boolean algebra (2 in particular) preserves orthocomplementation.

3. QUASIDISTRIBUTIVE LATTICES

For the characterization of a quasidistributive lattice, it is useful to define the *radical* Rad(L) of L to be the intersection of the family of all prime ideals in L. In a natural sense, Rad(L) provides an (inverse) measure of the number of two-valued homomorphisms on L: the more there are of these, the smaller Rad(L), and conversely.

We now prove our first characterization theorem:

Theorem 1. Let I be an ideal in a lattice L. Then the following are equivalent:

(1) $\mathrm{Rad}(L) \subseteq I$.
(2) Any $x \notin I$ is contained in a prime filter.
(3) For any $x \notin I$ there is a homomorphism $h: L \rightarrow 2$ such that $h(x) = 1$.
(4) There is a Boolean algebra B and a lattice homomorphism $f: L \rightarrow B$ such that $f^{-1}(0) \subseteq I$.

Proof. (1) $\Leftrightarrow$ (2). Assuming (1), for any $x \notin I$, we have $x \notin \mathrm{Rad}(L)$, i.e., there is a prime ideal Q such that $x \notin Q$. Then $P = L - Q$ is a prime filter and $x \in P$. Conversely, assuming (2), if $x \notin I$, then there is a prime filter P such that $x \in P$, and $Q = L - P$ is a prime ideal for which $x \notin Q$. It follows that $x \notin \mathrm{Rad}(L)$, so that $\mathrm{Rad}(L) \subseteq I$.

(2) $\Leftrightarrow$ (3) follows immediately from the bijective correspondence between two-valued homomorphisms and prime filters.

(2) $\Rightarrow$ (4). Let X be the set of prime filters in L and let B be the power set algebra of X. Then the map f: $L \to B$ given by $f(x) = \{P \in X: x \in P\}$ is a lattice homomorphism which, assuming (2), satisfies $f^{-1}(0) \subseteq I$.

(4) $\Rightarrow$ (3). Assuming (4), for any $x \notin I$ we have $f(x) \neq 0$. Since B is a Boolean algebra, there is a homomorphism g: $B \to 2$ such that $g(f(x)) = 1$; thus $h = g \circ f$: $L \to 2$ satisfies (3). ∎

Remark. Similar arguments to those above establish the (well-known) equivalence of the conditions: (1′) L is distributive; (2′) for any $x \not\leq y$ there is a prime filter containing x but not y; (3′) for any $x \neq y$ there is a homomorphism h: $L \to 2$ such that $h(x) \neq h(y)$; (4′) there is a Boolean algebra B and an injective lattice homomorphism f: $L \to B$.

By analogy with the distributive case, we shall call a lattice L with ideal I satisfying the equivalent conditions (1)–(4) of Theorem 1 an *I-quasidistributive lattice.*

It is immediate from (1) of Theorem 1 that if L is I-quasidistributive and $J \supseteq I$ is another ideal of L, then L is J-quasidistributive. If L is $\{0\}$-quasidistributive, we shall call it simply a *quasidistributive lattice*. Clearly any sublattice of a quasidistributive lattice will itself be quasidistributive, as will any power of a quasidistributive lattice. (Analogous results hold for I-quasidistributive lattices when I is nontrivial.)

Example. A simple quasidistributive lattice (that is not distributive) is given by the five-element lattice $\{0, a, b, c, 1\}$ with the partial ordering $0 < a < 1$, $0 < b < c < 1$. In that case, a B satisfying condition (4) of Theorem 1 is the four-element Boolean algebra obtained by identifying b and c.

In connection with condition (4) of Theorem 1, it is natural to attempt to characterize those ideals I which are kernels of homomorphisms to Boolean algebras. To do this, we introduce the property of I-maximality.

A filter F in L is said to be *I-maximal* if it is maximal with respect to the property of disjointness from I: thus $\{0\}$-maximal filters are the same as ultrafilters. Now we have our second characterization theorem:

Theorem 2. Let I be an ideal in a lattice L. Then the following are equivalent:

(1) I is the intersection of a (nonempty) family of prime ideals.

(2) Any $x \notin I$ is contained in a prime filter P such that $P \cap I = \varnothing$.

(3) For any $x \notin I$ there is a homomorphism h: $L \to 2$ such that $h(x) = 1$ and $I \subseteq h^{-1}(0)$.

(4) There is a Boolean algebra B and a lattice homomorphism $f: L \to B$ such that $I = f^{-1}(0)$.

(5) Every I-maximal filter is prime.

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) are proved in a similar way to the corresponding clauses of Theorem 1.

(2) $\Rightarrow$ (4). Let X be the set of prime filters in L disjoint from I and let B be the power set algebra of X. Then the map $f: L \to B$ given by $f(x) = \{P \in X: x \in P\}$ is a lattice homomorphism which, assuming (2), satisfies $I = f^{-1}(0)$.

(4) $\Rightarrow$ (5). Assume (4) and let F be an I-maximal filter in L. Since $F \cap I = \varnothing$ and $I = f^{-1}(0)$, it follows that $0 \notin f[F]$. The latter (using Zorn's lemma) is then contained in an ultrafilter P which (since B is a Boolean algebra) is also prime. It is now easily seen that $f^{-1}[P]$ is a prime filter containing F and disjoint from I (since $0 \notin P$). Since F was assumed to be I-maximal, F and $f^{-1}[P]$ coincide, so F is itself prime.

(5) $\Rightarrow$ (2). Assuming (5), let $x \notin I$. By Zorn's lemma, there is an I-maximal filter containing x which, by (5), is prime. ■

Remark. It follows from (1) $\Rightarrow$ (5) of this theorem that, if $\mathrm{Rad}(L) \neq L$, then every $\mathrm{Rad}(L)$-maximal filter is prime.

Call a lattice L with ideal I *strongly I-quasidistributive* if it satisfies the equivalent conditions of Theorem 2. Observe that L is strongly $\{0\}$-quasidistributive iff L is a quasidistributive lattice in the sense defined above. It also follows easily from the Remark following Theorem 1 that L is distributive iff L is strongly I-quasidistributive for every *principal* ideal I.

4. QUASIBOOLEAN ALGEBRAS

We now focus attention on ortholattices, which of course play a special role in quantum mechanics. It is appropriate to call a (strongly) I-quasidistributive ortholattice a (*strongly*) *I-quasiBoolean algebra.*

Examples. Ortholattices that define quasiBoolean (but not Boolean) algebras can be obtained by taking powers and subortholattices of the six-element, nondistributive ortholattice $\{0, a_1, a_2, b_1, b_2, 1\}$ with partial ordering $0 < a_1 < a_2 < 1$, $0 < b_1 < b_2 < 1$ and orthocomplementation $\perp$ defined by $0^\perp = 1$, $a_1{}^\perp = b_2$, $a_2{}^\perp = b_1$. Making the identifications $a_1 = a_2$ and $b_1 = b_2$ yields a B satisfying condition (4) of Theorem 1 for this ortholattice.

For any set S of elements in an (ortho)lattice, let $\vee S$ denote their join, provided it exists. A subset A of an ortholattice is called *disjointed* if $a \leq b^\perp$ for every pair a, b of distinct elements of A.

Our main result is the following general theorem about a class of quasiBoolean algebras obtainable as *maximal* quasiBoolean subalgebras of an ortholattice:

Theorem 3. Let A be a disjointed subset and I an ideal of an ortholattice L such that $A \cap I = \emptyset$ and $(\vee A)^{\perp} \in I$. Then the ortholattice defined by

$$L(A) = \{x \in L\colon \forall a \in A,\, a \leq x \text{ or } a \leq x^{\perp}\}$$

can be characterized as the largest subortholattice $M \subseteq L$ satisfying the following pair of conditions:

(a) Every element of A is an atom of M.
(b) M is an $I \cap M$-quasiBoolean algebra.

Proof. It is easy to see that $L(A)$ satisfies (a) and that, for each $a \in A$, $a^{\perp}$ generates a prime ideal in $L(A)$. So, defining $J = (\vee A)^{\perp}\!\downarrow \cap L(A)$, J is the intersection of the family of prime-in-$L(A)$ ideals $\{a^{\perp}\!\downarrow \cap L(A)\colon a \in A\}$. It follows from Theorem 2 that $L(A)$ is strongly J-quasiBoolean. Now given that $(\vee A)^{\perp} \in I$, equivalently that $(\vee A)^{\perp}\!\downarrow \subseteq I$, we have $J \subseteq I \cap L(A)$. Thus $L(A)$ must be $I \cap L(A)$-quasiBoolean, satisfying (b).

To complete the proof, we need only show that if $M \subseteq L$ satisfies (a) and (b), then $M \subseteq L(A)$. To this end, suppose M satisfies (a) and (b), but that there is an $x \in M$ such that $x \notin L(A)$, i.e., $a \nleq x$ and $a \nleq x^{\perp}$ for some $a \in A$. Since a was assumed to be an atom in M, it follows that $x \wedge a = x^{\perp} \wedge a = 0$, whence $x^{\perp} \vee a^{\perp} = x \vee a^{\perp} = 1$. Since $a \notin I \cap M$ (invoking $A \cap I = \emptyset$), by (b) (and Theorem 1) there is a prime filter P in M such that $a \in P$. Then since $x^{\perp} \vee a^{\perp} \in P$ and $x \vee a^{\perp} \in P$, we have $[x^{\perp} \in P$ or $a^{\perp} \in P]$ and $[x \in P$ or $a^{\perp} \in P]$. Hence $[x^{\perp} \in P$ and $x \in P]$ or $a^{\perp} \in P$. Recalling that $a \in P$, either alternative leads immediately to the contradiction $0 \in P$. ∎

Remark. Observe that if $I = (\vee A)^{\perp}\!\downarrow$, then the condition $A \cap I = \emptyset$ is redundant and $L(A)$ is also the largest subortholattice $M \subseteq L$ satisfying (a) of the theorem and the requirement that M be a *strongly* $I \cap M$-quasiBoolean algebra.

5. MAXIMAL QUASIBOOLEAN SUBALGEBRAS OF DEFINITE PROPERTIES

We now show how quasiBoolean algebras emerge as the natural structures for representing the definite properties of quantum systems. Our final task in the next section will be to use Theorem 3 to derive the different proposals in the literature for identifying those definite properties.

Consider a quantum system represented by a Hilbert space H whose state is represented at some moment by some (positive, Hermitian, trace-one) "density" operator D on H. (For simplicity, we assume throughout that H is finite dimensional.) Each projection operator P on H has eigenvalues 1 and 0 and defines a possible property, or proposition, of the system in state D, where the value 1 can be read as "true" and 0 as "false." (Henceforth, we reserve the notation P for projection operators, not to be confused with prime filters.) There is a well-known bijective correspondence between the set of projections on H and the set of subspaces of H. The set of all subspaces (hence projections) of H forms a complete, atomic ortholattice L_H with the partial ordering given by subspace inclusion, the meet of two subspaces given by their intersection, the join by the subspace they together generate, and where the orthocomplement of a subspace is the subspace orthogonal to it.

The following question now arises. Of the *possible* properties of the system, represented by L_H, which can be regarded as *actually* having simultaneously well-defined values (0 or 1) in state D? That is, which of the propositions in L_H can together be regarded as determinately true or false of the system when it is in state D?

Let L_D denote the subortholattice consisting of elements of L_H that have simultaneously definite values in state D. Then the different truth valuations on the set of properties in L_D are given by the various homomorphisms h: $L_D \to 2$. If H has nontrivial dimension, it is well known (and easy to see) that there do not exist *any* two-valued homomorphisms on L_H; equivalently, that $\mathrm{Rad}(L_H) = L_H$. Thus L_H itself is not a possible candidate for L_D, and we must seek some proper subortholattice of L_H in order to represent the (simultaneously) definite properties of our quantum system.

Clearly what must be done is to choose $L_D \subset L_H$ so that its radical $\mathrm{Rad}(L_D)$ is small enough to guarantee the existence of sufficiently many two-valued homomorphisms (i.e., prime filters) to capture the various possible truth valuations of the propositions in L_D. By Theorem 1, we see that a natural *mathematical* way to restrict $\mathrm{Rad}(L_D)$ is to demand that L_D form an $I \cap L_D$-quasiBoolean algebra with respect to some ideal I of L_H. We now show that one *physically* natural choice of I is forthcoming.

Consider the subset I_D of all projections P in L_H that are prescribed zero probability of being found on measurement to have value 1 in state D, i.e., those projections for which $\mathrm{Tr}(PD) = 0$. (We shall use the same notation for projection operators on H and the lattice elements in L_H to which they correspond, since no confusion will arise.) If P_D is the set of all spectral projections of D corresponding to its nonzero eigenvalues, then it is easy to see that $I_D = (\vee P_D)^{\perp}\downarrow$, so that I_D is indeed an ideal of L_H.

Now for any P in L_D that gets attributed *nonzero* probability of being found on measurement to have value 1 in state D, i.e., for any $P \notin I_D \cap$

L_D, there should exist a homomorphism h: $L_D \to 2$ sending P to 1. For otherwise it will not be possible to recover the probabilities prescribed by D for measurement results as measures over the set of possible valuations (homomorphisms into 2) on the subortholattice L_D. Since this subortholattice is supposed to represent the definite properties of our quantum system, it must have sufficient valuations so as to be consistent with what values it is possible to observe for the properties contained in L_D. Invoking Theorem 1, we see that the physically natural requirement on $L_D \subset L_H$ is, therefore, that L_D be an $I_D \cap L_D$-quasiBoolean algebra.

Indeed, it is natural to impose on L_D that it be as large a subortholattice of L_H as possible consistent with being an $I_D \cap L_D$-quasiBoolean algebra. For the problem at issue should really be one of discerning how much we can consistently assert about the simultaneously definite properties of a quantum system in an arbitrarily given state D. So the requirement becomes that L_D *be a maximal* $I_D \cap L_D$*-quasiBoolean subalgebra of* L_H.

6. RECOVERING PROPOSALS FOR A QUANTUM SYSTEM'S PROPERTIES

In general, there is no unique L_D satisfying the condition just stated, as shown by the various existing proposals in the literature for the set of definite properties of a quantum system—which all (as we shall shortly see) realize maximal $I_D \cap L_D$-quasiBoolean subalgebras of L_H. But with the aid of Theorem 3, it is now possible to *derive* each of these proposals, starting with different requirements on what elements of L_H should occur *as atoms* in L_D.

In all the proposals we shall consider, these atoms are not chosen arbitrarily, but are picked out by the state D of the system (hence our notation L_D). This is natural, since one wants the definite properties of a quantum system to be linked in some way to the dynamical evolution of its quantum state D. However, there is no unique way to set up such a link, i.e., to use D to pick out a set of atoms for L_D. Differences in the choice of atoms picked out by D reflect differences among the proposals as to what properties should be ascribed to a quantum system after it is subjected to, for example, a measurement interaction. Some indication of how these differences show up will be given after we have derived the various proposals using Theorem 3.

I. Suppose, first, that we require that the element $\vee P_D$ be an atom of L_D. Then since $\{\vee P_D\} \cap I_D = \emptyset$ and $(\vee P_D)^\perp \in I_D$, Theorem 3 yields

$$L_D = L_H(\{\vee P_D\}) = \{P \in L_H\colon \vee P_D \leq P \text{ or } \vee P_D \leq P^\perp\}$$

This (maximal) $I_D \cap L_D$-quasiBoolean subalgebra is easily seen to be generated by $\vee P_D$ and all atoms of L_H contained in $(\vee P_D)^\perp$. Furthermore, since I_D

$= (\vee P_D)^{\perp}\downarrow$, we know from the Remark following Theorem 3 that the L_D above is in fact a *strongly* $I_D \cap L_D$-quasiBoolean algebra.

Since $I_D = (\vee P_D)^{\perp}\downarrow$, an equivalent definition of this L_D is

$$L_D = \{P \in L_H: \mathrm{Tr}(PD) = 1 \text{ or } 0\}$$

which therefore amounts to the proposal that all and only those projections with particular values (0 or 1) that are *certain* to be found on measurement in state D correspond to definite properties in that state. This is the "orthodox" proposal advocated by von Neumann (1955, pp. 213–7) and, more explicitly, by Dirac (1958, pp. 46–7). It is perhaps too conservative a proposal, since it is what forces von Neumann to resort to his projection postulate to account for the definite results of measurements (not to mention a definite life state for Schrödinger's cat). It is therefore not surprising that other proposals have been offered.

II. Suppose, instead, that we require *each individual* (nonzero eigenvalue) spectral projection of D in the set P_D to be an atom of L_D. Then, since we still have $(\vee P_D)^{\perp} \in I_D$ and further $P_D \cap I_D = \varnothing$, and since the set P_D is disjointed by definition, Theorem 3 yields

$$L_D = L_H(P_D) = \{P \in L_H: \forall \underline{P} \in P_D, \underline{P} \leq P \text{ or } \underline{P} \leq P^{\perp}\}$$

which is readily verified (Clifton, 1995) to be the subortholattice of L_H generated by the elements of the set P_D and all atoms in $(\vee P_D)^{\perp}$ (again, it is *strongly* $I_D \cap L_D$-quasiBoolean as well). So we obtain a different maximal $I_D \cap L_D$-quasiBoolean subalgebra from our first example, but the difference only shows up in the different sets of atoms picked out by D that determine the two subalgebras via Theorem 3.

$L_H(P_D)$ is essentially the proposal made by van Fraassen (1991), Kochen (1985), and Dieks (1989), and is supposed to allow one to attribute enough properties to quantum systems and their subsystems (which will, generally, have mixed states, i.e., states for which $D^2 \neq D$) that von Neumann's ad hoc projection postulate becomes unnecessary.

Note that if D represents a pure state, so that $D^2 = D$, then D becomes simply a one-dimensional projection operator, and the two proposals $L_H(\{\vee P_D\})$ and $L_H(P_D)$ coincide (since $\{\vee P_D\} = P_D = \{D\}$).

III. Finally, there is another proposal for L_D due to Bub and Clifton (1995), which differs from these first two even in the case when D is pure. This proposal arguably recovers, as special cases, what Bohr's (1958) orthodoxy and Bohm's (1952) hidden variables would allow us to say about the definite properties of a (generally, composite) quantum system in a pure state D.

Let $\{R_i\}$ be the set of spectral projections of some observable represented by the Hermitian operator R, and (for pure D) define $D_{R_i} = (D \vee R_i^{\perp}) \wedge R_i$ for all i. Let $\{D_{R_j}\}$ be the set of all nonzero D_{R_i}. Since the set $\{R_i\}$ is disjointed, so is the set $\{D_{R_j}\}$. And, in this (pure) case, $I_D = D^{\perp}\downarrow$. So if we demand that each element of the set $\{D_{R_j}\}$ occur as an atom in L_D, then since $\{D_{R_j}\} \cap I_D = \emptyset$ and $(\vee_j D_{R_j})^{\perp} \in I_D$, Theorem 3 leaves us with

$$L_D = L_H(\{D_{R_j}\}) = \{P \in L_H: \forall j,\ D_{R_j} \leq P \text{ or } D_{R_j} \leq P^{\perp}\}$$

which (like our previous example) is easily verified to be the subortholattice generated by the set $\{D_{R_j}\}$ and all atoms of L_H in $(\vee_j D_{R_j})^{\perp}$. [In this case, we do not generally have $I_D = (\vee_j D_{R_j})^{\perp}\downarrow$, but L_D will always be strongly $(\vee_j D_{R_j})^{\perp}\downarrow \cap L_D$-quasiBoolean.]

$L_H(\{D_{R_j}\})$ will only agree with the first two proposals described above if D is an eigenstate of R, for then the set $\{D_{R_i}\}$ consists of only D itself.

Whatever D is, it will always be the case that $L_H(\{D_{R_j}\})$ will include the set $\{R_i\}$. The idea of this final proposal is then to say that, however the (pure) state D evolves, there is always one "preferred" observable R which has definite values at all times. If we choose R to be position in configuration space, it becomes possible to recover Bohm's hidden variable theory. On the other hand, if we choose R throughout the duration of a measurement interaction to be the macroscopic pointer observable on the classical measurement device used, we recover Bohr's views about how the properties one can meaningfully ascribe to a measured system depend on its context of measurement. (See Bub and Clifton (1995) for further discussion.)

It is clear that the physical differences between these different proposals for L_D reside solely in the different choices they make for the atoms, or equivalently (in these cases) the generators of L_D. Once these are fixed, the different proposals may be derived quite independently of the details of Hilbert space just as a consequence of our general theorem of Section 4 concerning quasiBoolean algebras obtainable as maximal subalgebras of an ortholattice. In the states D where the property ascriptions of the proposals differ, these differences will show up, in particular, as differences about what properties it is appropriate to attribute to quantum systems after they have undergone a measurement interaction.

To illustrate this, consider a simple case of ideal measurement of one quantum system, an "object" O represented by Hilbert space H_O, by another, an "apparatus" A represented by H_A. The initial state of the composite system $O + A$ will unitarily evolve into a post-measurement state of the form $\Psi = \sum_i c_i o_i \otimes a_i \in H_O \otimes H_A\ (= H_{OA})$, where the $\{c_i\}$ are expansion coefficients, the $\{o_i\}$ eigenstates of the observable O measured on the object system, and the $\{a_i\}$ eigenstates of the "pointer" observable A of the apparatus.

Consider, first, properties that get attributed to the composite system $O + A$, given its (pure) state P_Ψ (the projection onto the subspace generated by Ψ). In this case (as already noted), there is no difference between the first two proposals in the properties they attribute to $O + A$, i.e., both will regard those properties as represented by projections P such that $\mathrm{Tr}(PP_\Psi) = 1$ or 0. Generally none of these properties will correspond to projections that are the tensor product of a projection on H_O and one on H_A, so it will not be possible to say that the "object" and "apparatus" parts of the composite system have definite (nontrivial) properties—only that the composite system itself does.

But suppose we take the "preferred" observable R in the *third* proposal above to be $I \otimes A$ (where I is the identity operator on H_O)—either on the grounds that the apparatus pointer observable is an observable of a *classical* system which ought to be regarded *a priori* as having a definite value (Bohr), or because measurements with this pointer observable are ultimately reducible to measurements of position, which should always be regarded as having a definite value for any system (Bohm). Then the set $\{D_{R_j}\}$ (with $D = P_\Psi$) will be exactly those propositions picked out by the one-dimensional projections onto the subspaces generated by the vectors $\{o_i \otimes a_i\}$ in the expansion of Ψ, and so these propositions will automatically get included in $L_{H_{OA}}(\{D_{R_j}\})$. On this third proposal, then, it *will* make sense to say that the combined system has the property that its "object" part has a definite value for the observable O and its "apparatus" part has a definite pointer value. Note how this consequence of the third proposal follows almost directly from the choice of atoms it makes.

From the perspective of the composite system, it would seem that only the third proposal yields a satisfactory solution of the measurement problem, i.e., the problem of accounting for why quantum measurements yield definite results. But when we look at the two component systems O and A as separate systems, the second proposal for L_D (due to van Fraassen, Kochen, Dieks) *will* attribute the observables O and A definite values (at least when there is no degeneracy among the numbers $\{|c_i|^2\}$). This is because the component systems will be in mixed states D_O and D_A determined by P_Ψ through "reduction of the density matrix." Thus (in the absence of degeneracy) D_O (D_A) will have its (nonzero eigenvalue) spectral projections given by the projections onto the rays generated by the set $\{o_i\}$ ($\{a_i\}$), and therefore $L_{D_O} = L_{H_O}(P_{D_O})$ [$L_{D_A} = L_{H_A}(P_{D_A})$] will automatically include the corresponding propositions—just as a consequence of how this proposal chooses its atoms.

On the other hand, the first orthodox proposal (of von Neumann and Dirac) still makes a "coarse-grained" choice of atoms for L_{D_O} and L_{D_A} (viz. just $\vee P_{D_O}$ and $\vee P_{D_A}$), and so is unable to say anything more than that the projections of the subsystem O (A) that get attributed probability 1 or 0 by

D_O (D_A) define its definite properties. In particular, if the $\{o_i\}$ (or $\{a_i\}$) constitute a basis of H_O (H_A), then nothing but the trivial propositions 0 and I will get definite truth values for subsystem O (A).

In sum, what this simple illustration shows is that the choice of which elements one requires to be atoms in the (maximal) $I_D \cap L_D$-quasiBoolean subalgebra of definite properties of a quantum system makes all the difference to the account one gives of the properties that are instantiated in typical quantum measurement situations. In particular, the second and third proposals differ in an interesting way regarding the relation between the definite properties of the parts of a quantum system and the definite properties of the whole, though both manage to get beyond the measurement problem inherent in the first (orthodox) proposal by making a more "fine-grained" choice of atoms.

7. CONCLUSION

We have shown how to define a natural notion of "quasidistributive lattice," intermediate in nature between a lattice and a Boolean algebra, and given a number of alternative characterizations of such lattices, as well as of strongly quasidistributive lattices. These new notions, though completely general and independent of Hilbert space, serve to illuminate what seem initially to be quite different proposals in the literature for the set of definite properties of a quantum system in a given state.

In particular, we showed that these proposals differ physically only in regard to what the atoms of the maximal quasiBoolean subalgebra they select to represent definite properties should be taken to be. Once these are fixed, the subalgebra selected by each proposal is determined uniquely and independently of the details of Hilbert space. We hope that in bringing all these proposals together in this way, we thereby shed light on the problem of deciding which (if any) of the different proposals is best suited to the task of representing the definite properties of a quantum system, particularly in view of the quantum measurement problem.

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