Commutative Rings as Algebras of Values of Intensive Quantities

The concept of *commutative ring* (with identity) provides a basic link between algebra and geometry. Commutative rings arise naturally as *algebras of values of intensive quantities* over topological spaces. To understand this, we need first to distinguish between the idea of an *extensive* and an *intensive* quantity. *Extensive* quantities are *global* in that they are defined over extended regions of space. Examples include mass, weight, volume, and electrical charge. Extensive quantities are always *additive as quantities:* thus, for instance, 2 pounds + 2 pounds = 4 pounds. *Intensive* quantities, on the other hand, are defined only at a point, *locally,* that is, and are *not* (in general) additive as quantities in the way that extensive quantities are. Temperature, density, and pressure are intensive quantities which are not additive as quantities: thus, for example, on mixing two buckets of water each having a uniform temperature of 50 degrees one obtains a quantity of water at a temperature of 50, rather than 100, degrees. (But there are intensive quantities such as velocity and acceleration which are additive as quantities.)

While intensive quantities are not, in general, additive *as quantities*, the *numerical values* that they are customarily assigned (typically, real numbers) cannot only be added, but also multiplied. Thus, while it makes no sense to add a density to a velocity or to multiply a temperature by a pressure, adding or multiplying the numerical values (as real numbers) are perfectly well defined operations.

For example, consider the earth`s atmosphere **A**. There are many intensive quantities defined on **A** - temperature, pressure, density, (wind) velocity, etc. The real number values of these quantities varies continuously from point to point. In general, we can define a (continuously varying value of) *intensive quantity* on **A** to be a continuous function on **A** to the field $\mathbb{R}$ of real numbers. Intensive quantities construed in this way form an *algebra* in which *addition* and *multiplication* can be defined "pointwise": thus, given two intensive quantities f, g, the sum $f + g$ and the product fg are defined by setting, for each point x in **A,**

$$(f + g)(x) = f(x) + g(x) \qquad (fg)(x) = f(x)g(x).$$

In general, given a topological space **X**, we consider the set **C**(**X**) of continuous real-valued functions on **X**, with addition and multiplication defined pointwise as above. This turns **C**(**X**) into a *commutative ring,* the *ring of real-valued (continuously varying) intensive quantities over* **X.** We can also consider the subring **C***(**X**) of **C**(**X**) consisting of all *bounded* members of **C**(**X**), the ring of *bounded intensive quantities over* **X.** When **X** is *compact*, **C***(**X**) and **C**(**X**) coincide.

More generally, given any commutative topological ring **T**, the ring **C**(**X**, **T)** of continuous **T -** valued functions on **X** is called the *ring of* **T-***valued intensive quantities on* **X.**

Given a commutative ring, it is natural to raise the question as to whether it can be represented as a ring of intensive quantities (with values in some commutative topological ring) on some topological space.

It was *M. H. Stone* who provided the first answer to this question. In the celebrated *Stone Representation Theorem*, proved in the 1930s, he showed that each member of a certain class of rings, the so-called *Boolean rings,* is representable as a ring of intensive quantities - with values in a fixed simple topological ring - over a certain class of spaces - the *Boolean* or *Stone spaces.*

A *Boolean ring* is defined to a ring in which every element is *idempotent*, $x^2 = x$ for every x. The archetypal example of a Boolean ring is the ring, written **2,** of integers modulo 2, whose elements may be identified with the integers 0, 1. **2** is the simplest nontrivial (commutative) ring. A ring isomorphic to **2** will be called *rudimentary.* Clearly any rudimentary ring is a field. The term Boolean ring celebrates the 19th century English mathematician George Boole, who first demonstrated the importance of this mathematical structure in the analysis of logic.)

An elegant algebraic manipulation shows that *any Boolean ring is commutative.* That is, *universal idempotency implies commutativity.* To wit,

$$x + y = (x + y)^2 = x^2 + xy + yx + y^2 = x + y + xy + yx$$

Hence $0 = xy + yx$, so that $xy = -yx$. Putting $y = x$ then gives $x = x^2 = - x^2 = -x$. Thus $xy = -yx = yx$, establishing commutativity.

Now give the ring **2** the discrete topology, so turning it into a topological ring. For a topological space X, the ring **C(X, 2)** of **2**-valued intensive quantities on **X** is called the *characteristic ring* of **X.** It is easily seen that characteristic rings are always Boolean. If (and only if) **X** is connected, the characteristic ring of **X** is just the basic Boolean ring **2.** Stone`s idea was to represent an *arbitrary* Boolean ring as a characteristic ring of some type of topological space. Knowing that the characteristic rings of connected spaces are always basic, such spaces would necessarily have to be, in some sense, highly *disconnected.*

To this end Stone employed the concept - already introduced by topologists - of a *totally disconnected* space. A topological space is defined to be totally disconnected if its only connected subsets consist of single points. (This is about as close as a topological space can get to being discrete without neighbourhoods being reduced to single points). A totally disconnected compact Hausdorff space is called a *Boolean* space. It is easily shown that a Boolean space can also be characterized as a compact Hausdorff space with a base of clopen (i. e. simultaneously closed-and -open) sets.

With each Boolean ring R, Stone associated a Boolean space **St**(R), called its *Stone space,* and showed that R is isomorphic to the characteristic ring of **St**(R). The points of **St**(R) are the *prime ideals* of R, and the topology on **St**(R) is defined by taking closed sets to be subsets of **St**(R) of the form $\{P \in \mathbf{St}(R) : X \subseteq P\}$ for arbitrary subsets X of R.

Inversely, each Boolean space X gives rise to a Boolean ring $\mathbf{Cl}(X)$ called the *ring of clopens* of X. The elements of $\mathbf{Cl}(X)$ are the clopen subsets of X, with addition and multiplication operations given by:

$$U + V = (U \cup V) \setminus (U \cap V) \text{ (symmetric difference)} \qquad U \cdot V = U \cap V$$

The *Stone Representation Theorem* states that each Boolean ring is isomorphic to the chararacteristic ring of its Stone space and that each Boolean space is homeomorphic to the Stone space of its ring of clopens.

The Stone representation theorem establishes a duality between the category of Boolean rings and the category of Boolean spaces.

It is an immediate consequence of the Stone representation theorem that Boolean spaces are completely characterized by their characteristic rings, in the sense that, if the characteristic rings of two Boolean spaces are isomorphic, then the spaces themselves are homeomorphic.

In the late 1930s Gelfand and Kolmogorov showed that *compact Hausdorff* spaces are completely characterized by their rings of *real-valued* intensive quantities: if the rings of real-valued intensive quantities of two compact spaces are isomorrphic, then the spaces themselves are homeomorphic. This is the *Gelfand-Kolmogorov theorem.*

Essentially, Gelfand and Kolmogorov extended Stone`s procedure of associating a topological space with a Boolean ring to rings of the form **C(X)** for arbitrary compact Hausdorff spaces **X**. They then showed that, given a compact Hausdorff space **X,** the topological space so associated with **C(X)** is itself homeomorphic to **X.** The Gelfand-Kolmogorov theorem is an immediate consequence.

As later became clear, Gelfand, Kolmogorov and Stone's procedure of associating topological spaces be applied to *arbitrary commutative rings.* Given a commutaive ring R, the *structure space* or *maximal spectrum* **Max**(R) of R is defined as follows**.** The points of **Max**(R) are the *maximal ideals* of R, and the topology on **Max**(R) is defined by taking closed sets to be subsets of **Max**(R) of the form $\{P \in \mathbf{Max}(R) : X \subseteq P\}$ for arbitrary subsets X of R. We note in passing that, since in an arbitrary commutative ring, every maximal ideal is prime, and, in a Boolean ring, conversely, **Max**(R), for a Boolean ring R coincides with **St**(R).

Gelfand and Kolmogorov showed that, for points p in a compact Hausdorff space **X**, the set M_p = $\{f \in$ **C(X):** $f(p) = 0\}$ is a maximal ideal in **C(X)** (= **C*(X)**) hence an element of **Max(C(X)):** and that the map $p \mapsto M_p$ is a homeororphism of **X** with **Max(C(X, ℝ)).**

The Gelfand-Kolmogorov theorem leads to a duality between the categrories of compact Hausdorff spaces and the categories of rings of the form **C*(X)** for arbitrary topological spaces **X**, that is, rings of bounded intensive quantities. This result is less satisfactory than the Stone duality between the "clean-cut" categories of Boolean rings and Boolean spaces, since it naturally leaves unresolved the problem of providing an abstract characterization of rings of bounded intensive quantities. It turns out that such a characterization cannot be given in purely algebraic terms. It is necessary also to equip rings with an *order* structure and a *norm* naturally possessed by rings of bounded intensive quantities. The appropriate type of ordered normed ring is called a real C*-*algebra.* In 1940 Stone established what amounts to the duality between the category of compact Hausdorff spaces and the appropriately defined category of real C*-algebras.

Gelfand and his collaborators proceeded in another direction, replacing the real field by the complex field $\mathbb{C}$, so introducing rings (or algebras) of *complex-valued* intensive quantities. The abstract versions of these are called *commutative complex C**-algebras. Gelfand and Naimark established a duality between the category of commutative complex C*-algebras and the category of compact Hausdorff spaces. They also investigated *noncommutative* C*-algebras, proving the important representation theorem that every such algebra is isomorphic to an algebra of operators on a (complex) Hilbert space. Here noncommutativity is natural, since "multiplication" of operators corresponds to *composition of functions*, rather than to products of numbers.

To extend the representation of rings as rings of intensive quantities to *arbitrary* commutative rings, some new ideas are needed. The principal new ideas are those of a *bundle* and a *sheaf* on a topological space. For our purposes here these concepts can be introduced in the following way.

Consider the product space $X \times \mathbb{R}$ and the (continuous) projection map $\pi : X \times \mathbb{R} \to X$ given by $\pi((x,r)) = x$. Then the map $f^*: X \to X \times \mathbb{R}$ given by $f^*(x) = (x, f(x))$ is a continuous *section* of π, that is, $\pi \circ f^*$ is the identity map on X. Conversely, it is easily shown that each continuous section of π is of the form f^* for a unique $f \in C(X)$. The set $S(X)$ of continuous sections of π is then a ring with sum and product defined pointwise as for $C(X)$: for $s, t \in S(X)$, $s + t$ and st are given by $(s + t)(x) = (s(x) + t(x)), x)$ and $(st)(x) = (s(x)t(x), x)$ for $x \in X$. $S(X)$ is called the *ring of sections* of π. The map $f \mapsto f^*$ is an isomorphism between the rings $C(X)$ and $S(X)$.

It follows that every ring of the form $C(X)$, i.e. every ring of intensive quantities in the usual sense, is isomorphic to the ring of sections of a projection map. Thus the rings of sections of projection maps can be thought of as "generalized" rings of intensive quantities.

Similarly, if we replace $C(X)$ by the characteristic ring C(X, 2), then the latter is isomorphic to the ring of sections of the projection map $X \times 2 \to X$.

For each $x \in X$, the subset $P_x = \pi^{-1}(x)$ of $X \times \mathbb{R}$ is called the *fibre* or *stalk* of π. Clearly $P_x = \{x\} \times \mathbb{R}$ so that it can turned into a ring isomorphic to $\mathbb{R}$ in the obvious way. The ring operations on each P_x are *continuous* (with respect to the topology on $X \times \mathbb{R}$).

The projection map $\pi : X \times \mathbb{R} \to X$ and the rings P_x constitute a *bundle of rings over X.* In general, a *bundle* over X is just a continuous map p from some topological space E to X.

The subsets $P_x = p^{-1}(x)$ for $x \in X$ are called the *fibres* of p. A *section* of p is a (continuous) map $s: X \to E$ such that $p \circ s$ is the identity map on X. A *bundle of rings* over X is a bundle $p : E \to X$ such that each fibre P_x is a ring whose operations are continuous with respect to the topology on E. If each ring P_x is a ring of a certain fixed type, the bundle is called a bundle of rings of that type.

Given a bundle of rings $p : E \to X$, The set $\Gamma(E, p)$ of sections of p can be turned into a ring as follows: given $s, t \in \Gamma(E, p)$, and $x \in X$, we have $p(s(x)) = p(t(x)) = x$, i.e. $s(x), t(x) \in P_x$, so that $s(x) + t(x)$ and $s(x)t(x)$ are both defined in P_x. Accordingly we define $s + t$ and st by

$$(s + t)(x) = s(x) + t(x) \qquad (st)(x) = s(x)t(x) \quad \text{for } x \in X.$$

The resulting ring is called the *ring of sections* of the bundle (E, p). Each section s is a map on X taking values in a ring: to be precise, the value $s(x)$ at each $x \in X$ lies in the ring P_x. In the case of the projection bundle associated with the basic ring C(X) of intensive quantities on X all the rings P_x are isomorphic (in this case, to $\mathbb{R}$), and so the sections can be considered as `quantities`` with values in the `constant`` ring $\mathbb{R}$. For a general bundle of rings, the rings P_x in which the sections take values *varies* with the point x to which P_x is attached. Thus rings of sections of bundles of rings can be thought of as (still further) generalized rings of intensive quantities, only now in which the ring of values of the quantities varies with the point in the space at which the quantity is defined. The ring of sections of a bundle of rings can then, with some justice, be called the *ring of intensive quantities* of the bundle.

We have noted the (less than profound) facts that any ring of the form $C(X)$, as well as any characteristic ring, is isomorphic to the ring of sections of a bundle of rings. In the case of characteristic rings, each of the fibres of the associated bundle is a rudimentary ring. If we now bring the Stone Representation Theorem into the picture, we get the *non-trivial* result that *any Boolean ring is isomorphic to the ring of sections, or intensive quantities, of a bundle of rudimentary rings over a Boolean space.*

The bundle associated with the Stone Representation Theorem is the projection bundle π: $X \times 2 \rightarrow X$. In this case it happens that the map π has the special property - not possessed by all projections - of being a *local homeomorphism.* In general, a continuous map $p: E \rightarrow X$ is a local homeomrphism if each point $e \in E$ has an open neighbourhood U such that $p[U]$ is open in X and the restriction $p|U$ is a homeomorphism of U onto $p[U]$. It is easily seen that, for any discrete space D (in particular, when D = 2) the projection π: $X \times D \rightarrow X$ is a local homeomorphism. (For $(x, d) \in X \times D$, $X \times \{d\}$ is an open neighbourhood of (x, d) projecting homeomorphically onto $p[X \times \{d\}] = X$).

A bundle (of rings) $p: E \rightarrow X$ in which p is a local homeomorphism is called a *sheaf (of rings) on X.* The Stone Representation Theorem can accordingly be stated:

(*) *every Boolean ring is the ring of sections, or intensive quantities, of a sheaf of rudimentary rings on a Boolean space.*

The objective behind the sheaf representation of commutative rings is to represent each commutative ring as a ring of sections, or intensive quantities, of a sheaf of rings of some simple type.

It is a remarkable fact that the Stone Representation Theorem for Boolean rings in the form (*) above can be extended to arbitrary commutative rings. Call a ring *indecomposable* if 0 and 1 are its only idempotents. Clearly any field, and so any rudimentary ring, is indecomposable. The only indecomposable Boolean rings are rudimentary. C(X) is indecomposable iff X is connected. The *Pierce Representation Theorem* for commutative rings asserts that

every commutative ring is the ring of sections, or intensive quantities, of a sheaf of indecomposable rings on a Boolean space.

For a given ring *R*, the associated Boolean space is the Stone space of the Boolean ring of idempotents in *R*.

Commutative rings can also be represented as sheaves of *local* rings. A local ring is a (commutative) ring *R* with the following equivalent properties: (1) *R* has exactly one maximal ideal; (2) for any $x \in R$, either x or $1 - x$ is invertible. Any field, and so any rudimentary ring, is local.

Given a (commutative) ring *R*, we define the *spectrum* of *R*, Spec(*R*), to be the topological space whose underlying set is the set of prime ideals of and whose closed sets are defined to be all sets of prime ideals of the form $\{P: X \subseteq P\}$ for $X \subseteq R$. *Grothendieck's Representation Theorem* for commutative rings asserts that

any commutative ring R is the ring of sections, or intensive quantities, of a sheaf of local rings over Spec(R).