

ON COMPACT CARDINALS

by J. L. BELL in London (Great Britain)

Let κ be a cardinal and L a language. κ is said to be *L-compact* if whenever Σ is a set of sentences of L such that any subset of Σ of power $< \kappa$ has a model, so does Σ . If $\mathcal{L}$ is a class of languages, we say that κ is *$\mathcal{L}$ -compact* if κ is *L-compact* for each $L \in \mathcal{L}$.

In Section 1 of this paper we investigate the properties and relative sizes of $\mathcal{L}$ -compact cardinals for certain classes $\mathcal{L}$ of infinitary and second-order languages. In Section 2 we consider a natural extension of the notion of compactness to languages whose class of formulas is a proper class in the sense of MORSE-KELLEY set theory. Using a result of KUNEN [4], we show that there are no *L-compact* cardinals for any *second order* language L whose class of individual constants is a proper class.

§ 1. Results on $\mathcal{L}$ -compact cardinals

Notation. We write $\mathcal{L}^1$ for the class of all first order languages with equality, all of which are assumed to include a binary predicate symbol $\in$.

For cardinals κ, λ , we write $\mathcal{L}_{\kappa\lambda}$ for the class of all infinitary languages $L_{\kappa\lambda}$, where $L \in \mathcal{L}^1$.

$\mathcal{L}^2$ is the class of all second-order languages L^2 obtained from languages $L \in \mathcal{L}^1$ by adding a countable sequence $X_0, X_1, \dots$ of second-order variables, which are understood to range over all sets of individuals.

Definition. A cardinal κ is said to be

- (a) *strongly compact* if it is $\mathcal{L}_{\kappa\kappa}$ -compact;
- (b) *supercompact* if for each cardinal $\lambda \geq \kappa$ there is a κ -complete ultrafilter U over $P_\kappa(\lambda)$ (the family of all subsets of λ of power $< \kappa$) such that, for all $\xi < \lambda$,

$$\text{and, for all } f: P_\kappa(\lambda) \rightarrow \lambda, \quad \{x \in P_\kappa(\lambda): \xi \in x\} \in U$$

$$\{x \in P_\kappa(\lambda): f(x) \in x\} \in U \rightarrow (\exists \xi < \lambda) \{x \in P_\kappa(\lambda): f(x) = \xi\} \in U;$$

- (c) *extendable* if for all $\alpha > \kappa$ there is $\beta > \alpha$ and an elementary embedding j of $\langle R_\alpha, \in \rangle$ into $\langle R_\beta, \in \rangle$ such that κ is the first ordinal moved by j .

It is known [5] that every extendable cardinal is supercompact and that every supercompact cardinal is strongly compact. We first show that an extendable cardinal is a large supercompact cardinal.

Theorem 1. *Each extendable cardinal κ is a limit supercompact cardinal, i.e. there is a set of κ supercompact cardinals $< \kappa$.*

Proof. Let $sc(\xi)$ be a formula of the language of set theory expressing that ξ is a supercompact cardinal. By the reflection principle, we can find a limit ordinal $\alpha > \kappa$ such that, for all $\xi < \alpha$,

$$\langle R_\alpha, \in \rangle \models sc[\xi] \Leftrightarrow sc(\xi).$$

Now, since κ is extendable, there is a $\beta > \alpha$ and an elementary embedding j of $\langle R_\alpha, \in \rangle$ into $\langle R_\beta, \in \rangle$ such that κ is the first ordinal moved by j . By [5], the set

$$U = \{x \subseteq \kappa : \kappa \in j(x)\}$$

is a normal ultrafilter over κ . Now let $a = \{\xi < \kappa : \text{sc}(\xi)\}$. We claim that $a \in U$, which will prove the theorem since every member of U is evidently of power κ . By the choice of α , we have

$$a = \{\xi < \kappa : \langle R_\alpha, \in \rangle \models \text{sc}[\xi]\}.$$

Hence, since j is elementary, we have

$$j(a) = \{\xi < j(\kappa) : \langle R_\beta, \in \rangle \models \text{sc}[\xi]\}.$$

Since α is a limit ordinal and $\langle R_\beta, \in \rangle$ is elementarily equivalent to $\langle R_\alpha, \in \rangle$, β is also a limit ordinal. Hence, since κ is supercompact we have $\langle R_\alpha, \in \rangle \models \text{sc}[\kappa]$. It follows that $\kappa \in j(a)$, whence $a \in U$, as claimed. This completes the proof.

Corollary. *The least $\mathcal{L}^2$ -compact cardinal is a limit supercompact, hence also a limit strongly compact, cardinal.*

Proof. By a result of [2], the least $\mathcal{L}^2$ -compact cardinal is extendable, so the result follows immediately from Theorem 1.

We now consider $\mathcal{L}_{\omega_1\omega}$ and $\mathcal{L}_{\omega_1\omega_1}$ -compact cardinals. Our next result is related to the main theorem of [1] and is proved in a similar way. We leave the reader to supply the details of the proof, using the techniques of [1].

Theorem 2. *Let κ be a cardinal. Then the following conditions are equivalent.*

- (i) κ is $\mathcal{L}_{\omega_1\omega}$ -compact;
- (ii) κ is $\mathcal{L}_{\omega_1\omega_1}$ -compact;
- (iii) *for any transitive set M and any subset X of M with the property that any subset of X of power $< \kappa$ has non-empty intersection, there is a transitive set N and an elementary embedding j of $\langle M, \in \rangle$ into $\langle N, \in \rangle$ such that $\bigcap j''X \neq \emptyset$.*
- (iv) *for any cardinal λ , each κ -complete filter over λ can be extended to an ω_1 -complete ultrafilter over λ .*

The following result is an immediate consequence of Corollary 2.2 of [3].

Lemma. *Let μ_0 be the least (uncountable) measurable cardinal. Then, for any cardinal λ , each ω_1 -complete ultrafilter over λ is μ_0 -complete.*

Notice that this lemma together with Theorem 2 yields the conclusion that any $\mathcal{L}_{\omega_1\omega}$ -compact cardinal is $\mathcal{L}_{\mu_0\mu_0}$ -compact. For, if κ is $\mathcal{L}_{\omega_1\omega}$ -compact, then, by (iv) of Theorem 2 and the lemma, any κ -complete ultrafilter over a cardinal λ can be extended to a μ_0 -complete ultrafilter, and the required conclusion can then be obtained by a straightforward ultrapower argument.

Now let κ_0 be the least $\mathcal{L}_{\omega_1\omega}$ -compact cardinal and λ_0 the least strongly compact cardinal. It is clear from (iv) of Theorem 2 that $\mu_0 \leq \kappa_0$ and trivially $\kappa_0 \leq \lambda_0$. It is still unknown whether $\mu_0 < \lambda_0$ is provable in set theory.

Theorem 3. $\mu_0 < \lambda_0 \Rightarrow \mu_0 < \kappa_0$.

Proof. Suppose $\mu_0 = \kappa_0$. Then by (iv) of Theorem 2 and the lemma, each μ_0 -complete ultrafilter over a cardinal λ can be extended to a μ_0 -complete ultrafilter, and it is well known that this condition is equivalent to the strong compactness of μ_0 . Hence $\mu_0 = \lambda_0$, and the theorem follows.

In [6], VOPĚNKA and HRBÁČEK show that, if there is a cardinal κ such that, for any cardinal λ , each κ -complete filter over λ can be extended to a κ -complete ultrafilter (i.e. if κ is strongly compact), then the universe is not constructible from any set. Now it is quite easy to verify that the VOPĚNKA-HRBÁČEK proof only uses the fact that each κ -complete filter over λ can be extended to an ω_1 -complete ultrafilter over λ . But this is precisely condition (iv) of Theorem 2. Accordingly we have

Theorem 4. *If there is an $\mathcal{L}_{\omega_1\omega}$ -compact cardinal, then the universe is not constructible from any set.*

Corollary. *Let ZFM be the theory $\text{ZF} + \text{"there exists a measurable cardinal"}$. Then if ZFM is consistent, the statement " μ_0 is $\mathcal{L}_{\omega_1\omega}$ -compact" cannot be proved in ZFM.*

§ 2. Compactness in languages whose class of formulas is a proper class

Suppose $\mathcal{L}$ is a proper class of languages with a common stock of logical symbols. Consider the language UL whose class of non-logical symbols is given by the union of the sets of non-logical symbols of all members of $\mathcal{L}$ and whose logical symbols are the same as those of each member of $\mathcal{L}$. Evidently the class of non-logical symbols, and hence of formulas, of UL is a proper class in the sense of MORSE-KELLEY set theory. It is natural to call a cardinal κ *UL-compact for sets* if

- (*) whenever Σ is a set of sentences of UL and each subset of Σ of power $< \kappa$ has a model, then Σ has a model.

Clearly, then, if κ is UL -compact, it must be $\mathcal{L}$ -compact in the sense of § 1. Moreover, quite weak conditions on $\mathcal{L}$ are sufficient to ensure that the converse obtains, for example, if whenever $\mathcal{L}'$ is a subset of $\mathcal{L}$, we have $\text{UL}' \in \mathcal{L}$. This condition is met by all the classes of languages considered in this paper.

We propose to investigate what happens if we extend the condition (*) to all classes of sentences Σ of UL . Clearly, if $\mathcal{L}$ -compactness was a stringent condition, then this new condition is in general still more stringent. We shall see that, in fact, when $\mathcal{L}$ is the class of second-order languages, this condition is so stringent as to be unsatisfiable.

Let us define a *class language* to be a language $\mathbf{L}$ with one binary predicate symbol $\in$ in addition to the equality symbol $=$, and a proper class C of constant symbols. An *L-structure* is a triple¹⁾ $\mathfrak{A} = \langle A, E, F \rangle$ where A is a class, $E \subseteq A \times A$ and F is a function from C into A . If $c \in C$, we write $c^{\mathfrak{A}}$ for $F(c)$ as usual. The notion of satisfaction for formulas of $\mathbf{L}$ can now be defined in the customary way by interpreting $\in$ as E and c as $c^{\mathfrak{A}}$ for each $c \in C$. It is also easy to see that this definition of satisfaction can be formalized in MORSE-KELLEY set theory.

If κ is a cardinal and $\mathbf{L}$ is a class language, we say that $\mathbf{L}$ is *L-compact* if whenever Σ is a class of sentences of $\mathbf{L}$ such that each subset of Σ of power $< \kappa$ has a model, then Σ has a model.

Observe that the usual HENKIN-style completeness proof for first-order logic can be extended in a straightforward way to show that any consistent class of first-

¹⁾ $\mathfrak{A}$ is the customary notation in MORSE-KELLEY set theory for a finite sequence of

order sentences has a model. For the usual completeness proof to go through for a first order class language $\mathbf{L}$ it is evident that the following two conditions must be met:

- (1) there is a proper class of new constants available for adding to $\mathbf{L}$;
- (2) each consistent class of sentences of $\mathbf{L}$ can be extended to a consistent complete class of sentences.

Now (1) is not true in general, because the class C of constants of $\mathbf{L}$ might exhaust the whole universe. But we can always replace C by a proper class C' equipollent with C and such that the complement of C' is a proper class. The new language $\mathbf{L}'$ obtained in this way is equivalent to $\mathbf{L}$ and satisfies (1).

As for (2), the global axiom of choice yields an enumeration $\{\sigma_\alpha: \alpha \in \mathbf{ORD}\}^1$ of all sentences of $\mathbf{L}$. Starting with a consistent class of sentences Σ , we define by transfinite recursion a function f with domain $\mathbf{ORD}$ as follows:

$$f(\alpha) = \begin{cases} \sigma_\alpha & \text{if not } \Sigma \cup \{f(\beta): \beta < \alpha\} \vdash \neg \sigma_\alpha \\ \neg \sigma_\alpha & \text{if } \Sigma \cup \{f(\beta): \beta < \alpha\} \vdash \neg \sigma_\alpha \end{cases}$$

Then $\Sigma' = \Sigma \cup \{f(\alpha): \alpha \in \mathbf{ORD}\}$ is clearly a consistent complete class of sentences containing Σ .

From this discussion we immediately infer

Theorem 5. ω is $\mathbf{L}$ -compact for any first-order class language $\mathbf{L}$.

By contrast, however, we have

Theorem 6. Let $\mathbf{L}$ be a second-order class language. Then there are no $\mathbf{L}$ -compact cardinals.

Proof. First of all, it is clear that, if κ is $\mathbf{L}$ -compact for one second-order class language $\mathbf{L}$, it is $\mathbf{L}$ -compact for all second-order class languages $\mathbf{L}$. This is so because there is—assuming the global axiom of choice—a bijection between the classes of constants of any pair of second-order class languages.

Thus, without loss of generality, we may assume that $\mathbf{L}$ is a second-order class language whose class C of constants does not exhaust the whole universe V . By the global axiom of choice, there is a bijection F from C onto V . Let $\mathfrak{B}$ be the $\mathbf{L}$ -structure $\langle V, \in F \rangle$. For each $a \in V$, we write $\bar{a}$ for $F^{-1}(a)$; thus $\bar{a}$ is the constant of $\mathbf{L}$ which denotes a in $\mathfrak{B}$.

Now suppose, if possible, that κ is an $\mathbf{L}$ -compact cardinal. Let λ be the least regular cardinal $\geq \kappa$. Then since any cardinal $\geq \kappa$ is $\mathbf{L}$ -compact, so is λ . Let c be a constant not in C and let $\mathbf{L}'$ be the language obtained by adjoining the constant c to $\mathbf{L}$. Then, by the above remarks, λ is $\mathbf{L}'$ -compact.

Let Σ be the class of sentences of $\mathbf{L}'$ consisting of:

- (1) all sentences of $\mathbf{L}$ holding in $\mathfrak{B}$;
- (2) the sentences $\bar{\alpha} \in c$ for all $\alpha < \lambda$;
- (3) the sentence $c \in \lambda$.

We claim that each subset Σ' of Σ of power $< \lambda$ has a model. For let β be the supremum of all ordinals $\alpha < \lambda$ such that $\bar{\alpha}$ occurs in a sentence of type (2) in Σ' . Then, since

¹⁾ Here $\mathbf{ORD}$ is the class of all ordinals.

there are $< \lambda$ such α and λ is regular, we have $\beta < \lambda$. Accordingly, $\langle V, \in, F \cup \{\langle c, \beta \rangle\} \rangle$ is a model of Σ' .

Since λ is L' -compact, Σ has a model whose universe may be taken to be transitive since Σ contains the second-order sentence which asserts that $\in$ is a well-founded relation. Thus let $\mathfrak{A} = \langle A, \in, G \rangle$ be a transitive class model of Σ . Clearly the map $j: V \rightarrow A$ defined by $j(a) = \bar{a}^{\mathfrak{A}}$ for $a \in V$ is an elementary embedding of $\langle V, \in \rangle$ into $\langle A, \in \rangle$. Thus j is order-preserving on the ordinals and it follows that, for each ordinal α , we have $\alpha \leq j(\alpha)$. Accordingly A contains arbitrarily large ordinals, so, since it is transitive, it contains all the ordinals. Now put σ for the second-order sentence

$$\forall X_0 [\exists x \forall y [X_0(y) \rightarrow y \in x] \rightarrow \exists x \forall y [X_0(y) \leftrightarrow y \in x]].$$

σ says that every subset of an individual is (coextensive with) an individual. Now certainly σ holds in $\langle V, \in \rangle$, so it also holds in $\langle A, \in \rangle$. But it is well-known that the only transitive class model of $\mathbf{ZF} + \sigma$ containing all ordinals is V itself; it follows that $A = V$. Therefore j is an elementary embedding of $\langle V, \in \rangle$ into itself. Since $j(\lambda) = \bar{\lambda}^{\mathfrak{A}} > c^{\mathfrak{A}} > \bar{\alpha}^{\mathfrak{A}}$ for all $\alpha < \lambda$ it follows that $j(\lambda) > \lambda$, so that j is not the identity. But this contradicts a result in [4] which asserts that there are no non-trivial elementary embeddings of $\langle V, \in \rangle$ into itself.

This completes the proof.

Remark. Since the sentence characterizing well-foundedness and the sentence σ introduced in the proof of Theorem 6 are both Π_1^1 , it is clear that this proof actually establishes the following ostensibly stronger result: *For each cardinal κ there is a class Σ of Π_1^1 -sentences whose only non-logical constants are $\in$ and individual constants such that each subset of Σ of power $< \kappa$ has a model but Σ itself has no model.*

Added in proof: In the proof of Theorem 6, to justify the claim that the model $\mathfrak{A}$ may be taken to be transitive, it is also necessary to show that Σ contains a sentence σ' which asserts that each initial segment under $\in$ is a set. We may take σ' to be the second-order sentence

$$\forall x \exists X_0 \forall y [X_0(y) \leftrightarrow y \in x];$$

it is now not hard to see that $\sigma' \in \Sigma$.

References

- [1] BELL, J. L., On the relationship between weak compactness in $L_{\omega_1, \omega}$, L_{ω_1, ω_1} , and restricted second order languages. Arch. Math. Logik Grundle. Math. **15** (1972), 74–78.
- [2] MAGIDOR, M., On the role of supercompact and extendible cardinals in logic. Israel J. Math. **10** (1971), 147–157.
- [3] KEISLER, H. J., and A. TARSKI, From accessible to inaccessible cardinals. Fund. Math. **53** (1964), 225–308.
- [4] KUNEN, K., Elementary embeddings and infinitary combinatorics. J. Symb. Log. **36** (1971), 407–413.
- [5] REINHARDT, W. N., Conditions on natural models of set theory. Proc. 1967 U.C.L.A. Summer Institute in Set Theory.
- [6] VOPĚNKA, P., and K. HRBÁČEK, On strongly measurable cardinals. Bull. Acad. Polon. Sci., Sér. Sci., Math., Astron., Phys. **14** (1966), 587–591.

(Eingegangen am 29. März 1973)