

Chapter 3

COVER SCHEMES, FRAME-VALUED SETS AND THEIR POTENTIAL USES IN SPACETIME PHYSICS

John L. Bell

Department of Philosophy, University of Western Ontario, London, Ontario, Canada

ABSTRACT

In the present paper, the concept of a covering is presented and developed. The relationship between cover schemes, frames (complete Heyting algebras), Kripke models, and frame-valued set theory is discussed. Finally cover schemes and frame-valued set theory are applied in the context of Markopoulou's account of discrete spacetime as sets "evolving" over a causal set. We observe that Markopoulou's proposal may be effectively realized by working within an appropriate frame-valued model of set theory. We go on to show that, within this framework, cover schemes may be used to force certain conditions to prevail in the associated models: for example, rendering the universe timeless, obliterating a given event or forcing it to become the universe's "beginning".

PREAMBLE

The concept of *Grothendieck (pre)topology* or *covering* issued from the efforts of algebraic geometers to study "sheaf-like" objects defined on categories more general than the lattice of open sets of a topological space (see, e.g. [4]). A *Grothendieck pretopology* on a category $\mathcal{C}$ with pullbacks is defined by specifying, for each object U of $\mathcal{C}$, a set $P(U)$ of arrows to U called *covering families* satisfying appropriate category theoretic versions of the corresponding conditions for a family $\mathcal{A}$ of sets to cover a set U , namely: (i) $\{U\}$ covers U , (ii) if $\mathcal{A}$ covers U and $V \subseteq U$, then $\mathcal{A} \upharpoonright V = \{A \cap V : A \in \mathcal{A}\}$ covers V , and (iii) if $\mathcal{A}$ covers U and, for each $A \in \mathcal{A}$, $\mathcal{B}_A$ covers A , then $\bigcup_{A \in \mathcal{A}} \mathcal{B}_A$ covers U . In the present paper the covering concept—here called a *cover scheme*—is presented and developed in the simple case when the underlying category is a preordered set. The relationship between cover schemes, frames

(complete Heyting algebras), Kripke models, and frame-valued set theory is discussed. Finally cover schemes and frame-valued set theory are applied in the context of Markopoulou's [5] account of discrete spacetime as sets "evolving" over a causal set.

COVER SCHEMES ON PREORDERED SETS

A *preordered set* is a set equipped with a reflexive transitive relation $\leq$. Let $(P, \leq)$ be a fixed but arbitrary preordered set: we shall use letters p, q, r, s, t to denote elements of P . We write $p \equiv q$ for $(p \leq q \ \& \ q \leq p)$. A *meet* for a subset S of P is an element p of P such that $\forall q[\forall s \in S(q \leq s) \leftrightarrow q \leq p]$: if p and p' are both meets for S , then $p \equiv p'$. If the empty subset $\emptyset$ has a meet, any such meet m is necessarily a *largest* or *top* element of P , that is, satisfies $p \leq m$ for all p . We use the symbol 1 to denote a top element of P . A meet of a finite subset $\{p_1, \dots, p_n\}$ of P will be denoted by $p_1 \wedge \dots \wedge p_n$. P is a *lower semilattice* if each nonempty finite subset of P has a meet. A subset S of P is said to be a *sharpening* of, or to *sharpen*, a subset T of P , written $S \prec T$, if $\forall s \in S \exists t \in T(s \leq t)$. A *sieve* in P is a subset S such that $p \in S$ and $q \leq p$ implies $q \in S$. Each subset S of P generates a sieve $\bar{S}$ given by $\bar{S} = \{p : \exists s \in S(p \leq s)\}$.

A *cover scheme* on P is a map $\mathbf{C}$ assigning to each $p \in P$ a family $\mathbf{C}(p)$ of subsets of $p \downarrow = \{q : q \leq p\}$, called ($\mathbf{C}$ -)covers of p , such that, if $q \leq p$, any cover of p can be sharpened to a cover of q , i.e.,

$$S \in \mathbf{C}(p) \ \& \ q \leq p \rightarrow \exists T \in \mathbf{C}(q)[\forall t \in T \exists s \in S(t \leq s)]. \quad (\text{Cov})$$

If P is a lower semilattice, a *coverage* (see [3]) on P is a map $\mathbf{C}$ as above, satisfying, in place of (Cov), the condition

$$S \in \mathbf{C}(p) \ \& \ q \leq p \rightarrow S \wedge q = \{s \wedge q : s \in S\} \in \mathbf{C}(p).$$

A cover scheme $\mathbf{C}$ is said to be *normal* if every member of every $\mathbf{C}(p)$ is a sieve and whenever $S \in \mathbf{C}(p)$ and T is a sieve such that $S \subseteq T \subseteq p \downarrow$, we have $T \in \mathbf{C}(p)$. Any cover scheme $\mathbf{C}$ on P induces a normal cover scheme $\bar{\mathbf{C}}$ (called its *normalization*) defined by

$$\bar{\mathbf{C}}(p) = \{X \subseteq p \downarrow : X \text{ is a sieve} \ \& \ \exists S \in \mathbf{C}(p). S \subseteq X\}.$$

Notice that a normal cover scheme on a lower semilattice is always a coverage. For if $\mathbf{C}$ is such, then for $S \in \mathbf{C}(p)$ and $q \leq p$, any sharpening of S to a member of $\mathbf{C}(q)$ is easily seen to be included in $S \wedge q$, so that the latter is also in $\mathbf{C}(q)$.

Write $\text{Cov}(P)$ for the set of all cover schemes on P . There is a natural partial ordering $\triangleleft$ on $\text{Cov}(P)$ defined by

$$\mathbf{C} \triangleleft \mathbf{D} \leftrightarrow \forall p \, \mathbf{C}(p) \subseteq \mathbf{D}(p).$$

With this ordering $\text{Cov}(P)$ is a complete lattice in which the join $\bigvee_{i \in I} \mathbf{C}_i$ of any family $\{\mathbf{C}_i; i \in I\}$ is given by

$$\left(\bigvee_{i \in I} \mathbf{C}_i\right)(p) = \bigcup_{i \in I} \mathbf{C}_i(p).$$

There is also a natural *composition* $\star$ defined on $\text{Cov}(P)$. For $\mathbf{C}, \mathbf{D} \in \text{Cov}(P)$, $\mathbf{D} \star \mathbf{C}$ is defined by decreeing that $(\mathbf{D} \star \mathbf{C})(p)$ is to consist of all subsets of $p \downarrow$ of the form $\bigcup_{s \in S} T_s$, where $S \in \mathbf{C}(p)$ and, for each $s \in S$, $T_s \in \mathbf{D}(s)$. That $\mathbf{D} \star \mathbf{C}$ is a cover scheme on P may be verified (using the axiom of choice) as follows. Given $S \in \mathbf{C}(p)$, $\bigcup_{s \in S} T_s \in (\mathbf{D} \star \mathbf{C})(p)$ and $q \leq p$, there is $U \in \mathbf{C}(q)$ with $U \prec S$, so for each $u \in U$ there is $s(u) \in S$ for which $u \leq s(u)$. Then $T_{s(u)} \in \mathbf{D}(s(u))$ and we can choose $V_u \in \mathbf{D}(u)$ so that $V_u \prec T_{s(u)}$. Clearly $\bigcup_{u \in U} V_u \in (\mathbf{D} \star \mathbf{C})(q)$ and, since $V_u \prec T_{s(u)}$ for all $u \in U$, it follows immediately that $\bigcup_{u \in U} V_u \prec \bigcup_{s \in S} T_s$.

It is not hard to verify that $\star$ is associative and that with this operation $\text{Cov}(P)$ is actually a *quantale* (see, e.g., [6]) that is, for any $\mathbf{D}$, $\{\mathbf{C}_i; i \in I\}$ in $\text{Cov}(P)$,

$$\mathbf{D} \star \bigvee_{i \in I} \mathbf{C}_i = \bigvee_{i \in I} (\mathbf{D} \star \mathbf{C}_i) \quad \left(\bigvee_{i \in I} \mathbf{C}_i\right) \star \mathbf{D} = \bigvee_{i \in I} (\mathbf{C}_i \star \mathbf{D}).$$

Also the element $\mathbf{1} \in \text{Cov}(P)$ with $\mathbf{1}(p) = \{p\}$ acts as a quantal unit, since it is readily verified that $\mathbf{1} \star \mathbf{C} = \mathbf{C} \star \mathbf{1} = \mathbf{C}$ for all $\mathbf{C} \in \text{Cov}(P)$.

In this connection a *Grothendieck pretopology*—which we shall abbreviate simply to *pretopology*—on P may be identified as a cover scheme $\mathbf{C}$ on P satisfying $\mathbf{1} \triangleleft \mathbf{C}$ and $\mathbf{C} \star \mathbf{C} \triangleleft \mathbf{C}$, that is, $\{p\} \in \mathbf{C}(p)$ for all $p \in P$ and, if $S \in \mathbf{C}(p)$ and, for each $s \in S$, $T_s \in \mathbf{C}(s)$, then $\bigcup_{s \in S} T_s \in \mathbf{C}(p)$.

We observe that a normal pretopology $\mathbf{C}$ has the additional properties: (i) each $\mathbf{C}(p)$ is a filter of sieves in $p \downarrow$, that is, satisfies $S, T \in \mathbf{C}(p) \leftrightarrow S \in \mathbf{C}(p) \ \& \ T \in \mathbf{C}(p)$; (ii) $S \in \mathbf{C}(p) \ \& \ q \leq p \rightarrow S \cap q \downarrow \in \mathbf{C}(q)$. For (ii), we observe that $S \cap q \downarrow$, including as it does any sharpening of S to a member of $\mathbf{C}(q)$, is itself a member of $\mathbf{C}(q)$. As for (i), the “ $\rightarrow$ ” direction is obvious; conversely, if $S, T \in \mathbf{C}(p)$, then $S \cap T = S \cap \bigcup_{t \in T} (t \downarrow) = \bigcup_{t \in T} (S \cap t \downarrow)$. But from (ii) we have $S \cap t \downarrow \in \mathbf{C}(t)$ for every $t \in T$, whence $\bigcup_{t \in T} (S \cap t \downarrow) \in \mathbf{C}(p)$, and so $S \cap T \in \mathbf{C}(p)$.

A normal pretopology is also called a *Grothendieck topology*. A normal cover scheme satisfying (i) and (ii) is called a *regular* cover scheme.

Each cover scheme $\mathbf{C}$ generates a pretopology, and a Grothendieck topology in the following way. First, define $\mathbf{C}^n$ for $n \in \omega$ recursively by $\mathbf{C}^0 = \mathbf{1}$ and $\mathbf{C}^{n+1} = \mathbf{C} \star \mathbf{C}^n$. Now put $\mathbf{G} = \bigvee_{n \in \omega} \mathbf{C}^n$. Then $\mathbf{G}$ is a pretopology, for obviously $\mathbf{1} \twoheadrightarrow \mathbf{G}$, and

$$\mathbf{G} \star \mathbf{G} = \bigvee_{n \in \omega} \mathbf{C}^n \star \bigvee_{m \in \omega} \mathbf{C}^m = \bigvee_{n \in \omega} \bigvee_{m \in \omega} \mathbf{C}^{m+n} = \bigvee_{n \in \omega} \mathbf{C}^n = \mathbf{G}.$$

Also $\mathbf{C} \twoheadrightarrow \mathbf{G}$, and $\mathbf{G}$ is evidently the $\leftarrow$ -least pretopology. $\mathbf{G}$ is called the *pretopology generated by $\mathbf{M}$* . The normalization $\overline{\mathbf{G}}$ of $\mathbf{G}$ is then a Grothendieck topology called the *Grothendieck topology generated by $\mathbf{C}$* .

Now let $\mathbf{M}$ be a map assigning to each $p \in P$ a subset $\mathbf{M}(p)$ of subsets of $p \downarrow$. Since $\text{Cov}(P)$ is a complete lattice, there is a $\leftarrow$ -least cover scheme $\mathbf{C}$ such that $\mathbf{M}(p) \subseteq \mathbf{C}(p)$ for all p . $\mathbf{C}$ is called the cover scheme *generated by $\mathbf{M}$* ; the pretopology and Grothendieck topology generated in turn by $\mathbf{C}$ are said to be *generated by $\mathbf{M}$* .

There are several naturally defined cover schemes on P which also happen to be pretopologies. First, each sieve A in P determines two cover schemes $\mathbf{C}_A$ and $\mathbf{C}^A$ defined by

$$S \in \mathbf{C}_A(p) \leftrightarrow p \in A \cup S \quad S \in \mathbf{C}^A(p) \leftrightarrow p \downarrow \cap A \subseteq S:$$

these are easily shown to be pretopologies. Notice that $\emptyset \in \mathbf{C}_A(p) \leftrightarrow p \in A$ and $\emptyset \in \mathbf{C}^A(p) \leftrightarrow p \downarrow \cap A = \emptyset$.

Next, we have the *dense cover scheme* **Den** given by:

$$S \in \mathbf{Den}(p) \leftrightarrow \forall q \leq p \exists s \in S \exists r \leq s (r \leq q): \quad (*)$$

it is a straightforward exercise to show that this is a pretopology. When S is a sieve, the above condition (*) is easily seen to be equivalent to the familiar condition of *density below p* : that is, $\forall q \leq p \exists s \in S (s \leq q)$.

Note that the following are equivalent for any cover scheme $\mathbf{C}$: (a) $\mathbf{C} \twoheadrightarrow \mathbf{Den}$, (b) $\emptyset \notin \mathbf{C}(p)$ for all p . For since $\emptyset \notin \mathbf{Den}(p)$, (a) clearly implies (b). Conversely, assume (b), and let $S \in \mathbf{C}(p)$. Then for each $q \leq p$ there is $T \in \mathbf{C}(q)$ for which $\forall t \in T \exists s \in S (t \leq s)$. Since (by (b)) $T \neq \emptyset$, we may choose $t_0 \in T$ and $s_0 \in S$ for which $t_0 \leq s_0$. Since $t_0 \leq q$, and $q \leq p$ was arbitrary, it follows that S satisfies the condition (*) above for membership in $\mathbf{Den}(p)$. This gives (a).

Finally, we have the *Beth cover scheme* **Bet**. This is defined as follows. First we define a *road* from p to be a maximal linearly preordered subset of $p \downarrow$: clearly any road from p contains p . Let us call a *rome* over p any subset of $p \downarrow$ intersecting every road from p . Now the Beth coverage has $\mathbf{Bet}(p) =$ collection of all romes over p . Let us check first that **Bet** is a cover scheme. Suppose that S is a rome over p and $q \leq p$. We claim that

$$T = \{t \leq q : \exists s \in S (t \leq s)\}$$

is a rome over q . For let Y be any road from q ; then, by Zorn's lemma, Y may be extended to a road X from p . We note that since $X \cap q \downarrow$ is linearly preordered and includes Y , it must

coincide with Y . Since S is a rome over p , there must be an element $s \in S \cap X$. Since also $q \in Y \subseteq X$, we have $s \leq q$ or $q \leq s$. If $s \leq q$, then $s \in X \cap q \downarrow = Y$ and $s \in T$, so that $s \in Y \cap T$. If $q \leq s$, then $q \in T$; since $q \in Y$, it follows that $q \in Y \cap T$. So in either case $Y \cap T \neq \emptyset$; therefore T is a rome over q . Since clearly also $T \prec S$, we have shown that **Bet** is a cover scheme.

To show that **Bet** is a pretopology, we observe first that, for any p , $\{p\}$ is a rome over p . Now suppose that we are given: a rome S over p , for each $s \in S$, a rome T_s over s , and a road X from p . Then $s \in X \cap S$ for some s : we claim that $X \cap s \downarrow$ is a road from s . For suppose $t \leq s$ is comparable with every member of $X \cap s \downarrow$; now since $s \in X$, for each $x \in X$ either $s \leq x$ or $x \leq s$. In the first case $t \leq x$; in the second t is comparable with x by assumption. Hence t is comparable with every member of X , and so $t \in X$. Accordingly $X \cap s \downarrow$ is, as claimed, a road from s ; as such, it must meet the rome T_s , so X meets $\bigcup_{s \in S} T_s$, and the latter is therefore a rome over p . So **Bet** is indeed a pretopology.

Since clearly $\emptyset \notin \mathbf{Bet}(p)$ for any p , it follows from what we have noted above that **Bet** $\nprec$ **Den**, a fact that can also be easily verified directly.

Any preordered set $(P, \leq)$ generates a *free lower semilattice* $\tilde{P}$ which may be described as follows. The elements of $\tilde{P}$ are the finite subsets of P ; the preordering on $\tilde{P}$ is the *refinement* relation $\sqsubseteq$, that is, for $F, G \in \tilde{P}$,

$$F \sqsubseteq G \leftrightarrow \forall q \in G \exists p \in F (p \leq q).$$

The meet operation $\wedge$ in $\tilde{P}$ is set-theoretic union; the canonical embedding of P into $\tilde{P}$ is the map $p \mapsto \{p\}$. Notice also that $\emptyset$ is the unique top element of $\tilde{P}$.

Now, suppose we are given a cover scheme $\mathbf{C}$ on P . This induces a cover scheme $\tilde{\mathbf{C}}$ on $\tilde{P}$ defined in the following way. We start by setting $\tilde{\mathbf{C}}(\emptyset) = \{\{\emptyset\}\}$. Now fix a nonempty finite subset F of P , take any nonempty subset $\{p_1, \dots, p_n\}$ of F and any $S_1 \in \mathbf{C}(p_1), \dots, S_n \in \mathbf{C}(p_n)$. Define

$$S_1 \bullet \dots \bullet S_n = \{\{s_1, \dots, s_n\} \cup F : s_1 \in S_1, \dots, s_n \in S_n\}.$$

We decree that $\tilde{\mathbf{C}}(F)$ is to consist of all sets of the form $S_1 \bullet \dots \bullet S_n$, for $S_1 \in \mathbf{C}(p_1), \dots, S_n \in \mathbf{C}(p_n)$, and all nonempty finite subsets $\{p_1, \dots, p_n\}$ of F .

Let us check that $\tilde{\mathbf{C}}$ is a cover scheme on $\tilde{P}$. To begin with, the unique cover $\{\emptyset\}$ of $\emptyset$ is clearly sharpenable to any cover $S_1 \bullet \dots \bullet S_n$ of any nonempty member of $\tilde{P}$. Now suppose that $S_1 \bullet \dots \bullet S_n$ is a $\tilde{\mathbf{C}}$ -cover of a nonempty member F of $\tilde{P}$ and that $G = \{q_1, \dots, q_m\} \sqsubseteq F$. Then for each $1 \leq i \leq m$ there is $q_i \in G$ for which $q_i \leq p_i$, hence $T_i \in \mathbf{C}(q_i)$ with $T_i \prec S_i$. Clearly $T_1 \bullet \dots \bullet T_m \in \tilde{\mathbf{C}}(G)$. Also

$T_1 \bullet \dots \bullet T_n \prec S_1 \bullet \dots \bullet S_n$. For, given $t_1 \in T_1, \dots, t_n \in T_n$, then since $T_i \prec S_i$ for each i , there are $s_1 \in S_1, \dots, s_n \in S_n$ for which $t_1 \leq s_1, \dots, t_n \leq s_n$, whence $\{t_1, \dots, t_n\} \cup G \subseteq \{s_1, \dots, s_n\} \cup F$. So $\tilde{\mathbf{C}}$ satisfies the conditions of a cover scheme.

The normalization $\tilde{\mathbf{C}}$ of $\tilde{\mathbf{C}}$ is then a coverage on $\tilde{P}$ called the coverage on $\tilde{P}$ induced by $\mathbf{C}$.

We next show how cover schemes give rise to complete Heyting algebras, or frames (see, e.g. [3]).

A *Heyting algebra* is a lattice L with top and bottom elements $1, 0$ such that, for any elements $x, y \in L$, there is an element—denoted by $x \Rightarrow y$ —of L such that, for any $z \in L$,

$$z \leq x \Rightarrow y \text{ iff } z \wedge x \leq y.$$

Thus $x \Rightarrow y$ is the *largest* element z such that $z \wedge x \leq y$. So in particular, if we write $\neg x$ for $x \Rightarrow 0$, then $\neg x$ is the largest element z such that $x \Rightarrow z = 0$: it is called the *pseudocomplement* of x . A *Boolean algebra* is a Heyting algebra in which $\neg \neg x = x$ for all x , or equivalently, in which $x \vee \neg x = 1$ for all x .

If we think of the elements of a (complete) Heyting algebra as “truth values”, then $0, 1, \wedge, \vee, \neg, \Rightarrow, \perp, \top$ represent “true”, “false”, “and”, “or”, “not” and “implies”, “there exists” and “for all”, respectively. The laws satisfied by these operations in a general Heyting algebra correspond to those of *intuitionistic logic*. In Boolean algebras the counterpart of the law of excluded middle also holds.

A basic fact about *complete* Heyting algebras is that the following identity holds in them:

$$x \wedge \bigvee_{i \in I} y_i = \bigvee_{i \in I} (x \wedge y_i) \quad (*)$$

And conversely, in any complete lattice satisfying (*), defining the operation $\Rightarrow$ by $x \Rightarrow y = \perp \{z : z \wedge x \leq y\}$ turns it into a Heyting algebra.

In view of this result a complete Heyting algebra is frequently defined to be a complete lattice satisfying (*). A complete Heyting algebra is briefly called a *frame*.

Now we associate a frame with each cover scheme on P . First, we define $\widehat{P}$ to be the set of sieves in P partially ordered by inclusion: $\widehat{P}$ is then a frame—the *completion*² of P —in which joins and meets are just set-theoretic unions and intersections, and in which the operations $\Rightarrow$ and $\neg$ are given by

$$I \Rightarrow J = \{p : I \cap p \downarrow \subseteq J\} \quad \neg I = \{p : I \cap p \downarrow = \emptyset\}.$$

Given a cover scheme $\mathbf{C}$ on P , a sieve I in P is said to be *$\mathbf{C}$ -closed* if

² Writing $\mathbf{Lat}$ for the category of complete lattices and join preserving homomorphisms, $\widehat{P}$ is in fact the object in $\mathbf{Lat}$ freely generated by P .

$$\exists S \in \mathbf{C}(p)(S \subseteq I) \rightarrow p \in I.$$

We write $\widehat{\mathbf{C}}$ for the set of all $\mathbf{C}$ -closed sieves in P , partially ordered by inclusion.

Lemma. If $I \in \widehat{P}$, $J \in \widehat{\mathbf{C}}$, then $I \Rightarrow J \in \widehat{\mathbf{C}}$.

Proof. Suppose that $I \in \widehat{P}$, $J \in \widehat{\mathbf{C}}$, and $S \subseteq I \Rightarrow J$ with $S \in \mathbf{C}(p)$. Define $U = \{q \in I : \exists s \in S. q \leq s\}$. Then $U \subseteq J$. If $q \in I \cap p \downarrow$, then there is $T \in \mathbf{C}(q)$ for which $T \prec S$. Then for any $t \in T$, there is $s \in S$ for which $t \leq s$, whence $t \in U$. Accordingly $T \subseteq U \subseteq J$. Since J is a $\mathbf{C}$ -closed, it follows that $q \in J$. We conclude that $I \cap p \downarrow \subseteq J$, whence $p \in p \downarrow \subseteq I \Rightarrow J$. Therefore $I \Rightarrow J$ is $\mathbf{C}$ -closed. $\square$

It follows from the lemma that $\widehat{\mathbf{C}}$ is a frame. For clearly an arbitrary intersection of $\mathbf{C}$ -closed sieves is $\mathbf{C}$ -closed. So $\widehat{\mathbf{C}}$ is a complete lattice. In view of the lemma the implication operation in $\widehat{P}$ restricts to one in $\widehat{\mathbf{C}}$, making $\widehat{\mathbf{C}}$ a Heyting algebra, and so a frame.

Proposition 1. Suppose that $\mathbf{C}$ is a pretopology. Then (i) the bottom element of $\widehat{\mathbf{C}}$ is $\mathbf{0} = \{p : \emptyset \in \mathbf{C}(p)\}$, (ii) the $\mathbf{C}$ -closed sieve generated by a sieve A (that is, the smallest $\mathbf{C}$ -closed sieve containing A) is $\{p : \exists S \in \mathbf{C}(p). S \subseteq A\}$, (iii) the join operation in $\widehat{\mathbf{C}}$ is given by $\bigvee_{i \in I} J_i = (\bigcup_{i \in I} J_i)^*$. If $\mathbf{C}$ is a Grothendieck topology, then (iv) for any sieve $S \subseteq p \downarrow$, $p \in S^* \leftrightarrow S \in \mathbf{C}(p)$.

Proof. Suppose that $\mathbf{C}$ is a pretopology. Then $\mathbf{0}$ is a $\mathbf{C}$ -closed sieve. For it is easily seen to be a sieve; and it is $\mathbf{C}$ -closed because if $S \in \mathbf{C}(p)$ and $S \subseteq \mathbf{0}$, then $\emptyset \in \mathbf{C}(s)$ for each $s \in S$, whence $\emptyset = \bigcup_{s \in S} \emptyset \in \mathbf{C}(p)$, and so $p \in \mathbf{0}$. Finally, $\mathbf{0} \subseteq I$ for any $\mathbf{C}$ -closed sieve I , for if $\emptyset \in \mathbf{C}(p)$, then from $\emptyset \subseteq I$ we infer $p \in I$. This gives (i). As for (ii), suppose given a sieve A . Then $A \subseteq A^*$ follows from $\{p\} \in \mathbf{C}(p)$. A^* is a sieve, since if $p \in A^*$ and $q \leq p$, then there is $S \in \mathbf{C}(p)$ for which $S \subseteq A$, and $T \in \mathbf{C}(q)$ sharpening S ; clearly $T \subseteq A$ also, whence $q \in A^*$. And A^* is $\mathbf{C}$ -closed, since if $S \subseteq A^*$ with $S \in \mathbf{C}(p)$, then for each $s \in S$ there is $T_s \in \mathbf{C}(s)$ with $T_s \subseteq A$; it follows that $\bigcup_{s \in S} T_s \subseteq A$ and $\bigcup_{s \in S} T_s \in \mathbf{C}(p)$, whence $p \in A^*$. Part (iii) is an immediate consequence of (ii). Finally, if $\mathbf{C}$ is a Grothendieck topology and $S \subseteq p \downarrow$ is a sieve, then $p \in S^* \leftrightarrow \exists T \in \mathbf{C}(p). T \subseteq S \leftrightarrow S \in \mathbf{C}(p)$, i.e. (iv). $\square$

We observe parenthetically that $\widehat{\mathbf{Den}}$ is a Boolean algebra. To establish this it suffices to show that, for any $I \in \widehat{\mathbf{Den}}$, $\neg\neg I \subseteq I$. Now since $\emptyset \notin \mathbf{Den}(p)$, it follows from (i) of the proposition above that the bottom element of $\widehat{\mathbf{Den}}$ is $\emptyset$, so that, for any $I \in \widehat{\mathbf{Den}}$, $\neg I = \{p : I \cap p \downarrow = \emptyset\}$, whence $\neg\neg I = \{p : \forall q \leq p \exists r \leq q. r \in I\}$. But it easily checked that the defining condition for I to be a member of $\widehat{\mathbf{Den}}$ is precisely that, if $\forall q \leq p \exists r \leq q. r \in I$, then $p \in I$. That is, $\neg\neg I \subseteq I$.

Cover schemes on P correspond to certain self-maps on $\widehat{P}$ called (weak) nuclei. A weak nucleus on a frame H is a finite-meet-preserving map $j : H \rightarrow H$ such that $j(1) = 1$ and $a \leq$

$j(a)$ for any $a \in H$. If in addition $j(j(a)) \leq j(a)$ (so that $j(j(a)) = j(a)$) for all $a \in H$, j is called a *nucleus* on H .

Proposition 2. Let $\mathbf{C}$ be a cover scheme on P . For each $I \in \widehat{P}$ let I^* be the least $\mathbf{C}$ -closed sieve containing I . Then the map $k_{\mathbf{C}}: I \mapsto I^*$ is a nucleus on $\widehat{P}$.

Proof. Clearly $I \subseteq I^*$ and $I^{**} = I^*$. It remains to be shown that, for $I, J \in \widehat{P}$, $(I \cap J)^* = I^* \cap J^*$. Since $*$ is obviously inclusion-preserving, $(I \cap J)^* \subseteq I^* \cap J^*$. For the reverse inclusion, note first that $I \in \widehat{\mathbf{C}} \leftrightarrow I^* = I$. Given $I, J \in \widehat{P}$, define $K = I \Rightarrow (I \cap J)^*$. By the Lemma above, $K \in \widehat{\mathbf{C}}$, so that $K^* = K$. Now $J^* \subseteq K$ since

$$J \cap I \subseteq (I \cap J)^* \rightarrow J \subseteq [I \Rightarrow (I \cap J)^*] = K,$$

whence $J^* \subseteq K^* \subseteq K$. Similarly, if we define $L = K \Rightarrow (I \cap J)^*$, then $I^* \subseteq L$. It follows that

$$I^* \cap J^* \subseteq K \cap L = K \cap [K \Rightarrow (I \cap J)^*] \subseteq (I \cap J)^*. \quad \square$$

Inversely, any *weak* nucleus j on $\widehat{P}$ determines a *regular* cover scheme $\mathbf{D}_j$ on P , given by $S \in \mathbf{D}_j(p) \leftrightarrow p \in j(S)$.

Let us check that $\mathbf{D}_j$ is indeed a regular cover scheme. To do this it suffices to show that each $\mathbf{D}_j(p)$ is a filter of sieves and that, if $S \in \mathbf{D}_j(p)$, and $q \leq p$, then $S \cap q \downarrow \in \mathbf{D}_j(q)$. The first of these properties follows immediately from the fact that j preserves finite intersections, and the second from the observation that, if $S \in \mathbf{D}_j(p)$, and $q \leq p$, then $p \in j(S)$, so that $q \in j(S)$, and $q \in q \downarrow \subseteq j(q \downarrow)$, whence $q \in j(S) \cap j(q \downarrow) = j(S \cap q \downarrow)$, i.e. $S \cap q \downarrow \in \mathbf{D}_j(q)$.

When j is a *nucleus*, $\mathbf{D}_j$ is a *Grothendieck topology*. For under this assumption, if $S \in \mathbf{D}_j(p)$ and $T_s \in \mathbf{D}_j(s)$ for each $s \in S$, then $s \in j(T_s)$ for each $s \in S$, and it follows that

$$S \subseteq \bigcup_{s \in S} j(T_s) \subseteq j\left(\bigcup_{s \in S} T_s\right)$$

so that

$$p \in j(S) \subseteq j\left(j\left(\bigcup_{s \in S} T_s\right)\right) = j\left(\bigcup_{s \in S} T_s\right)$$

i.e., $\bigcup_{s \in S} T_s \in \mathbf{D}_j(p)$.

The correspondences $\mathbf{C} \mapsto k_{\mathbf{C}}$ and $j \mapsto \mathbf{D}_j$ between Grothendieck topologies on P and nuclei on $\widehat{P}$ are mutually inverse. For if $\mathbf{C}$ is a Grothendieck topology on P , then, by Proposition I (iv) we have

$$S \in \mathbf{D}_{k_{\mathbf{C}}}(p) \leftrightarrow p \in k_{\mathbf{C}}(S) = S^* \leftrightarrow S \in \mathbf{C}(p),$$

whence $\mathbf{D}_{k_c} = \mathbf{C}$. And, for a nucleus j on $\widehat{P}$, we have, using Proposition 1(ii),

$$\begin{aligned} k_{\mathbf{D}_j}(I) &= \text{least } \mathbf{D}_j\text{-closed sieve} \supseteq I \\ &= \{p : \exists S \in \mathbf{D}_j(p). S \subseteq I\} \\ &= \{p : \exists S \subseteq I. p \in j(S)\} \\ &= j(I), \end{aligned}$$

whence $k_{\mathbf{D}_j} = j$.

COVER SCHEMES AND FRAMES

The relationship between cover schemes on a preordered set and (weak) nuclei on its completion can be extended to cover schemes on partially ordered sets and general frames. Accordingly let H be a frame: we write $\bigvee, \bigwedge, \Rightarrow$ for the join, meet and implication operations, respectively, in H . The partially ordered set $(P, \leq)$ is said to be *dense* in H if P is a subset of H , the partial ordering on P is the restriction to P of that of H , and either of the two following equivalent conditions is satisfied: (i) for any $a \in H$, $a = \bigvee\{p : p \leq a\}$ (ii) for any $a, b \in H$, $a \leq b \Leftrightarrow \forall p[p \leq a \rightarrow p \leq b]$. The canonical example of a frame in which P is dense is the frame $\widehat{P}$ described in section I: here each $p \in P$ is identified with the $p\downarrow \in \widehat{P}$. $\widehat{P}$ is easily seen to have the property that in it, for any $S \subseteq P$, $p \leq \bigwedge S$ iff $p \in S$.

Now fix a frame H in which P is dense and a cover scheme $\mathbf{C}$ on P . An element $a \in H$ is said to *cover* an element $p \in P$ if there exists a cover S of p for which $\bigwedge S \leq a$. A $\mathbf{C}$ -element of H is one which dominates every element of P that it covers—that is, an element $a \in H$ satisfying

$$\forall p \in P[(\exists S \in \mathbf{C}(p)) \bigvee S \leq a \rightarrow p \leq a].$$

We write $H_{\mathbf{C}}$ for the set of all $\mathbf{C}$ -elements of H . It is evident that $H_{\mathbf{C}}$ is closed under the meet operation of H . Notice that $\mathbf{C}$ -elements and $\overline{\mathbf{C}}$ -elements coincide (recalling that $\overline{\mathbf{C}}$ is the normalization of $\mathbf{C}$.)

The canonical H -cover scheme $\mathbf{C}_H$ on P is given by

$$S \in \mathbf{C}_H(p) \Leftrightarrow \bigvee S = p.$$

Clearly $\mathbf{C}_H$ is a pretopology, and every element of H is a $\mathbf{C}_H$ -element.

Corresponding to the Lemma of §I, we have:

Lemma. If $a \in H, b \in H_{\mathbf{C}}$, then $a \Rightarrow b \in H_{\mathbf{C}}$.

Proof. Suppose $a \in H, b \in H_{\mathbf{C}}$, $S \in \mathbf{C}(p)$ and $\bigvee S \leq (a \Rightarrow b)$. Writing U for $\{q : q \leq a \ \& \ \exists s \in S(q \leq s)\}$, we have

$$\bigvee U \leq \bigvee\{s \wedge q : s \in S, q \leq a\} = \bigvee S \wedge \bigvee\{q : q \leq a\} = \bigvee S \wedge a \leq b.$$

Now if $q \leq p \wedge a$, there is $T \in \mathbf{C}(q)$ sharpening S . Then

$$t \in T \rightarrow t \leq a \ \& \ \exists s \in S(t \leq s),$$

so that $T \subseteq U$, and therefore $\bigvee T \leq \bigvee U \leq b$. Since $b \in H_C$, it follows that $q \leq b$. Hence $q \leq p \wedge a \rightarrow q \leq b$, so that $p \wedge a \leq b$ and $p \leq (a \Rightarrow b)$. We conclude that $(a \Rightarrow b) \in H_C$. $\square$

It follows from the lemma that H_C is itself a frame.

The nucleus on H associated with the cover scheme $\mathbf{C}$ on P is the map $j = k_C: H \rightarrow H$ defined by

$$j(a) = \bigwedge \{x \in H_C : a \leq x\}.$$

That j is a nucleus results from the following observations. Evidently j is order preserving, maps H onto H_C , is the identity on H_C , and satisfies $j(1) = 1$ and $a \leq j(a)$ for all $a \in A$. Also it is easily shown that $j(j(a)) = j(a)$. Finally, j preserves finite meets. For clearly $j(a \wedge b) \leq j(a) \wedge j(b)$ since j is order preserving. For the reverse inequality, consider first the element $u = (a \Rightarrow j(a \wedge b))$: this is, by the Lemma above, an element of H_C , so that $j(u) = u$. Also $j(b) \leq u$. For from $b \wedge a \leq j(a \wedge b)$ we deduce $b \leq (a \Rightarrow j(a \wedge b)) = u$, whence $j(b) \leq j(u) = u$. Similarly, $v = ((a \Rightarrow j(a \wedge b)) \Rightarrow j(a \wedge b))$ is an element of H_C and $j(a) \leq v$. Therefore

$$j(a) \wedge j(b) \leq v \wedge u \leq j(a \wedge b),$$

as required.

Notice that the nucleus associated with a cover scheme coincides with that associated with its normalization.

Accordingly we have shown that each cover scheme on P determines a nucleus on H . Conversely, we can show that any *weak nucleus* on H determines a cover scheme on P . For, starting with a weak nucleus j on H , define the map $\mathbf{D}_j$ on P by

$$\mathbf{D}_j(p) = \{S \subseteq p \downarrow : p \leq j(\bigvee S)\}.$$

Then $\mathbf{D}_j$ is a cover scheme on P . For suppose $q \leq p$ and $S \in \mathbf{D}_j(p)$. Then $q \leq p \leq j(\bigvee S)$; since $q \leq j(q)$ and j preserves finite meets, it follows that

$$q \leq j(q) \wedge j(\bigvee S) = j(q \wedge \bigvee S) = j(\bigvee \{s \wedge q : s \in S\}). \quad (*)$$

Now define $T \subseteq q \downarrow$ by

$$T = \{t : t \leq q \ \& \ \exists s \in S(t \leq s)\}.$$

We claim that T is a $(\mathbf{D}_j)$ cover of q sharpening S . That T sharpens S is evident from its definition. To see that it is a cover of q we observe that, if $s \in S$, then

$$s \wedge q = \bigvee \{t : t \leq s \wedge q\} = \bigvee \{t : t \leq s \ \& \ t \leq q\} \leq \bigvee T.$$

Therefore $\bigvee \{s \wedge q : s \in S\} \leq \bigvee T$, so that, by (*),

$$q \leq j(\bigvee \{s \wedge q : s \in S\}) \leq j(\bigvee T),$$

that is, T covers q .

When j is a *nucleus*, the associated cover scheme $\mathbf{D}_j$ is actually a *pretopology*. For in any case $\{p\} \in \mathbf{D}_j(p)$. Moreover, if j is a nucleus, $S \in \mathbf{D}_j(p)$ and $T_s \in \mathbf{D}_j(s)$ for each $s \in S$, then

$$p \leq j(\bigvee S) \leq j(\bigvee_{s \in S} j(\bigvee T_s)) \leq j(j(\bigvee_{s \in S} \bigvee T_s)) = j(\bigvee_{s \in S} T_s).$$

Therefore $\bigcup_{s \in S} T_s \in \mathbf{D}_j(p)$, and $\mathbf{D}_j$ is a pretopology.

Starting with a weak nucleus j , we obtain the corresponding cover scheme $\mathbf{D}_j$. The latter in turn determines a nucleus j^* given by

$$j^*(a) = \bigwedge \{x \in H_{\mathbf{D}_j} : a \leq x\}.$$

Now by definition, we have

$$\begin{aligned} x \in H_{\mathbf{D}_j} &\leftrightarrow \forall p[(\exists S \in \mathbf{D}_j(p)) \bigvee S \leq x \rightarrow p \leq x] \\ &\leftrightarrow \forall p[(\exists S \subseteq p \downarrow)(p \leq j(\bigvee S) \ \& \ \bigvee S \leq x) \rightarrow p \leq x] \quad (a) \\ &\leftrightarrow \forall p[p \leq j(x) \rightarrow p \leq x] \quad (b) \\ &\leftrightarrow j(x) = x. \end{aligned}$$

(To see the equivalence between (a) and (b), we need to establish the equivalence between $(\exists S \subseteq p \downarrow)(p \leq j(\bigvee S) \ \& \ \bigvee S \leq x)$ and $p \leq j(x)$. Clearly the first of these implies the second. As for the converse, if $p \leq j(x)$, then since $p \leq j(p)$, we have

$$p \leq j(x) \wedge j(p) = j(x \wedge p) = j(\bigvee S),$$

where $S = \{q : q \leq x \wedge p\}$. Then $S \subseteq p \downarrow$, $p \leq j(\bigvee S)$ and $\bigvee S \leq x$, and the first statement follows.) Accordingly

$$j^*(a) = 1 \{x \in H : a \leq x \ \& \ j(x) = x\}. \quad (*)$$

j^* is called the nucleus³ generated by the weak nucleus j ; it is easily deduced from (*) that when j is a nucleus, j^* and j coincide.

The generation of nuclei by weak nuclei can itself be seen as an instance of a nuclear operation. For consider the set $W(H)$ of all weak nuclei on H . When $W(H)$ is partially ordered pointwise in the obvious way, it becomes a frame with implications, joins, and meets given by the following specifications: $(j \Rightarrow k)(a) = \bigwedge_{b \geq a} (j(b) \Rightarrow k(b))$ and for $S \subseteq W(H)$,

$$(\bigvee S)(a) = \bigvee_{s \in S} s(a), \quad (\bigwedge S)(a) = \bigwedge_{s \in S} s(a).$$

The subset $N(H)$ of $W(H)$ consisting of all nuclei can be shown to be a sublocale (see [3]) of $N(H)$, that is, it is closed under arbitrary meets in $W(H)$ and is such that $(j \Rightarrow k) \in N(H)$ whenever $j \in W(H)$, $k \in N(H)$. That being the case, the map $\varphi: W(H) \rightarrow N(H)$ defined by

$$\varphi(j) = \bigwedge \{k \in N(H) : j \leq k\}$$

is a nucleus on $W(H)$, and it is easily shown that $\varphi(j) = j^*$. So the generation of nuclei by weak nuclei is precisely the action of the nucleus φ .

Now start with a cover scheme $\mathbf{C}$ on P , obtain the associated nucleus $k_{\mathbf{C}}$ on H , and consider its associated cover scheme $\mathbf{D}_{k_{\mathbf{C}}} = \mathbf{C}^*$ on P . By definition we have, for $S \subseteq p \downarrow$,

$$\begin{aligned} S \in \mathbf{C}^*(p) &\leftrightarrow p \leq j_{\mathbf{C}}(\bigvee S) \\ &\leftrightarrow p \leq 1 \{x \in H_{\mathbf{C}} : \bigvee S \leq x\} \\ &\leftrightarrow \forall x \in H_{\mathbf{C}} (\bigvee S \leq x \rightarrow p \leq x) \end{aligned}$$

Recalling the definition of $H_{\mathbf{C}}$, we see immediately that this last assertion is implied by $S \in \mathbf{C}(p)$, so that always $\mathbf{C}(p) \subseteq \mathbf{C}^*(p)$. The reverse inclusion will hold, and so $\mathbf{C}$ will coincide with $\mathbf{C}^*$, precisely when the cover scheme $\mathbf{C}$ is *saturated*, that is, coincides with its *saturate*, which we next proceed to define.

The $(H-)$ *saturate* $\tilde{\mathbf{C}}$ of a cover scheme $\mathbf{C}$ on P is defined by

$$\tilde{\mathbf{C}}(p) = \{S \subseteq p \downarrow : \forall x \in H_{\mathbf{C}} (\bigvee S \leq x \rightarrow p \leq x)\}.$$

Then $\tilde{\mathbf{C}}$ is a cover scheme. For if $S \in \tilde{\mathbf{C}}(p)$ and $q \leq p$, consider the subset T of $p \downarrow$ defined by

$$T = \{t \leq q : \exists s \in S (t \leq s)\}.$$

It is easily shown that $\bigvee T = (\bigvee S) \wedge q$. Now if $x \in H_{\mathbf{C}}$ and $1T \leq x$, then $\bigvee S \wedge q \leq x$, whence $\bigvee S \leq (q \Rightarrow x)$. But since x is an element of $H_{\mathbf{C}}$, so, by the lemma, is $q \Rightarrow x$, and since $S \in \tilde{\mathbf{C}}(p)$, it follows that $p \leq (q \Rightarrow x)$. Thus $q = p \wedge q \leq x$. Accordingly $T \in \tilde{\mathbf{C}}(q)$, and T obviously sharpens S . This shows that $\tilde{\mathbf{C}}$ is indeed a cover scheme.

³ It can be verified directly that j^* is a nucleus.

It is readily shown that any cover scheme associated with a nucleus (as opposed to a weak nucleus) is saturated. Observe that, when H is $\widehat{P}$, every coverage on P is saturated, since in that case H_C is $\widehat{\mathbf{C}}$ and so we have, using Proposition I.1 (iv),

$$S \in \widetilde{\mathbf{C}}(p) \leftrightarrow \forall I \in \widehat{\mathbf{C}}[S \subseteq I \rightarrow p \in I] \leftrightarrow p \in S^* \leftrightarrow S \in \mathbf{C}(p).$$

To sum up, each weak nucleus on H gives rise to a cover scheme on P and the cover scheme associated with a nucleus is saturated. Conversely, each cover scheme gives rise to a nucleus. This establishes mutually inverse correspondences between nuclei and saturated cover schemes.

Given $a \in H$, we define the nuclei j_a, j^a on H by

$$j_a(x) = a \vee x \quad j^a(x) = a \Rightarrow x.$$

The associated cover schemes (easily seen to be pretopologies) on P are given by:

$$\begin{aligned} S \in \mathbf{C}_a(p) &\leftrightarrow p \leq a \vee \bigvee S \\ S \in \mathbf{C}^a(p) &\leftrightarrow p \wedge a \leq \bigvee S. \end{aligned}$$

Notice that

$$\begin{aligned} p \leq a &\leftrightarrow \emptyset \in \mathbf{C}_a(p) \\ p \leq \neg a &\leftrightarrow \emptyset \in \mathbf{C}^a(p). \end{aligned}$$

The *double negation operation* $\neg\neg$ is a nucleus on H , whose associated cover scheme is precisely the dense cover scheme **Den** (which accordingly is also known as the *double negation cover scheme*). An argument similar to the one above showing that **Den** is a Boolean algebra establishes that H_{Den} is a Boolean algebra: it is in fact the complete Boolean algebra of $\neg\neg$ -closed elements of H .

Proposition. Let j be a weak nucleus on H . Then the following are equivalent: (a) $j(0) = 0$ (b) $j \leq \neg\neg$ (in the pointwise ordering of $\mathcal{W}(H)$) (c) $\emptyset \notin \mathbf{D}_j(p)$ for all p .

Proof. If $j \leq \neg\neg$ then $j0 \leq \neg\neg 0 = 0$. Conversely if $j0 = 0$ then, for any $a \in H$, $j(a) \wedge \neg a \leq j(a) \wedge j(\neg a) = j(a \wedge \neg a) = j0 = 0$. So $j(a) \leq \neg\neg a$. Finally, we have

$$\begin{aligned} j0 = 0 &\leftrightarrow 0 \in H_{\mathbf{D}_j} \leftrightarrow \forall p[(\exists S \in \mathbf{D}_j(p)) \bigvee S = 0 \rightarrow p = 0] \\ &\leftrightarrow \forall p[\neg(\exists S \in \mathbf{D}_j(p)) \bigvee S = 0] \\ &\leftrightarrow \forall p[\emptyset \notin \mathbf{D}_j(p)]. \quad \square \end{aligned}$$

COVER SCHEMES AND KRIPKE MODELS

We revert to the assumption that $(P, \leq)$ is a preordered set. Recall that a *presheaf* on P is an assignment, to each $p \in P$, of a set $\mathbf{F}(p)$ and to each pair (p, q) with $q \leq p$ of a map $\mathbf{F}_{pq}: \mathbf{F}(p) \rightarrow \mathbf{F}(q)$ in such a way that $\mathbf{F}_{pp}$ is the identity on $\mathbf{F}(p)$ and, for $r \leq q \leq p$, $\mathbf{F}_{pr} = \mathbf{F}_{qr} \circ \mathbf{F}_{pq}$. The set $V(\mathbf{F}) = \bigcup_{p \in P} \mathbf{F}(p)$ is called the *universe* of $\mathbf{F}$. A *Kripke model* based on P is a presheaf $\mathbf{K}$ for which $\mathbf{K}(p) \subseteq \mathbf{K}(q)$ whenever $q \leq p$ and each $\mathbf{K}_{pq}$ is the corresponding insertion map. Put more simply, a Kripke model based on P is a map $\mathbf{K}$ from P to a family of sets satisfying $\mathbf{K}(p) \subseteq \mathbf{K}(q)$ whenever $q \leq p$. A Kripke model $\mathbf{K}$ based on P may be regarded as a set “evolving” or “growing” over P : each $\mathbf{K}(p)$ may be thought of as the “state” of the evolving set $\mathbf{K}$ at “stage” p .

Now suppose that we are given a cover scheme $\mathbf{C}$ on P . A Kripke model $\mathbf{K}$ based on P satisfying

$$\mathbf{K}(p) = \bigcap_{s \in S} \mathbf{K}(s)$$

for any $p \in P$, $S \in \mathbf{C}(p)$ is said to be *compatible with $\mathbf{C}$* . (When P is directed downward, that is, whenever each pair of elements of P has a lower bound, and $\mathbf{C}$ is a pretopology on P , a Kripke model compatible with $\mathbf{C}$ is nothing other than a *$\mathbf{C}$ -sheaf*.)

Each Kripke model $\mathbf{K}$ based on P induces a Kripke model $\widetilde{\mathbf{K}}$ based on the free lower semilattice $\widetilde{P}$ generated by P by setting

$$\widetilde{\mathbf{K}}(\emptyset) = \emptyset \quad \widetilde{\mathbf{K}}(\{p_1, \dots, p_n\}) = \mathbf{K}(p_1) \cup \dots \cup \mathbf{K}(p_n).$$

If, further, $\mathbf{K}$ is compatible with the cover scheme $\mathbf{C}$ on P , then $\widetilde{\mathbf{K}}$ is compatible with the cover scheme on $\widetilde{P}$ induced by P (and hence also with the associated coverage on $\widetilde{P}$.) For suppose that $\mathbf{K}$ is in fact compatible with the cover scheme $\mathbf{C}$ on P . Given $F \in \widetilde{P}$, a nonempty subset $\{p_1, \dots, p_n\}$ of F , and $S_1 \in \mathbf{C}(p_1), \dots, S_n \in \mathbf{C}(p_n)$, we have

$$\begin{aligned}
\bigcap_{X \in S_1 \bullet \dots \bullet S_n} \widetilde{K}(X) &= \bigcap_{s_1 \in S_1, \dots, s_n \in S_n} \widetilde{K}(\{s_1, \dots, s_n\} \cup F) \\
&= \bigcap_{s_1 \in S_1, \dots, s_n \in S_n} (K(s_1) \cup \dots \cup K(s_n) \cup \bigcup_{p \in F} K(p)) \\
&= \bigcap_{s_1 \in S_1} K(s_1) \cup \dots \cup \bigcap_{s_n \in S_n} K(s_n) \cup \bigcup_{p \in F} K(p) \\
&= K(p_1) \cup \dots \cup K(p_n) \cup \bigcup_{p \in F} K(p) \\
&= \widetilde{K}(F).
\end{aligned}$$

Now suppose that the cover scheme $\mathbf{C}$ is in fact a *pretopology*. Then any Kripke model K based on P induces a Kripke model $K_{\mathbf{C}}$ also based on P but in addition compatible with $\mathbf{C}$ given by

$$K_{\mathbf{C}}(p) = \bigcup_{S \in \mathbf{C}(p)} \bigcap_{s \in S} K(s),$$

that is,

$$a \in K_{\mathbf{C}}(p) \leftrightarrow \exists S \in \mathbf{C}(p) \forall s \in S. a \in K(s).$$

We note that $K(p) \subseteq K_{\mathbf{C}}(p)$ for every p . It is easily checked that this defines a Kripke model over P ; let us confirm its compatibility with $\mathbf{C}$. It suffices to show that, given $S \in \mathbf{C}(p)$, we have $\bigcap_{s \in S} K_{\mathbf{C}}(s) \subseteq K_{\mathbf{C}}(p)$. Indeed, if $a \in \bigcap_{s \in S} K_{\mathbf{C}}(s)$, then for each $s \in S$ there is $T_s \in \mathbf{C}(s)$ with $a \in \bigcap_{t \in T_s} K(t)$. Writing $T = \bigcup_{s \in S} T_s$, we then have $T \in \mathbf{C}(p)$ and $a \in \bigcap_{t \in T} K(t)$. It follows that $a \in K_{\mathbf{C}}(p)$, as required.

Let us now examine some special cases. Let U be a subset of the universe V of K , and let U^* be the sieve $\{p: U \subseteq K(p)\}$. Now consider the Kripke model K^U compatible with $\mathbf{C}^{U^*}$ induced by K . For arbitrary $p \in P$, we have $S = p \downarrow \cap U^* \in \mathbf{C}^{U^*}(p)$ and $U \subseteq K(s) \subseteq K^U(s)$ for every $s \in S$. Hence $U \subseteq \bigcap_{s \in S} K^U(s) = K^U(p)$. Thus, under these conditions, U is a subset of every $K^U(p)$. In other words, the passage from K to K^U forces U to be included in the state of K^U at each stage. (Note that if $U^* = \emptyset$ then K^U assumes the constant value V .)

Again let U be a subset of V ; this time define U^+ to be the sieve $\{p: U \cap K(p) \neq \emptyset\}$. Now consider the Kripke model K_U compatible with $\mathbf{C}_{U^+}$ induced by K . Then for any p we have

$$U \cap K(p) \neq \emptyset \rightarrow p \in U^+ \rightarrow \emptyset \in \mathbf{C}_{U^+}(p) \rightarrow K_U(p) = \bigcap_{s \in \emptyset} K_U(s) = V.$$

That is, the passage from $\mathbf{K}$ to $\mathbf{K}_U$ forces each state of $\mathbf{K}_U$, apart from those already maximal, to be disjoint from U . (Notice that if $U^+ = P$, then $\mathbf{K}_U$ assumes the constant value V .)

We next turn to *logic* in Kripke models. Each Kripke model $\mathbf{K}$ based on P , with universe V , determines a map $\widehat{\mathbf{K}} : V \rightarrow \widehat{P}$ given by

$$\widehat{\mathbf{K}}(v) = \{p : v \in \mathbf{K}(p)\}.$$

This extends naturally to a homomorphism—also denoted by $\widehat{\mathbf{K}}$ —of the free Heyting algebra $F(V)$ generated by V into $\widehat{P}$. Think of the members of $F(V)$ as the formulas of intuitionistic propositional logic generated by the members of V regarded as propositional atoms. Introduce the familiar *forcing* relation $\Vdash_{\mathbf{K}}$ between P and $F(V)$ by defining

$$p \Vdash_{\mathbf{K}} \varphi \leftrightarrow p \in \widehat{\mathbf{K}}(\varphi). \quad (*)$$

Then the fact that $\widehat{\mathbf{K}} : F(V) \rightarrow \widehat{P}$ is a homomorphism of Heyting algebras translates into the usual rules for “Kripke semantics”, namely

- $p \Vdash_{\mathbf{K}} \varphi \wedge \psi \leftrightarrow p \Vdash_{\mathbf{K}} \varphi \ \& \ p \Vdash_{\mathbf{K}} \psi$
- $p \Vdash_{\mathbf{K}} \varphi \vee \psi \leftrightarrow p \Vdash_{\mathbf{K}} \varphi \ \text{or} \ p \Vdash_{\mathbf{K}} \psi$
- $p \Vdash_{\mathbf{K}} \varphi \Rightarrow \psi \leftrightarrow \forall q \leq p [q \Vdash_{\mathbf{K}} \varphi \rightarrow q \Vdash_{\mathbf{K}} \psi]$
- $p \Vdash_{\mathbf{K}} \neg \varphi \leftrightarrow \forall q \leq p \ q \nVdash_{\mathbf{K}} \varphi$

Equally, the map $\widehat{\mathbf{K}} : V \rightarrow \widehat{P}$ extends to a frame homomorphism (i.e., a map preserving top elements, $\wedge$, and $\perp$)—again denoted by $\widehat{\mathbf{K}}$ —of the *free frame* $\Phi(V)$ generated by V . Think of the members of $\Phi(V)$ as the formulas of *infinitary* intuitionistic propositional logic generated by the members of V regarded as propositional atoms. Such a formula φ is said to be *geometric* if it is generated from propositional atoms by applying just $\wedge$ and $\perp$. Introducing the forcing relation $\Vdash_{\mathbf{K}}$ between P and $\Phi(V)$ as in (*) above, the fact that $\widehat{\mathbf{K}} : \Phi(V) \rightarrow \widehat{P}$ is a frame homomorphism translates into the semantical rules for *geometric* formulas:

- $p \Vdash_{\mathbf{K}} \varphi \wedge \psi \leftrightarrow p \Vdash_{\mathbf{K}} \varphi \ \& \ p \Vdash_{\mathbf{K}} \psi$
- $p \Vdash_{\mathbf{K}} \bigvee_{i \in I} \varphi_i \leftrightarrow \exists i \in I \ p \Vdash_{\mathbf{K}} \varphi_i$

Now suppose that $\mathbf{C}$ is a pretopology on P . It is then easily seen that $\mathbf{K}$ is compatible with $\mathbf{C}$ iff each $\widehat{\mathbf{K}}(v)$ is a $\mathbf{C}$ -closed sieve. So if $\mathbf{K}$ is compatible with $\mathbf{C}$, the resulting map $\widehat{\mathbf{K}} : V \rightarrow \widehat{\mathbf{C}}$ can be extended to a homomorphism, which we shall denote by $\widehat{\mathbf{K}}_{\mathbf{C}}$, of $F(V)$ into $\widehat{\mathbf{C}}$. Introducing the forcing relation $\Vdash_{\mathbf{K}, \mathbf{C}}$ between P and $F(V)$ by

$$p \Vdash_{K, \mathbf{C}} \varphi \leftrightarrow p \in \widehat{K_{\mathbf{C}}}(\varphi), \quad (**)$$

we find that the fact that $\widehat{K_{\mathbf{C}}} : \Phi(V) \rightarrow \widehat{\mathbf{C}}$ is a homomorphism translates into the rules of “Beth-Kripke-Joyal” semantics for $\Vdash_{K, \mathbf{C}}$ (see, e.g., [4]), viz.,

- $p \Vdash_{K, \mathbf{C}} \varphi \wedge \psi \leftrightarrow p \Vdash_{K, \mathbf{C}} \varphi \ \& \ p \Vdash_{K, \mathbf{C}} \psi$
- $p \Vdash_{K, \mathbf{C}} \varphi \vee \psi \leftrightarrow \exists S \in \mathbf{C}(p) \forall s \in S [s \Vdash_{K, \mathbf{C}} \varphi \text{ or } s \Vdash_{K, \mathbf{C}} \psi]$
- $p \Vdash_{K, \mathbf{C}} \varphi \Rightarrow \psi \leftrightarrow \forall q \leq p [q \Vdash_{K, \mathbf{C}} \varphi \rightarrow q \Vdash_{K, \mathbf{C}} \psi]$
- $p \Vdash_{K, \mathbf{C}} \neg \varphi \leftrightarrow \forall q \leq p [q \Vdash_{K, \mathbf{C}} \varphi \rightarrow \emptyset \in \mathbf{C}(q)].$

We verify the second and fourth of these. We have, using Proposition I 1. (iii),

$$p \Vdash_{K, \mathbf{C}} \varphi \vee \psi \leftrightarrow p \in \widehat{K_{\mathbf{C}}}(\varphi \vee \psi) = \widehat{K_{\mathbf{C}}}(\varphi) \vee \widehat{K_{\mathbf{C}}}(\psi) \text{ (in } \widehat{\mathbf{C}})$$

$$\leftrightarrow \exists S \in \mathbf{C}(p). S \subseteq \widehat{K_{\mathbf{C}}}(\varphi) \cup \widehat{K_{\mathbf{C}}}(\psi)$$

$$\leftrightarrow \exists S \in \mathbf{C}(p) \forall s \in S. s \in \widehat{K_{\mathbf{C}}}(\varphi) \vee s \in \widehat{K_{\mathbf{C}}}(\psi)$$

$$\leftrightarrow \exists S \in \mathbf{C}(p) \forall s \in S [s \Vdash_{K, \mathbf{C}} \varphi \text{ or } s \Vdash_{K, \mathbf{C}} \psi].$$

Also, using Proposition I 1. (ii) we have

$$p \Vdash_{K, \mathbf{C}} \neg \varphi \leftrightarrow p \in \widehat{K_{\mathbf{C}}}(\neg \varphi) = \neg \widehat{K_{\mathbf{C}}}(\varphi) = (\widehat{K_{\mathbf{C}}}(\varphi) \Rightarrow \mathbf{0})$$

$$\leftrightarrow \forall q \leq p [q \in \widehat{K_{\mathbf{C}}}(\varphi) \rightarrow \emptyset \in \mathbf{C}(q)]$$

$$\leftrightarrow \forall q \leq p [q \Vdash_{K, \mathbf{C}} \varphi \rightarrow \emptyset \in \mathbf{C}(q)].$$

Since $\widehat{\mathbf{Den}}$ is a Boolean algebra it follows that, when K is compatible with $\mathbf{Den}$, $p \Vdash_{K, \mathbf{Den}} \varphi \vee \neg \varphi$ for every p , i.e., classical logic prevails in the Kripke model associated with $\widehat{K_{\mathbf{Den}}}$.

When K is compatible with $\mathbf{C}$, the map $\widehat{K} : V \rightarrow \widehat{\mathbf{C}}$ can be extended to a frame homomorphism, which we shall again denote by $\widehat{K_{\mathbf{C}}}$, of $\Phi(V)$ into $\widehat{\mathbf{C}}$. Introduce the forcing relation $\Vdash_{K, \mathbf{C}}$, now between P and $\Phi(V)$, by the same equivalence (**) as above. When $\mathbf{C}$ is a Grothendieck topology, a straightforward inductive argument shows that, for any geometric formula φ ,

$$p \Vdash_{K, \mathbf{C}} \varphi \leftrightarrow \exists S \in \mathbf{C}(p) \forall s \in S. s \Vdash_K \varphi. \quad (\dagger)$$

This may be applied to “force” any given set Σ of geometric formulas to become true in a Kripke model. For, starting with a Kripke model K , let A be the sieve $\{p: \forall \sigma \in \Sigma. p \Vdash_K \sigma\}$. Let $\mathbf{G}$ be the Grothendieck topology generated by the coverage $\mathbf{C}^A$: it is easily verified that a sieve $S \subseteq p \downarrow$ satisfies the same condition for membership in $\mathbf{G}(p)$ as in $\mathbf{C}^A(p)$, viz., $p \downarrow \cap A \subseteq S$. Now by $(\dagger)$ we have, for each $\sigma \in \Sigma$,

$$p \Vdash_{K, \mathbf{G}} \sigma \leftrightarrow \exists S \in \mathbf{G}(p) \forall s \in S. s \Vdash_K \sigma. \quad (\ddagger)$$

If we take S to be $p \downarrow \cap A$, then evidently $S \in \mathbf{G}(p)$ and $\forall s \in S. s \Vdash_K \sigma$. It now follows from $(\ddagger)$ that $p \Vdash_{K, \mathbf{G}} \sigma$ for every $\sigma \in \Sigma$ and every $p \in P$. In this sense $\mathbf{G}$ “forces” all the members of Σ to be true in the Kripke model associated with $\widehat{K}_{\mathbf{G}}$.

COVER SCHEMES AND FRAME-VALUED SET THEORY

We now set about relating what has been done so far to frame-valued set theory. Associated with each frame H is an H -valued model $V^{(H)}$ of (intuitionistic) set theory (see, e.g. [1] or [2]): we recall some of its principal features.

- Each of the members of $V^{(H)}$ —the H -valued sets—is a map from a subset of $V^{(H)}$ to H .
- Corresponding to each sentence σ of the language of set theory (with names for all elements of $V^{(H)}$) is an element $\llbracket \sigma \rrbracket = \llbracket \sigma \rrbracket^H \in H$ called its *truth value in $V^{(H)}$* . These “truth values” satisfy the following conditions. For $a, b \in V^{(H)}$,

$$\begin{aligned} \llbracket b \in a \rrbracket &= \bigvee_{c \in \text{dom}(a)} \llbracket b = c \rrbracket \wedge a(c) \\ \llbracket b = a \rrbracket &= \bigwedge_{c \in \text{dom}(a) \cup \text{dom}(b)} (\llbracket c \in b \rrbracket \leftrightarrow \llbracket c \in a \rrbracket) \\ \llbracket \sigma \wedge \tau \rrbracket &= \llbracket \sigma \rrbracket \wedge \llbracket \tau \rrbracket, \text{ etc.} \\ \llbracket \exists x \varphi(x) \rrbracket &= \bigvee_{a \in V^{(H)}} \llbracket \varphi(a) \rrbracket \\ \llbracket \forall x \varphi(x) \rrbracket &= \bigwedge_{a \in V^{(H)}} \llbracket \varphi(a) \rrbracket \end{aligned}$$

A sentence σ is *valid*, or *holds*, in $V^{(H)}$, written $V^{(H)} \models \sigma$, if $\llbracket \sigma \rrbracket = 1$, the top element of H . The truth value $\llbracket \sigma \rrbracket$ “measures” the degree or extent to which σ holds: the larger $\llbracket \sigma \rrbracket$ is, the “truer” σ is. In particular, when $\llbracket \sigma \rrbracket = 1$, σ is ‘universally’ or ‘absolutely’ true, and when $\llbracket \sigma \rrbracket = 0$, σ is ‘universally’ or ‘absolutely’ false.

- The axioms of intuitionistic Zermelo-Fraenkel set theory are valid in $V^{(H)}$. Accordingly the category $\text{Set}^{(H)}$ of sets constructed within $V^{(H)}$ is a topos: in fact $\text{Set}^{(H)}$ can be shown to be equivalent to the topos of canonical sheaves on H .
- There is a canonical embedding $x \mapsto \hat{x}$ of the universe V of sets into $V^{(H)}$ satisfying

$$\llbracket u \in \hat{x} \rrbracket = \bigvee_{y \in x} \llbracket u = \hat{y} \rrbracket \quad \text{for } x \in V, u \in V^{(H)}$$

$$x \in y \leftrightarrow V^{(H)} \models \hat{x} \in \hat{y}, \quad x = y \leftrightarrow V^{(H)} \models \hat{x} = \hat{y} \quad \text{for } x, y \in V$$

$$\varphi(x_1, \dots, x_n) \leftrightarrow V^{(H)} \models \varphi(\hat{x}_1, \dots, \hat{x}_n) \quad \text{for } x_1, \dots, x_n \in V \text{ and restricted } \varphi$$

(Here a formula φ is *restricted* if all its quantifiers are restricted, i.e. can be put in the form $\forall x \in y$ or $\exists x \in y$.)

It follows from the last of these assertions that the canonical representative $\hat{H}$ of H is a Heyting algebra in $V^{(H)}$. The *canonical prime filter* in $\hat{H}$ is the H -set Φ_H defined by

$$\text{dom}(\Phi_H) = \{\hat{a} : a \in H\}, \quad \Phi_H(\hat{a}) = a \quad \text{for } a \in H.$$

Clearly $V^{(H)} \models \Phi_H \subseteq \hat{H}$, and it is easily verified that

$$V^{(H)} \models \Phi_H \text{ is a (proper) prime filter}^4 \text{ in } \hat{H}.$$

It can also be shown that Φ_H is V -generic in the sense that, for any subset $A \subseteq H$,

$$V^{(H)} \models \bigvee \hat{A} \in \Phi_H \leftrightarrow \Phi_H \cap \hat{A} \neq \emptyset.$$

Moreover, for any $a \in H$ we have $\llbracket \hat{a} \in \Phi_H \rrbracket = a$, and in particular, for any sentence σ , $\llbracket \sigma \rrbracket = \llbracket \llbracket \sigma \rrbracket \in \Phi_H \rrbracket$. Thus $V^{(H)} \models \sigma \leftrightarrow V^{(H)} \models \llbracket \sigma \rrbracket \in \Phi_H$ —in this sense Φ_H is the filter of “true” sentences in $V^{(H)}$.

This suggests that we define a *truth set* in $V^{(H)}$ to be an H -set F for which

$$V^{(H)} \models F \text{ is a filter in } \hat{H} \text{ such that } F \supseteq \Phi_H.$$

There is a natural bijective correspondence between truth sets in $V^{(H)}$ and weak nuclei on H . With each weak nucleus j on H we associate the H -set T_j defined by $\text{dom}(T_j) = \text{dom}(\Phi_H)$ and $T_j(\hat{a}) = j(a)$ for $a \in H$. It is easily verified that T_j is a truth set—the requirement that T_j be a filter corresponds exactly to the condition that j preserve finite meets and that it contain

⁴ We recall that a filter F in a lattice is *prime* if $x \vee y \in F$ implies $x \in F$ or $y \in F$.

Φ_H to the condition that j satisfy $a \leq j(a)$. Inversely, given a truth set F in $V^{(H)}$, we define the map $j_F: H \rightarrow H$ by $j_F(a) = \llbracket \hat{a} \in F \rrbracket$. Again, it is readily verified that j_F is a weak nucleus on H . These correspondences are evidently mutually inverse and in fact establish an isomorphism between the frame $W(H)$ of weak nuclei on H and the internal frame of filters in $\widehat{H}$ containing Φ_H . Under this isomorphism *nuclei* correspond precisely to *reflexive truth sets*, that is, truth sets satisfying the additional condition (evidently met by Φ_H)

$$V^{(H)} \models \llbracket \hat{a} \in F \rrbracket \in F \rightarrow \hat{a} \in F.$$

It is of interest to examine the familiar case in which H is a complete *Boolean algebra* B . In this case the canonical prime filter Φ_B is an *ultrafilter* in $\widehat{B}$, so that, in $V^{(B)}$, the only filters in $\widehat{B}$ containing Φ_B —the only truth sets—are Φ_B itself and $\widehat{B}$. It follows that, for truth sets F and G in $V^{(B)}$

$$V^{(B)} \models F = G \leftrightarrow [\hat{0} \in F \leftrightarrow \hat{0} \in G].$$

In other words, the truth value $\llbracket \hat{0} \in F \rrbracket$, which can be an arbitrary member of B , determines the identity of F . This means that truth sets in $V^{(B)}$, and so equally weak nuclei on B , are in bijective correspondence with the members of B . In fact it is readily shown directly that any weak nucleus on a Boolean algebra B is of the form j_a for some $a \in B$. For given a weak nucleus j on B , observe: $\neg x \leq j(\neg x)$, whence $\neg j(\neg x) \leq \neg \neg x = x$. Also $j(x) \wedge j(\neg x) = j(x \wedge \neg x) = j(0)$, whence $j(x) \leq j(\neg x) \Rightarrow j(0) = \neg j(\neg x) \vee j(0) \leq x \vee j(0)$. But clearly $x \vee j(0) \leq j(x)$, so that $j(x) = x \vee j(0)$.

Consider now the special case in which H is the completion $\widehat{P}$ of a preordered set P . We have already established a bijective correspondence between Grothendieck topologies on P and nuclei on $\widehat{P}$. This leads in turn to a bijective correspondence between Grothendieck topologies on P and reflexive truth sets in $V^{(\widehat{P})}$. Explicitly, this correspondence assigns to each Grothendieck topology $\mathbf{C}$ on P the reflexive truth set $T_{\mathbf{C}}$ in $V^{(\widehat{P})}$ given by $T_{\mathbf{C}}(S) = S^*$ for $S \in \widehat{P}$, and to each reflexive truth set F in $V^{(\widehat{P})}$ the Grothendieck topology $\mathbf{C}_F$ on P defined by $S \in \mathbf{C}_T(p) \leftrightarrow p \in \llbracket \hat{S} \in T \rrbracket$.

The topos $\mathbf{Sd}^{(\widehat{P})}$ of sets in $V^{(\widehat{P})}$ is, as we have observed, equivalent to the topos of canonical sheaves on $\widehat{P}$, which is itself, as is well known, equivalent to the topos $\mathbf{Sd}^{P^{op}}$ of presheaves on P . Moreover, Grothendieck topologies on P are known (see [4]) to correspond bijectively to internal Lawvere-Tierney topologies—that is, internal nuclei—on the truth-value object Ω in $\mathbf{Sd}^{P^{op}}$. How this fact related to the representation of Grothendieck topologies as reflexive truth sets in $V^{(\widehat{P})}$? It turns out that in a general $V^{(H)}$ there is a natural bijection between truth sets/reflexive truth sets and weak nuclei/nuclei on $\Omega = \{u: u \subseteq \hat{1}\}$.

The representation of Grothendieck topologies as truth sets in $V^{(H)}$, while equivalent to that through Lawvere-Tierney topologies, seems especially perspicuous.

The *forcing* relation $\Vdash_P$ in $V^{(\bar{P})}$ between sentences and elements of P is defined by

$$p \Vdash_P \sigma \leftrightarrow p \in \llbracket \sigma \rrbracket^{\bar{P}}.$$

Note that we then have

$$\llbracket \sigma \rrbracket^{\bar{P}} = \{p : p \Vdash_P \sigma\}.$$

$\Vdash_P$ satisfies the usual rules governing Kripke semantics for predicate sentences, viz.,

- $p \Vdash_P \phi \wedge \psi \leftrightarrow p \Vdash_P \phi \ \& \ p \Vdash_P \psi$
- $p \Vdash_P \phi \vee \psi \leftrightarrow p \Vdash_P \phi \ \text{or} \ p \Vdash_P \psi$
- $p \Vdash_P \phi \rightarrow \psi \leftrightarrow \forall q \leq p [q \Vdash_P \phi \rightarrow q \Vdash_P \psi]$
- $p \Vdash_P \neg \phi \leftrightarrow \forall q \leq p \ q \not\Vdash_P \phi$
- $p \Vdash_P \forall x \phi \leftrightarrow p \Vdash_P \phi(a)$ for every $a \in V^{(\bar{P})}$
- $p \Vdash_P \exists x \phi \leftrightarrow p \Vdash_P \phi(a)$ for some $a \in V^{(\bar{P})}$.

We note also that $\Vdash_P$ is *persistent* in the sense that, if $p \Vdash_P \phi$ and $q \leq p$, then $q \Vdash_P \phi$.

If $\mathbf{C}$ be a pretopology on P , the forcing relation $\Vdash_{\mathbf{C}}$ in the model $V^{(\bar{\mathbf{C}})}$ is similarly defined by

$$p \Vdash_{\mathbf{C}} \sigma \leftrightarrow p \in \llbracket \sigma \rrbracket_{\bar{\mathbf{C}}}.$$

As for Kripke models, this relation can be shown to satisfy the rules of Beth-Kripke-Joyal semantics, viz.,

- $p \Vdash_{\mathbf{C}} \phi \wedge \psi \leftrightarrow p \Vdash_{\mathbf{C}} \phi \ \& \ p \Vdash_{\mathbf{C}} \psi$
- $p \Vdash_{\mathbf{C}} \phi \vee \psi \leftrightarrow \exists S \in \mathbf{C}(p) \ \forall s \in S [s \Vdash_{\mathbf{C}} \phi \ \text{or} \ s \Vdash_{\mathbf{C}} \psi]$
- $p \Vdash_{\mathbf{C}} \phi \Rightarrow \psi \leftrightarrow \forall q \leq p [q \Vdash_{\mathbf{C}} \phi \rightarrow q \Vdash_{\mathbf{C}} \psi]$
- $p \Vdash_{\mathbf{C}} \neg \phi \leftrightarrow \forall q \leq p [q \Vdash_{\mathbf{C}} \phi \rightarrow \emptyset \in \mathbf{C}(p)]$
- $p \Vdash_{\mathbf{C}} \forall x \phi \leftrightarrow p \Vdash_{\mathbf{C}} \phi(a)$ for every $a \in V^{(\bar{\mathbf{C}})}$
- $p \Vdash_{\mathbf{C}} \exists x \phi \leftrightarrow \exists S \in \mathbf{C}(p) \ \forall s \in S \ s \Vdash_{\mathbf{C}} \phi(a)$ for some $a \in V^{(\bar{\mathbf{C}})}$.

POTENTIAL APPLICATIONS OF COVER SCHEMES, KRIPKE MODELS, AND FRAME-VALUED SET THEORY IN SPACETIME PHYSICS

In spacetime physics any set C of events—a *causal set*—is taken to be partially ordered by the relation $\leq$ of *possible causation*: for $p, q \in C$, $p \leq q$ means that q is in p 's future light

cone. In her groundbreaking paper [5] Fotini Markopoulou proposes that the causal structure of spacetime itself be represented by “sets evolving over C ”—that is, in essence, by the topos $\mathbf{Set}^C$ of presheaves on C^{op} . To enable what she has done to be the more easily expressed within the framework presented here, we will reverse the causal ordering, that is, C will be replaced by C^{op} , and the latter written as P —which will, moreover, be required to be no more than a *preordered* set. Thus P is a set of events preordered by the relation $\leq$, where $p \leq q$ is intended to mean that p is in q ’s future light cone—that q *could* be the cause of p . In requiring that $\leq$ be no more than a preordering—in dropping, that is, the antisymmetry of $\leq$ —we are, in physical terms, allowing for the possibility that the universe is of Gödelian type, containing closed timelike lines.

Specifically, then, we fix a preordered set $(P, \leq)$, which we shall call the *universal causal set*; its members will be called *events* and $p \leq q$ understood to mean that p is in q ’s causal future, or q ’s future light cone, in short, that p is a possible *effect* of q . (Thus, for each event p , the set $p \downarrow$ is p ’s future light cone.) Markopoulou, in essence, suggests that viewing the universe “from the inside” amounts to placing oneself within the topos of presheaves or “evolving universe” $\mathbf{Set}^{P^{\text{op}}}$. Since, as we have already observed, $\mathbf{Set}^{P^{\text{op}}}$ is equivalent to the topos of sets in $V^{(\hat{P})}$, Markopoulou’s proposal may be effectively realized by working within $V^{(\hat{P})}$. Let us do so, writing for simplicity H for $\hat{P}$: we think of $V^{(H)}$ as an *evolving universe*, and describing what the universe looks like “from the inside” will then amount to reporting the view from $V^{(H)}$. Each sentence σ of the language of set theory will be construed as an *assertion* concerning the evolving universe $V^{(H)}$.

The fact that each truth value $\llbracket \sigma \rrbracket^H$ (which we shall normally abbreviate to $\llbracket \sigma \rrbracket$) is a sieve in P —that is, satisfies $p \in \llbracket \sigma \rrbracket$ and $q \leq p \rightarrow q \in \llbracket \sigma \rrbracket$ —may be understood as asserting that truth values in the evolving universe are “closed under potential effects”, or “causally closed”.

The forcing relation $\Vdash_P$ (which we will usually write simply as $\Vdash$) defined in the previous section now links events p and assertions σ : $p \Vdash \sigma$ will be taken to mean that σ *holds* as a result of (the occurrence of) event p , or that p *induces* the assertion σ to hold. The persistence of $\Vdash$ —i.e. the fact that, if $p \Vdash \sigma$ and $q \leq p$, then $q \Vdash \sigma$ —amounts to the observation that, once an event p deduces an assertion to hold, that assertion continues to hold throughout p ’s causal future⁵.

Define the set $K \in V^{(H)}$ by $\text{dom}(K) = \{\hat{p} : p \in P\}$ and $K(\hat{p}) = p \downarrow$. Then, in $V^{(H)}$, K is a subset of $\hat{P}$ and for $p \in P$, $\llbracket \hat{p} \in K \rrbracket = p \downarrow$. K is the counterpart in $V^{(H)}$ of the “evolving” set *Past* Markopoulou defines by $\text{Past}(p) = p \downarrow$, with insertions as transition maps. ($\hat{P}$, incidentally, is the $V^{(H)}$ -counterpart of the constant presheaf on P with value P which

⁵ It follows that assertions must be taken as being implicitly in the past tense: “such and such *was* the case”.

Markopoulou calls *World*.) Accordingly the “causal past” of any “event” p is represented by the truth value in $V^{(H)}$ of the statement $\hat{p} \in K$. The fact that, for any $p, q \in P$ we have

$$q \Vdash_P \hat{p} \in K \leftrightarrow q \leq p$$

may be construed as asserting that the events in the causal future of an event p are precisely those forcing (the canonical representative of) p to be a member of K . For this reason we shall call K the causal set in $V^{(H)}$.

If we identify each $p \in P$ with $p \downarrow \in H$, P may then be regarded as a subset of H so that, in $V^{(H)}$, $\hat{P}$ is a subset of $\hat{H}$. It is not hard to show that, in $V^{(H)}$, K generates the canonical prime filter Φ_H in $\hat{H}$. Using the V -genericity of Φ_H , and the density of P in H , one can show that $\llbracket \sigma \rrbracket = \llbracket \exists p \in K. p \leq \llbracket \sigma \rrbracket \rrbracket$, so that, with moderate abuse of notation,

$$V^{(H)} \models [\sigma \leftrightarrow \exists p \in K. p \Vdash \sigma].$$

That is, in $V^{(H)}$, a sentence holds precisely when it is forced to do so at some “causal past stage” in K . This establishes the centrality of the causal set K —and, correspondingly, that of the “evolving” set *Past*—in determining the truth of sentences “from the inside”, that is, inside the universe $V^{(H)}$.

Markopoulou also considers the complement of *Past*. In the present setting, this is the $V^{(H)}$ -set $\neg K$ —the complement of the causal set K —for which $\llbracket \hat{p} \in \neg K \rrbracket = \llbracket p \notin K \rrbracket = \neg p \downarrow$. Markopoulou calls (*mutatis mutandis*) the events in $\neg p \downarrow$ those *beyond p 's causal horizon*, in that no observer at p can ever receive “information” from any event in $\neg p \downarrow$. Since clearly we have

$$q \Vdash \hat{p} \in \neg K \leftrightarrow q \in \neg p \downarrow, \quad (\star)$$

it follows that the events beyond the causal horizon of an event p are precisely those forcing (the canonical representative of) p to be a member of $\neg K$. In this sense $\neg K$ reflects, or “measures” the causal structure of P .

In this connection it is natural to call $\neg \neg p \downarrow = \{q : \forall r \leq q \exists s \leq r. s \leq p\}$ the *causal horizon* of p : it consists of those events q for which an observer placed at p could, in its future, receive information from any event in the future of an observer placed at q . Since

$$q \Vdash \hat{p} \in \neg \neg K \leftrightarrow q \in \neg \neg p \downarrow,$$

it follows that *the events within the causal horizon of an event are precisely those forcing (the canonical representative of) p to be a member of $\neg \neg K$.*

It is easily shown that $\neg K$ is empty (i.e. $V^{(H)} \models \neg K = \emptyset$) if and only if P is *directed downwards* in the sense that for any $p, q \in P$ there is $r \in P$ for which $r \leq p$ and $r \leq q$; that is,

if the future light cones of any pair of events have nonempty intersection or “overlap”. This holds in the case, considered by Markopoulou, of *discrete Newtonian time evolution*—in the present setting, the case in which P is the opposite $\mathbb{N}^{op}$ of the totally ordered set $\mathbb{N}$ of natural numbers. Here the corresponding complete Heyting algebra H is the family of all downward-closed sets of natural numbers. Interestingly, in this case, the causal set K is *neither finite nor actually infinite*.

To see this, first note that, for any natural number n , we have, $\llbracket \neg(\hat{n} \in \neg K) \rrbracket = \mathbb{N}$. It follows that $V^{(H)} \models \neg\neg\forall n \in \hat{\mathbb{N}}. n \in K$. But, working in $V^{(H)}$, if $\forall n \in \hat{\mathbb{N}}. n \in K$, then K is not finite, so if K is finite, then $\neg\forall n \in \hat{\mathbb{N}}. n \in K$, and so $\neg\neg\forall n \in \hat{\mathbb{N}}. n \in K$ implies the non-finiteness of K .

But, in $V^{(H)}$, K is not actually infinite. For (again working in $V^{(H)}$), if K were actually infinite (i.e., if there existed an injection of $\hat{\mathbb{N}}$ into K), then the statement

$$\forall x \in K \exists y \in K. x > y$$

would also have to hold in $V^{(H)}$. But calculating that truth value gives:

$$\begin{aligned} \llbracket \forall x \in K \exists y \in K. x > y \rrbracket &= \bigcap_{m \in \mathbb{N}^{op}} [m \downarrow \Rightarrow \bigcup_{n \in \mathbb{N}^{op}} n \downarrow \cap \llbracket \hat{m} > \hat{n} \rrbracket] \\ &= \bigcap_m [m \downarrow \Rightarrow \bigcup_{n < m} n \downarrow] \\ &= \bigcap_m [m \downarrow \Rightarrow (m+1) \downarrow] \\ &= \bigcap_m (m+1) \downarrow \\ &= \emptyset \end{aligned}$$

So $\forall x \in K \exists y \in K. x > y$ is false in $V^{(H)}$ and therefore K is not actually infinite.

In other words, in evolving Newtonian spacetime, the set K representing past time is potentially, but not actually infinite: this is, in essence, what Kant asserted of time.

In order to formulate an observable causal *quantum theory* Markopoulou considers the possibility of introducing a *causally evolving algebra of observables*. This amounts to specifying a presheaf of C^* -algebras on P , which, in the present framework, corresponds to specifying a set A in $V^{(H)}$ satisfying

$$V^{(H)} \models A \text{ is a } C^*\text{-algebra.}$$

The “internal” C^* -algebra A is then subject to the intuitionistic internal logic of $V^{(H)}$: any theorem concerning C^* -algebras—provided only that it be constructively proved—automatically applies to A . Reasoning with A is more direct and simpler than reasoning with $\mathcal{A}$.

This same procedure of “internalization” can be performed with any causally evolving object: each such object of type T corresponds to a set S in $V^{(H)}$ satisfying

$$V^{(H)} \models S \text{ is of type } T.$$

Internalization may also be applied in the case of the presheaves *Antichains* and *Graphs* considered by Markopoulou. Here, for each event p , *Antichains*(p) consists of all sets of causally unrelated events in *Past*(p), while *Graphs*(p) is the set of all graphs supported by elements of *Antichains*(p). In the present framework *Antichains* is represented by the $V^{(H)}$ -set $Anti = \{X \subseteq \hat{P} : X \text{ is an antichain}\}$ and *Graphs* by the $V^{(H)}$ -set $Grph = \{G : \exists X \in A . G \text{ is a graph supported by } X\}$. Again, both *Anti* and *Grph* can be readily handled using the internal intuitionistic logic of $V^{(H)}$.

Finally let us examine the role of cover schemes on causal sets. Suppose we are given a cover scheme **C** on the universal causal set P . Each **C**-cover of an event p may be thought of as a “sampling” of the events in p ’s causal future, a “survey” of p ’s potential effects—in a word, a *survey of p* . Using this language the defining condition (**Cov**) for cover schemes laid down in section I becomes: *for any survey S of a given event p , and any event q which is a possible effect of p , there exists a survey of q each event in which is the possible effect of some event in S .*

As we have seen, cover schemes may be used to force certain conditions to prevail in the associated models. Let us consider, for example, the cover scheme **Den** in P . We know that the associated frame **Den** is a Boolean algebra—let us denote it by B . The corresponding causal set K_B in $V^{(B)}$ then has the property

$$\llbracket \hat{p} \in K_B \rrbracket = \neg\neg p \downarrow;$$

so that,

$$q \Vdash_B \hat{p} \in K_B \leftrightarrow q \in \neg\neg p \downarrow$$

$$\leftrightarrow q \text{ is in } p\text{'s causal horizon.}$$

Comparing this with (★) above, we see that moving to the universe $V^{(B)}$ —“Booleanizing” it, so to speak—amounts to replacing causal futures by causal horizons. When P is linearly ordered, as for example in the case of Newtonian time, the causal horizon of any event coincides with the whole of P , B is the two-element Boolean algebra **2**, so that $V^{(B)}$ is just the universe V of “static” sets. In this case, then, the effect of “Booleanization” is to render the universe timeless.

The universes associated with the cover schemes C^A and C_A seem also to have a rather natural physical meaning. Consider, for instance the case in which A is the sieve $p \downarrow$ —the causal future of p . In the associated universe $V^{(C^A)}$ the corresponding causal set K^A satisfies

$$\llbracket \hat{q} \in K^A \rrbracket = \text{least } C^A\text{-closed sieve containing } q$$

so that, in particular

$$\begin{aligned} \llbracket \hat{p} \in K^A \rrbracket &= \text{least } \mathbf{C}^A\text{-closed sieve containing } p \\ &= P. \end{aligned}$$

This means that, for every event q ,

$$q \Vdash_{\mathbf{C}^A} \hat{p} \in K^A.$$

Comparing this with $(\star)$, we see that in $V^{(\mathbf{C}^A)}$ that every event has been “forced” into p ’s causal future: in short, that p now marks the “beginning” of the universe as viewed from inside $V^{(\mathbf{C}^A)}$.

Similarly, we find that the causal set K_A in the universe $V^{(\mathbf{C}_A)}$ has the property

$$q \leq p \rightarrow \forall r [r \Vdash_{\mathbf{C}_A} \hat{q} \in \neg K_A].$$

a comparison with $(\star)$ above reveals that, in $V^{(\mathbf{C}^A)}$, every event—including p itself—has been placed beyond p ’s causal horizon. In effect, the event p has been obliterated, effaced from the universe—like the extraordinary events in H.G. Wells’ *The Man Who Could Work Miracles*, the event p never occurred!

As a final possibility consider the universe $V^{(\tilde{P})}$ associated with the free lower semilattice $\tilde{P}$ generated by P . In this case the elements of $\tilde{P}$ are finite sets of events, preordered by the relation $\sqsubseteq$: for $F, G \in \tilde{P}$, $F \sqsubseteq G$ iff every event in G is in the causal past of an event in F .

The empty set of events is the top element of $\tilde{P}$. The causal set $\tilde{K}$ in $V^{(\tilde{P})}$ has the property that its complement $\neg \tilde{K}$ is empty (so that, in this universe, the light cones of any pair of “events” overlap) and $\hat{\emptyset}$ is an initial event in the sense that $F \Vdash_{\tilde{P}} \hat{\emptyset} \in \tilde{K}$ for every “event”

F . In this case passage to the new universe $V^{(\tilde{P})}$ preserves the original causal relations in the sense that

$$\{q\} \Vdash_{\tilde{P}} \{\hat{p}\} \in \tilde{K} \leftrightarrow q \Vdash_P \hat{p} \in K.$$

In other words, in passing to the new universe the initial event $\hat{\emptyset}$ and the new light cone overlaps have been “freely adjoined” to the original universe.

REFERENCES

- [1] Bell, J. L. *Boolean-Valued Models and Independence Proofs in Set Theory*. Oxford University Press, 1977.

- [2] Grayson, R. J. Heyting-valued models for intuitionistic set theory. In: Applications of Sheaves. Springer Lecture Notes in Mathematics **753**, 1979.
- [3] Johnstone, P.T. Stone Spaces. Cambridge University Press, 1982.
- [4] Mac Lane, S. and Moerdijk, I. Sheaves in Geometry and Logic. Springer-Verlag, 1992.
- [5] Markopoulou, F. The internal description of a causal set: What the universe looks like from the inside. arXiv:gr-qc/9811053 v2 18 Nov 1999.
- [6] Rosenthal, K. I. Quantales and their Applications. Longman, 1990.