Lectures on the Foundations of Mathematics

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THE PHILOSOPHY OF MATHEMATICS.

THE CLOSE CONNECTION BETWEEN mathematics and philosophy has long been recognized by practitioners of both disciplines. The apparent timelessness of mathematical truth, the exactness and objective nature of its concepts, its applicability to the phenomena of the empirical world—explicating such facts presents philosophy with some of its subtlest problems. Let me begin by reminding you of some celebrated past attempts made by philosophers and mathematicians to explicate the nature of mathematics.

Classical Views on the Nature of Mathematics.

Plato (*c.*428–347 B.C.) included mathematical entities—numbers and the objects of pure geometry such as points, lines, and circles—among the well-defined, independently existing eternal objects he called *Forms*. It is the fact that mathematical statements refer to these definite Forms that enables such statements to be true (or false). Mathematical statements about the empirical world are true to the extent that sensible objects resemble or manifest the corresponding Forms. Plato considered mathematics not as an idealization of aspects of the empirical world, but rather as *a direct description of reality*, that is, the world of Forms as apprehended by reason.

Plato's pupil and philosophical successor *Aristotle* (384–322 B.C.), on the other hand, rejected the notion of Forms being separate from empirical objects, and maintained instead that *the Forms constitute parts of objects.* Forms are grasped by the mind through a process of *abstraction* from sensible objects, but they do not thereby attain an autonomous existence detached from these latter. Mathematics arises from this process of abstraction; its subject matter is the body of idealizations engendered by this process; and mathematical rigour arises directly from the simplicity of the properties of these idealizations. Aristotle rejected the concept of *actual* (or completed) *infinity*, admitting only *potential infinity*, to wit, that of a totality which, while finite at any given time, grows beyond any preassigned bound, e.g. the sequence of natural numbers or the process of continually dividing a line.

Leibniz (1646-1716) divided all true propositions, including those of mathematics, into two types: *truths of fact*, and *truths of reason*, also known as *contingent* and *analytic* truths, respectively. According to Leibniz, true mathematical propositions are truths of reason and their truth is therefore just logical truth: their denial would be logically impossible. Mathematical propositions do not have a special "mathematical" content—as they did for Plato and Aristotle—and so true mathematical propositions are true in all possible worlds, that is, they are *necessarily* true. [On the other hand, empirical propositions containing mathematical terms such as 2 *cats* + 3 *cats* = 5 *cats* are true because they hold in the *actual* world, and, according to Leibniz, this is the case only because the actual world is the "best possible" one. Thus, despite the fact that 2 + 3 = 5 is true in all possible worlds, 2 *cats* + 3 *cats* = 5 *cats* could be false in some world.] Leibniz attached particular importance to the *symbolic* aspects of mathematical reasoning. His program of developing a *characteristica universalis* centered around the idea of devising a method of representing thoughts by means of arrangements of characters and signs in such a way that relations among thoughts are reflected by similar relations among their representing signs. Leibniz may be considered a forerunner of both the logicists and, in some sense, the formalists.

Immanuel Kant (1724–1804) introduced a new classification of (true) propositions: *analytic*, and nonanalytic, or *synthetic*, which he further subdivided into empirical, or *a posteriori*, and nonempirical, or *a priori*. Synthetic a priori propositions are not dependent on sense perception, but are necessarily true in the sense that, if *any* propositions about the empirical world are true, *they* must be true. According to Kant, mathematical propositions are synthetic a priori because they ultimately involve reference to *space and time*. Kant attached particular importance to the idea of *a priori construction* of mathematical objects. He distinguishes sharply between mathematical *concepts* which, like noneuclidean geometries, are merely internally consistent, and mathematical *objects* whose construction is made possible by the fact that perceptual space and time have a certain inherent structure. Thus, on this reckoning, $2 + 3 = 5$ is to be regarded ultimately as an assertion about a certain construction, carried out in time and space, involving the succession and collection of units. The *logical possibility* of an arithmetic in which $2 + 3 \neq 5$ is not denied; it is only asserted that the correctness of such an arithmetic would be incompatible with the structure of perceptual space and time. So for Kant the propositions of pure arithmetic and geometry are necessary, but *synthetic a priori*. *Synthetic,* because they are ultimately about the structure of space and time, revealed through the objects that can be constructed there. And *a priori* because the structure of space and time provides the universal preconditions rendering possible the perception of such objects. On this reckoning, *pure mathematics* is the analysis of the structure of pure space and time, free from empirical material, and *applied mathematics* is the analysis of the structure of space and time, augmented by empirical material. Like Aristotle, Kant distinguishes between *potential* and *actual infinity*. However, Kant does not regard actual infinity as being a logical impossibility, but rather, like non-Euclidean geometry, as an *idea of reason*, internally consistent but neither perceptible nor constructible. Kant may be considered a forerunner of the intuitionists.

I come now to some more recent views on the nature of mathematics. To begin with:

Logicism

The Greeks had developed mathematics as a rigorous demonstrative science, in which geometry occupied central stage. But they lacked an abstract conception of *number*: this in fact only began to emerge in the Middle Ages under the stimulus of Indian and Arabic mathematicians, who brought about the liberation of the number concept from the dominion of geometry. The seventeenth century witnessed two decisive innovations which mark the birth of modern mathematics. The first of these was introduced by Descartes and Fermat, who, through their invention of coordinate geometry, succeeded in correlating the then essentially separate domains of algebra and geometry, so paving the way for the emergence of modern mathematical analysis. The second great innovation was, of course, the development of the infinitesimal calculus by Leibniz and Newton.

However, a price had to be paid for these achievements. In fact, they led to a considerable diminution of the deductive rigour on which the certainty of Greek mathematics had rested. This was especially true in the calculus, where the rapid development of spectacularly successful new techniques for solving previously intractable problems excited the imagination of mathematicians to such an extent that they frequently threw logical caution to the winds and allowed themselves to be carried away by the spirit of adventure. A key element in these techniques was the concept of *infinitesimal quantity* which, although of immense fertility, was logically somewhat dubious. By the end of the eighteenth century a somewhat more circumspect attitude to the cavalier use of these techniques had begun to make its appearance, and in the nineteenth century serious steps began to be taken to restore the tarnished rigour of mathematical demonstration. The situation (in 1884) was summed up by Frege in a passage from his *Foundations of Arithmetic:*

> After deserting for a time the old Euclidean standards of rigour, mathematics is now returning to them, and even making efforts to go beyond them. In arithmetic, it has been the tradition to reason less strictly than in geometry. The discovery of higher analysis only served to confirm this tendency; for considerable, almost insuperable, difficulties stood in the way of any rigorous treatment of these subjects, while at the same time small

> reward seemed likely for the efforts expended in overcoming them. Later developments, however, have shown more and more clearly that in mathematics a mere moral conviction, supported by a mass of successful applications, is not good enough. Proof is now demanded of many things that formerly passed as self-evident. Again and again the limits to the validity of a proposition have been in this way established for the first time. The concepts of function, of continuity, of limit and of infinity have been shown to stand in need of sharper definition. Negative and irrational numbers, which had long since been admitted into science, have had to admit to a closer scrutiny of their credentials. In all directions these same ideals can be seen at work—rigour of proof, precise delimitation of extent of validity, and as a means to this, sharp definition of concepts.

Both Frege and Dedekind were concerned to supply mathematics with rigorous definitions. They believed that the central concepts of mathematics were ultimately *logical* in nature, and, like Leibniz, that truths about these concepts should be established by purely logical means. For instance, Dedekind asserts (in the Preface to his *The Nature and Meaning of Numbers*, 1888) that

> I consider the number concept [to be] entirely independent of the notions or intuitions of space and time … an immediate result from the laws of thought.

Thus, if we make the traditional identification of logic with the laws of thought, Dedekind is what we would now call a *logicist* in his attitude toward the nature of mathematics. Dedekind's "logicism" embraced *all* mathematical concepts: the concepts of number—natural, rational, real, complex—and geometric concepts such as continuity: in fact, it was the imprecision surrounding the concept of continuity that impelled him to embark on the program of critical analysis of mathematical concepts. As a practicing mathematician Dedekind brought a certain latitude to the conception of what was to count as a "logical" notion—a law of thought—as is witnessed by his remark that… we are led to consider the ability of the mind to relate things to things, to let a thing correspond to a thing, or to represent a thing by a thing, an ability without which no thinking is possible. This idea of *correspondence* or *functionality,* taken by Dedekind as fundamental, was to become the central concept of *category theory.* Dedekind was not particularly concerned with providing precise formulation of the logical principles supporting his reasoning, believing that reference to self-evident "laws of thinking" would suffice. Dedekind's logicism was accordingly of a less thoroughgoing and painstaking nature than that of his contemporary Frege, whose name, together with Bertrand Russell's, is virtually synonymous with logicism. In his logical analysis of the concept of number, Frege undertook to fashion in exacting detail the symbolic language within which his analysis was to be presented. Frege's analysis is presented in three works:

Begriffsschrift (1879): Concept-Script, a symbolic language of pure thought modelled on the language of arithmetic.

Grundlagen (1884): The Foundation of Arithmetic, a logico-mathematical investigation into the concept of number.

Grundgesetze (1893, 1903): Fundamental Laws of Arithmetic, derived by means of concept-script.

In the *Grundgesetze* Frege refines and enlarges the symbolic language first introduced in the *Begriffsschrift* so as to undertake, in full formal detail, the analysis of the concept of number, and the derivation of the fundamental laws of arithmetic. The logical universe of *Grundgesetze* comprises two sorts of entity: *functions*, and *objects*. Any function f associates with each value ξ of its argument an object $f(\xi)$: if this object is always one of the two *truth values* **0** (false) or **1** (true), then f is called a *concept* or *propositional function,* and when $f(\xi) = \mathbf{1}$ we say that ξ *falls under* the concept f. If two functions f and g assign the same objects to all possible values of their arguments, we should naturally say that they have the same *course of values*; if f and g are concepts, we would say that they both have the same *extension*. Frege's decisive step in the *Grundgesetze* was to introduce a new kind of object expression—which we shall write as $\hat{f}$ —to symbolize the course of values of f and to lay down as a basic principle the assertion

$$\hat{f} = \hat{g} \leftrightarrow \forall\xi\ [f(\xi) = g(\xi)]. \qquad (1)$$

Confining attention to concepts, this may be taken as asserting that *two concepts have the same extension exactly when the same entities fall under them.*

The notion of the extension of a concept underpins Frege's definition of number, which in the *Grundlagen* he had argued persuasively should be taken as a measure of the size of a concept's extension. [It is helpful to think of the extension of a concept as the class of all entities that fall under it, so that, for example, the extension of the concept *red* is the class of all red objects. However, it is by no means necessary to identify extensions with classes; all that needs to be known about extensions is that they are objects satisfying (1).] He introduced the term *equinumerous* for the relation between two concepts that obtains when the fields of entities falling under each can be put in biunique correspondence. He then defined cardinal number by stipulating that the cardinal number of a concept F is the extension of the concept *equinumerous with the concept F.* In this way a number is associated with a *second-order* concept—a concept about concepts. Thus, if we write $\nu(F)$ for the cardinal number of F so defined, and $F \approx G$ for *the concept F is equinumerous with the concept G*, then it follows from (1) that

$$\nu(F) = \nu(G) \leftrightarrow F \approx G. \qquad (2)$$

And then the natural numbers can be defined as the cardinal numbers of the following concepts:

N_0: $x \neq x$	$0 = \nu(N_0)$
N_1: $x = 0$	$1 = \nu(N_1)$
N_2: $x = 0 \vee x = 1$	$2 = \nu(N_2)$

Etc.

In a technical *tour-de-force* Frege established that the natural numbers so defined satisfy the usual principles expected of them.Unfortunately, in 1902 Frege learned of *Russell's paradox*, which can be derived from his principle (1) and shows it to be *inconsistent.* Russell's paradox, as formulated for sets or classes in the previous chapter, can be seen to be attendant upon the usual supposition that *any property determines a unique class*, to wit, the class of all objects possessing that property (its "extension"). To derive the paradox in Frege's system, classes are replaced by Frege's extensions: we define the concept R by

$$R(x) \leftrightarrow \exists F[x = \hat{F} \wedge \neg F(x)]$$

(in words: *x falls under the concept R exactly when x is the extension of some concept under which it does not fall*). Now write r for the extension of R, i.e.,

$$r = \hat{R}.$$

Then

$$R(r) \leftrightarrow \exists F[r = \hat{F} \wedge \neg F(r)]. \qquad (3)$$

Now suppose that $R(r)$ holds. Then, for some concept F,

$$r = \hat{F} \wedge \neg F(r).$$

But then

$$\hat{F} = r = \hat{R},$$

and so we deduce from (1) that

$$\forall x[F(x) \leftrightarrow R(x)].$$

Since $\neg F(r)$, it follows that $\neg R(r)$. We conclude that

$$R(r) \rightarrow \neg R(r).$$

Conversely, assume $\neg R(r)$. Then

$$r = \hat{R} \wedge \neg R(r),$$

and so *a fortiori*

$$\exists F[r = \widehat{F} \wedge \neg F(r)].$$

It now follows from the definition of R that $\neg R(r)$. Thus we have shown that

$$\neg R(r) \rightarrow R(r).$$

We conclude that Frege's principle (1) yields the contradiction

$$R(r) \leftrightarrow \neg R(r).$$

Thus Frege's system in the *Grundgesetze* is, as it stands, inconsistent. Later investigations, however, have established that the definition of the natural numbers and the derivation of the basic laws of arithmetic can be salvaged by suitably restricting (1) so that it becomes consistent, leaving the remainder of the system intact. In fact it is only necessary to make the (consistent) assumption that the extensions of a certain special type of concept—the *numerical concepts* (a numerical concept is one expressing equinumerosity with some given concept)—satisfy (1). Alternatively, one can abandon extensions altogether and instead take the cardinal number $\nu(F)$ as a primitive notion, governed by equivalence (2). In either case the whole of Frege's derivation of the basic laws of arithmetic can be recovered.

Where does all this leave Frege's (and Dedekind's) claim that arithmetic can be derived from logic? Both established beyond dispute that arithmetic can be formally or logically derived from principles which involve no explicit reference to spatiotemporal intuitions. In Frege's case the key principle involved is that certain concepts have extensions satisfying (1). But although this principle involves no reference to spatiotemporal intuition, it can hardly be claimed to be of a purely logical nature. For it is an *existential* assertion and one can presumably conceive of a world devoid of the objects ("extensions") whose existence is asserted. It thus seems fair to say that, while Frege (and Dedekind) did succeed in showing that the concept and properties of number are "logical" in the sense of being independent of spatiotemporal intuition, they did not (and it would appear could not) succeed in showing that these are "logical" in the stronger Leibnizian sense of holding in every possible world.

The logicism of *Bertrand Russell* was in certain respects even more radical than that of Frege, and closer to the views of Leibniz. In *The Principles of Mathematics* (1903) he asserts that mathematics and logic are *identical.* To be precise, he proclaims at the beginning of this remarkable work that

Pure mathematics is the class of all propositions of the form "p implies q" where p and q are propositions ... and neither p nor q contains any constants except logical constants.

Thus at the time this was asserted Russell was what could be described as an "implicational logicist".

The monumental, and formidably recondite[1] *Principia Mathematica,* written during 1910–1913 in collaboration with *Alfred North Whitehead* (1861–1947), contains a complete system of pure mathematics, based on what were intended to be purely logical principles, and formulated within a precise symbolic language. One may get an idea of just how difficult this work is by quoting the following extract from a review of it in a 1911 number of the London magazine *The Spectator:*

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A central concern of *Principia Mathematica* was to avoid the so-called *vicious circle paradoxes,* such as those of Russell and Grelling-Nelson—mentioned in the previous chapter—which had come to trouble mathematicians concerned with the ultimate soundness of their discipline. Another is *Berry's paradox*, in one form of which we consider the phrase *the least integer not definable in less than eleven words*. This phrase defines, in less than eleven words (ten, actually), an integer which satisfies the condition stated, that is, of not being definable in less than eleven words. This is plainly self-contradictory.

If we examine these paradoxes closely, we find that in each case a term is defined by means of an implicit reference to a certain class or domain which contains the term in question, thereby generating a vicious circle. Thus, in Russell's paradox, the defined entity, that is, the class *R* of all classes not members of themselves is obtained by singling out, from the class *V* of all classes *simpliciter*, those that are not members of themselves. That is, *R* is defined in terms of *V*, but since *R* is a member of *V*, *V* cannot be obtained without being given *R* in advance. Similarly, in the Grelling-Nelson paradox, the definition of the adjective *heterological* involves considering the concept *adjective* under which *heterological* itself falls. And in the Berry paradox, the term *the least integer not definable in less than eleven words* involves reference to the class of all English phrases, including the phrase defining the term in question.

Russell's solution to these problems was to adopt what he called the *vicious circle principle* which he formulated succinctly as: *whatever involves* all *of a collection must not be one of a collection.* This injunction has the effect of excluding, not just self-contradictory entities of the above sort, but *all* entities whose definition is in some way circular, even those, such as the class of all classes which are members of themselves, the adjective "autological", or the least integer definable in less than eleven words, the supposition of whose existence does not appear to lead to contradiction. It may be noted that the self-contradictory nature of the "paradoxical" entities we have described derives as much from the occurrence of *negation* in their definitions as it does from the circularity of those definitions.

The vicious circle principle suggests the idea of arranging classes or concepts (propositional functions) into distinct *types* or *levels,* so that, for instance, any class may only contain classes (or individuals) of lower level as members, and a propositional function can have only (objects or) functions of lower level as possible arguments. The idea of stratifying classes into types had also occurred to Russell in connection with his analysis of classes as genuine *pluralities*, as opposed to *unities.* On this reckoning, one starts with individual objects (lowest type), pluralities of these comprise the entities of next highest type, pluralities of these pluralities the entities of next highest type, etc. Thus the evident distinction between individuals and pluralities is "projected upwards" to produce a hierarchy of types.

Under the constraints imposed by this theory, one can no longer form the class of all possible classes as such, but only the class of all classes of a given level. The resulting class must then be of a higher type than each of its members, and so cannot be a member of itself. Thus Russell's paradox cannot arise. The Grelling-Nelson paradox is blocked because the property of heterologicality, which involves self-application, is inadmissible. Unfortunately, however, this simple theory of types does not circumvent paradoxes such as Berry's, because in these cases the defined entity is clearly of the same level as the entities involved in its definition. To avoid paradoxes of this kind Russell was therefore compelled to introduce a further "horizontal" subdivision of the totality of entities at each level, into what he called *orders*, and in which the *mode of definition* of these entities is taken into account. The whole apparatus of types and orders is called the *ramified theory of types* and forms the backbone of the formal system of *Principia Mathematica.*

To convey a rough idea of how Russell conceived of orders, let us confine attention to propositional functions taking only individuals (type 0) as arguments. Any such function which can be defined without application of quantifiers to any variables other than individual variables is said to be of *first order*. For example, the propositional function *everybody loves x* is of first order. Then *second order* functions are those whose definition involves application of quantifiers to nothing more than individual and first order variables, and similarly for third, fourth,..., n^{th} order functions. Thus *x has all the first-order qualities that make a great philosopher* represents a function of second order and first type.

Distinguishing the order of functions enables paradoxes such as Berry's to be dealt with. There the word *definable* is incorrectly taken to cover not only definitions in the usual sense, that is, those in which no functions occur, but also definitions involving functions of all orders. We must instead insist on specifying the orders of all functions figuring in these definitions. Thus, in place of the now illegitimate *the least integer not definable in less than eleven words* we consider *the least integer not definable in terms of functions of order* n *in less than eighteen words*. This integer is then indeed *not* definable in terms of functions of order *n* in less than eighteen words, but *is* definable in terms of functions of order *n*+1 in less than eighteen words. There is no conflict here.

While the ramified theory of types circumvents all known paradoxes (and can in fact be proved consistent from some modest assumptions), it turns out to be too weak a system to support unaided the development of mathematics. To begin with, one cannot prove within it that there is an infinity of natural numbers, or indeed that each natural number has a distinct successor. To overcome this deficiency Russell was compelled to introduce an *axiom of infinity*, to wit, that there exists a level containing infinitely many entities. As Russell admitted, however, this can hardly be considered a principle of logic, since it is certainly possible to conceive of circumstances in which it might be false. In any case, even augmented by the axiom of infinity, the ramified theory of types proves inadequate for the development of the basic theory of the real numbers. For instance, the theorem that every bounded set of real numbers has a least upper bound, upon which the whole of mathematical analysis rests, is not derivable without further *ad hoc* strengthening of the theory, this time by the assumption of the so-called *axiom of reducibility*. This asserts that any propositional function of any order is equivalent to one of first order in the sense that the same entities fall under them. Again, this principle can hardly be claimed to be a fact of logic.

Various attempts have been made to dispense with the axiom of reducibility, notably that of *Frank Ramsey* (1903–1930). His idea was to render the whole apparatus of orders superfluous by eliminating quantifiers in definitions. Thus he proposed that a universal quantifier be regarded as indicating a conjunction, and an existential quantifier a disjunction, even though it may be impossible in practice to write out the resulting expressions in full. On this reckoning, then, the statement ***Citizen Kane*** *has all the qualities that make a great film* would be taken as an abbreviation for something like ***Citizen Kane*** *is a film, brilliantly directed, superbly photographed, outstandingly performed, excellently scripted, etc.* For Ramsey, the distinction of orders of functions is just a complication imposed by the structure of our language and not, unlike the hierarchy of types, something inherent in the way things are. For these reasons he believed that the simple theory of types would provide an adequate foundation for mathematics.

What is the upshot of all this for Russell's logicism? There is no doubt that Russell and Whitehead succeeded in showing that mathematics can be derived within the ramified theory of types from the axioms of infinity and reducibility. This is indeed no mean achievement, but, as Russell admitted, the axioms of infinity and reducibility seem to be at best contingent truths. In any case it seems strange to have to base the truth of mathematical assertions on the proviso that there are infinitely many individuals in the world. Thus, like Frege's, Russell's attempted reduction of mathematics to logic contains an irreducible mathematical residue.

Formalism

In 1899 David Hilbert published his epoch-making work *Grundlagen der Geometrie* ("Foundations of Geometry"). Without introducing any special symbolism, in this work Hilbert formulates an absolutely rigorous axiomatic treatment of Euclidean geometry, revealing the hidden assumptions, and bridging the logical gaps, in traditional accounts of the subject. He also establishes the *consistency* of his axiomatic system by showing that they can be interpreted (or as we say, possess a *model*) in the system of real numbers. Another important property of the axioms he demonstrated is their *categoricity*, that is, the fact that, up to isomorphism they have exactly *one* model, namely, the usual 3-dimensional space of real number triples. Although in this work Hilbert was attempting to show that geometry is entirely self-sufficient *as a deductive* *system*—in this connection one recalls his famous remark: *one must be able to say at all times, instead of points, lines and planes—tables, chairs, and beer mugs*—he nevertheless thought, as did Kant, that geometry is ultimately *the logical analysis of our intuition of space.* This can be seen from the fact that as an epigraph for his book he quotes Kant's famous remark from the *Critique of Pure Reason:*

> Human knowledge begins with intuitions, goes from there to concepts, and ends with ideas.

The great success of the method Hilbert had developed to analyze the deductive system of Euclidean geometry—we might call it the *rigorized axiomatic method*, or the *metamathematical method*—emboldened him to attempt later to apply it to pure mathematics as a whole, thereby securing what he hoped to be perfect rigour for all of mathematics. To this end Hilbert elaborated a subtle philosophy of mathematics, later to become known as *formalism*, which differs in certain important respects from the logicism of Frege and Russell and betrays certain Kantian features. Its flavour is well captured by the following quotation from an address he made in 1927:

> No more than any other science can mathematics be founded on logic alone; rather, as a condition for the use of logical inferences and the performance of logical operations, something must already be given to us in our faculty of representation, certain extralogical concrete objects that are intuitively present as immediate experience prior to all thought. If logical inference is to be reliable, it must be possible to survey these objects completely in all their parts, and the fact that they occur, that they differ from one another, and that they follow each other, or are concatenated, is immediately given intuitively, together with the objects, as something that can neither be reduced to anything else, nor requires reduction. This is the basic philosophical position that I regard as requisite for mathematics and, in general, for all scientific thinking, understanding, and communication. And in mathematics, in particular, what we consider is the concrete signs themselves, whose shape, according to the conception we have adopted, is immediately clear and recognizable. This is the very least that must be presupposed, no scientific thinker can dispense with it, and therefore everyone must maintain it, consciously or not.

Thus, at bottom, Hilbert, like Kant, wanted to ground mathematics on the description of concrete spatiotemporal configurations, only Hilbert restricts these configurations to *concrete signs* (such as inscriptions on paper). No inconsistencies can arise within the realm of concrete signs, since precise descriptions of concrete objects are always mutually compatible. In particular, within the mathematics of concrete signs, actual infinity cannot generate inconsistencies since, as for Kant, this concept cannot describe any concrete object. On this reckoning, the soundness of mathematics thus issues ultimately, not from a *logical* source, but from a *concrete* one, in much the same way as the consistency of truly reported empirical statements is guaranteed by the concreteness of the external world.

Yet Hilbert also thought that adopting this position would not require the abandonment of the infinitistic mathematics of Cantor and others which had emerged in the nineteenth century and which had enabled mathematics to make such spectacular strides. He accordingly set himself the task of accommodating infinitistic mathematics within a mathematics restricted to the deployment of finite concrete objects. Thus *Hilbert's program*, as it came to be called, had as its aim the provision of a new foundation for mathematics not by reducing it to logic, but instead by *representing its essential form within the realm of concrete symbols.* As the quotation above indicates, Hilbert considered that, in the last analysis, the completely reliable, irreducibly self-evident constituents of mathematics are *finitistic,* that is, concerned just with finite manipulation of surveyable domains of concrete objects, in particular, mathematical symbols presented as marks on paper. Mathematical propositions referring only to concrete objects in this sense he called *real*, or *concrete*, propositions, and all other mathematical propositions he considered as possessing an *ideal*, or *abstract* character. Thus, for example, $2 + 2 = 4$ would count as a real proposition, while *there exists an odd perfect number* would count as an ideal one.

Hilbert viewed ideal propositions as akin to the ideal lines and points "at infinity" of projective geometry. Just as the use of these does not violate any truths of the "concrete" geometry of the usual Cartesian plane, so he hoped to show that the use of ideal propositions—in particular, those of Cantor's set theory—would never lead to falsehoods among the real propositions, that, in other words, such use *would never contradict any self-evident fact about concrete objects.* Establishing this by strictly concrete, and so unimpeachable means was the central aim of Hilbert's program. In short, its objective was to prove classical mathematical reasoning *consistent.* With the attainment of this goal, mathematicians would be free to roam unconstrained within "Cantor's Paradise" (in Hilbert's memorable phrase—he actually asserted that "no one will ever be able to expel us from the paradise that Cantor has created for us.") This was to be achieved by setting classical mathematics out as a purely formal system of symbols, devoid of meaning—here it should be emphasized that Hilbert was *not* claiming that (classical) mathematics *itself* was meaningless, only that the formal system representing it was to be so regarded—and then showing that no proof in the system can lead to a false assertion, e.g. 0 = 1. This, in turn, was to be done by employing the *metamathematical* technique of replacing each abstract classical proof of a real proposition by a concrete, finitistic proof. Since, plainly, there can be no concrete proof of the real proposition 0 = 1, there can be no classical proof of this proposition either, and so classical mathematical reasoning is consistent.

As is well known, Gödel rocked Hilbert's program by demonstrating, through his celebrated *Incompleteness Theorems*, that *there would always be real propositions provable by ideal means which cannot be proved by concrete means.* He achieved this by means of an ingenious modification of the ancient *Liar paradox*. To obtain the liar paradox in its most transparent form, one considers the sentence *this sentence is false*. Calling this sentence *A,* it is clear that *A* is true if and only if it is false, that is, *A asserts its own falsehood.* Now Gödel showed that, if in *A* one replaces the word *false* by the phrase *not concretely provable*, then the resulting statement *B* is *true* —i.e., provable by ideal means—but *not concretely provable*. This is so because, as is easily seen, *B* actually asserts its own concrete unprovability in just the same way as *A* asserts its own falsehood. And by extending these arguments Gödel also succeeded in showing that the *consistency of arithmetic* cannot be proved by concrete means.

Accordingly there seems to be no doubt that Hilbert's program for establishing the consistency of mathematics (and in particular, of arithmetic) *in its original, strict form* was shown by Gödel to be unrealizable. However, Gödel himself thought that the program for establishing the consistency of arithmetic might be salvageable through an enlargement of the domain of objects admitted into finitistic metamathematics. That is, by allowing finite manipulations of suitably chosen *abstract* objects in addition to the concrete ones Gödel hoped to strengthen finitistic metamathematics sufficiently to enable the consistency of arithmetic to be demonstrable within it. In 1958 he achieved his goal, constructing a consistency proof for arithmetic within a finitistic, but not strictly concrete, metamathematical system admitting, in addition to concrete objects (numbers), abstract objects such as functions, functions of functions, etc., over finite objects. So, although Hilbert's program cannot be carried out in its original form, for arithmetic at least Gödel showed that it can be carried out in a weakened form by countenancing the use of suitably chosen abstract objects.

As for the doctrine of "formalism" itself, this was for Hilbert (who did not use the term, incidentally) not the claim that mathematics could be *identified* with formal axiomatic systems. On the contrary, he seems to have regarded the role of formal systems as being to provide distillations of mathematical practice of a sufficient degree of precision to enable their formal features to be brought into sharp focus. The fact that Gödel succeeded in showing that certain features (e.g. consistency) of these logical distillations could be *expressed,* but *not demonstrated* by finitistic means does not undermine the essential cogency of Hilbert's program.

Intuitionism.

A third tendency in the philosophy of mathematics to emerge in the twentieth century, *intuitionism*, is largely the creation of *L.E.J. Brouwer* (1882-1966). Like Kant, Brouwer held the idealist view that mathematical concepts are admissible only if they are adequately grounded in *intuition* and that mathematical theories are significant only if they concern entities which are constructed out of something given immediately in intuition. In *Intuitionism and Formalism* (1912), while admitting that the emergence of noneuclidean geometry had discredited Kant's view of space, he maintained, in opposition to the logicists (whom he called "formalists") that arithmetic, and so all mathematics, must derive from the *intuition of time*. In his own words:

> Neointuitionism considers the falling apart of moments of life into qualitatively different parts, to be reunited only while remaining separated by time, as the fundamental phenomenon of the human intellect, passing by abstracting from its emotional content into the fundamental phenomenon of mathematical thinking, the intuition of the bare two-oneness. This intuition of two-oneness, the basal intuition of mathematics, creates not only the numbers one and two, but also all finite ordinal numbers, inasmuch as one of the elements of the two-oneness may be thought of as a new two-oneness, which process may be repeated indefinitely; this gives rise still further to the smallest infinite ordinal ω . Finally this basal intuition of mathematics, in which the connected and the separate, the continuous and the discrete are united, gives rise immediately to the intuition of the linear continuum, i.e., of the "between", which is not exhaustible by the interposition of new units and which can therefore never be thought of as a mere collection of units. In this way the apriority of time does not only qualify the properties of arithmetic as synthetic a priori judgments, but it does the same for those of geometry, and not only for elementary two- and three-dimensional geometry, but for non-euclidean and *n*-dimensional geometries as well. For since Descartes we have learned to reduce all these geometries to arithmetic by means of coordinates.

For Brouwer, intuition meant essentially what it did to Kant, namely, the mind's apprehension of what it has itself constructed; on this view, the only acceptable mathematical proofs are *constructive*. A constructive proof may be thought of as a kind of "thought experiment" —the performance, that is, of an experiment in imagination. According to *Arend Heyting* (1898–1980), a leading member of the intuitionist school,

> Intuitionistic mathematics consists … in mental constructions; a mathematical theorem expresses a purely empirical fact, namely, the success of a certain construction. "2 + 2 = 3 + 1" must be read as an abbreviation for the statement "I have effected the mental construction indicated by '2 + 2' and '3 + 1' and I have found that they lead to the same result."

From passages such as these one might infer that for intuitionists mathematics is a purely subjective activity, a kind of introspective reportage, and that each mathematician has a personal mathematics. Certainly they reject the idea that mathematical thought is dependent on any special sort of language, even, occasionally, claiming that, at bottom, mathematics is a "languageless activity". Nevertheless, the fact that intuitionists evidently regard mathematical theorems as being valid for all intelligent beings indicates that for them mathematics has, if not an objective character, then at least a *transsubjective* one.

The major impact of the intuitionists' program of constructive proof has been in the realm of *logic*. Brouwer maintained, in fact, that the applicability of traditional logic to mathematics

> was caused historically by the fact that, first, classical logic was abstracted from the mathematics of the subsets of a definite finite set, that, secondly, an a priori existence independent of mathematics was ascribed to the logic, and that, finally, on the basis of this supposed apriority it was unjustifiably applied to the mathematics of infinite sets.

Thus Brouwer held that much of modern mathematics is based, not on sound reasoning, but on an illicit extension of procedures valid only in the restricted domain of the finite. He therefore embarked on the heroic course of setting the whole of existing mathematics aside and starting afresh, using only concepts and modes of inference that could be given clear intuitive justification. He hoped that, once enough of the program had been carried out, one could discern the logical laws that intuitive, or constructive, mathematical reasoning actually obeys, and so be able to compare the resulting *intuitionistic,* or *constructive, logic* with classical logic. (This is not to say that Brouwer was primarily interested in *logic,* far from it: indeed, his distaste for formalization led him not to take very seriously subsequent codifications of intuitionistic logic.)

The most important features of intuitionistic mathematical reasoning are that *an existential statement can be considered affirmed only when an instance is produced.* (In this connection one recalls Hermann Weyl's remarking of nonconstructive existence proofs that "they inform the world that a treasure exists without disclosing its location.")As a consequence—*a disjunction can be* *considered affirmed only when an explicit one of the disjuncts is demonstrated.* A striking consequence of this is that, as far as properties of (potentially) infinite domains are concerned, *neither the classical law of excluded middle nor the law of strong reductio ad absurdum can be accepted without qualification.* To see this, consider for example the existential statement *there exists an odd perfect number* (i.e., an odd number equal to the sum of its proper divisors) which we shall write as $\exists nP(n)$. Its contradictory is the statement $\forall n\neg P(n)$. Classically, the law of excluded middle then allows us to affirm the disjunction

$$\exists nP(n) \vee \forall n\neg P(n) \qquad (1)$$

Constructively, however, in order to affirm this disjunction we must *either* be in a position to affirm the first disjunct $\exists nP(n)$, i.e., to possess, or have the means of obtaining, an odd perfect number, *or* to affirm the second disjunct $\forall n\neg P(n)$, i.e. to possess a demonstration that no odd number is perfect. Since at the present time mathematicians have neither of these, the disjunction (1), and *a fortiori* the law of excluded middle is not constructively admissible.

[It might be thought that, if in fact the second disjunct in (1) is *false*, that is, not every number falsifies P, then we can actually find a number satisfying P by the familiar procedure of testing successively each number 0, 1, 2, 3,... and breaking off when we find one that does: in other words, that from $\neg\forall n\neg P(n)$ we can infer $\exists nP(n)$. Classically, this is perfectly correct, because the *classical* meaning of $\neg\forall n\neg P(n)$ is "$P(n)$ will not as a matter of *fact* be found to fail for every number n." But *constructively* this latter statement has no meaning, because it presupposes that every natural number *has already been constructed* (and checked for whether it satisfies P). Constructively, the statement must be taken to mean something like "we can derive a contradiction from the supposition that we could prove that $P(n)$ failed for every n." From this, however, we clearly cannot extract a guarantee that, by testing each number in turn, we shall eventually find one that satisfies P. So we see that the law of strong reductio ad absurdum also fails to be constructively admissible.

Thus we see that constructive reasoning differs from its classical counterpart in that it attaches a stronger meaning to some of the logical operators. It has become customary, following Heyting, to explain this stronger meaning in terms of the primitive relation *a is a proof of p,* between mathematical constructions a and mathematical assertions p. To assert the *truth* of p is to assert that one has a construction a such that a is a proof of p. [Here by *proof* we are to understand a mathematical construction that establishes the assertion in question, *not* a derivation in some formal system. For example, a proof of 2 + 3 = 5 in this sense consists of successive constructions of 2, 3 and 5, followed by a construction that adds 2 and 3, finishing up with a construction that compares the result of this addition with 5.]

The meaning of the various logical operators in this scheme is spelt out by specifying how proofs of composite statements depend on proofs of their constituents. Thus, for example,

> a is a proof of $p \wedge q$ means: a is a pair (b, c) consisting of a proof b of p and c of q;
> a is a proof of $p \vee q$ means: a is a pair (b, c) consisting of a natural number b and a construction c such that, if $b = 0$, then c is a proof of p, and if $b \neq 0$, then c is a proof of q;
> a is a proof of $p \rightarrow q$ means: a is a construction that converts any proof of p into a proof of q;
> a is a proof of $\neg p$ means: a is a construction that shows that no proof of p is possible.

It is readily seen that, for example, the law of excluded middle is not generally true under this ascription of meaning to the logical operators. For a proof of $p \vee \neg p$ is a pair (b,c) in which c is either a proof of p or a construction showing that no proof of p is possible, and there is nothing inherent in the concept of mathematical construction that guarantees, for an arbitrary proposition p, that either will ever be produced.

If we compare the law of excluded middle with Euclid's fifth postulate, then intuitionistic logic may be compared with *neutral* geometry—geometry, that is, without the fifth postulate—and classical logic to Euclidean geometry. Just as noneuclidean geometry revealed a "strange new universe", so intuitionistic logic has allowed new features of the logico-mathematical landscape—invisible through the lens of classical logic—to be discerned. Intuitionistic logic has proved to be a subtle instrument, more delicate than classical logic, for investigating the mathematical world. Famously, Hilbert remarked, in opposition to intuitionism, that "to deny the mathematician the use of the law of excluded middle would be to deny the astronomer the use of a telescope or the boxer the use of his fists." But with experience in using the refined machinery of intuitionistic logic one comes to regard Hilbert's simile as inappropriate. A better one might be: to deny the mathematician the use of the law of excluded middle would be to deny the surgeon the use of a butcher knife, or the general the use of nuclear weapons. Or, as Kreisel has observed, denying the use of the law of excluded middle might be compared to denying the nonabelian group theorist the use of the commutative law.

Despite the fact that Logicism, Intuitionism and Formalism cannot be held to provide complete accounts of the nature of mathematics, each gives expression to an important partial truth about that nature: *Logicism,* that mathematical truth and logical demonstration go hand in hand; *Intuitionism*, that mathematical activity proceeds by the performance of mental constructions, and finally *Formalism,* that the results of these constructions are presented through the medium of formal symbols.

Foundational Schemes for Mathematics.

In attempting to provide a foundation for mathematics, should one take as the primary datum its seemingly *objective* character, namely, the fact, assented to in one way or another by every mathematician and first given systematic articulation by Plato, that its contents are in some sense objectively *true* or *correct?* Or should one follow Kant in taking the essence of mathematics to be the generation of *certainty* in the mind? That, in the end, mathematical knowledge is self-knowledge of the human mind. Ontology—or epistemology? Realism, or idealism?

A useful place to begin our discussion of foundational schemes for mathematics is with ***John Mayberry***'s ringing endorsement of realism. In his paper *What is Required of a Foundation for Mathematics?,* he asserts that an account of the foundations of mathematics must specify the following four things:

1. The primitive concepts in which other mathematical concepts are to be defined.
2. The rules governing the laying down of definitions.
3. The ultimate premises of proofs.
4. The rules allowing the advance from premises to desired conclusions.

According to Mayberry the primitive concepts falling under 1. are those of *Cantorian set theory*, while the ultimate premises falling under 3. are those of (second-order) *Zermelo-Fraenkel set theory:* these are claimed to possess the self-evident character required of "ultimate premises". This self-evidence derives, according to Mayberry, from the very concept of set itself as "an extensional plurality of determinate size, composed of properly distinguished objects."

For Mayberry, a foundation for mathematics must provide compelling evidence for the *truth* of mathematical assertions. The truth of a mathematical proposition, in his view, does not depend in any way "on the correctness, formal or otherwise, of any purported proof of it, or even on the possibility of a proof." He urges that we should be "Platonists" in at least the following sense:

> "Considerations of human activities and capacities, actual or idealized, have no place in the foundations of mathematics, and we must therefore make every effort to exclude them from the elements, principles and methods upon which we intend to base our mathematics."

Mayberry thus advocates an entirely *objective* approach to the foundations of mathematics. While the requirement of objectivity alone does not demand the use of *set theory,* it must be recognized that for the past century or so, set theory has furnished the official foundation for mathematics—indeed, for the majority of mathematicians (who admittedly don't reflect much on the matter)—the phrase "foundations of mathematics" is virtually synonymous with "set theory". Set theory has offered ultimate answers to *ontological* questions such as "what is a number?", "what is a function?" and the like. And most mathematicians, if asked to identify the ultimate basis for the truth of mathematical propositions, would, if pressed, respond that the proposition in question is provable from the axioms of some suitable set theory.

Mayberry repudiates the idea that any *axiomatic* theory could serve as a foundation for mathematics in his sense:

> "The idea that any first-order theory could fulfill that role [of being a foundation for mathematics] is simply incoherent. Indeed, *no axiomatic theory, formal or informal, of first or higher order, can logically play a foundational role in mathematics.* Here, of course, I mean an axiomatic theory in the conventional. modern sense in which group theory, ring theory, general topology, category theory, topos theory, the theory of complete ordered fields and the theory of simply infinite sytems are all axiomatic theories. Surely it is obvious that you cannot *use* the axiomatic method to *explain* what the axiomatic method is. Since any would-be replacement for set theory as the foundation for mathematics must supply a convincing account of axiomatic definition, it cannot, on pain of circularity, itself be presented by means of an axiomatic definition.
> The *fons et origo* of all confusion here is the view that set theory is just another axiomatic theory and the universe of sets just another mathematical structure. The universe of sets is not a structure: it is the world that all mathematical structures inhabit, the sea in which they all swim."

Elegant as Mayberry's pelagic metaphor may be, one may still ask: what is it about the set concept that has conferred on it this ultimate authority? Perhaps the following:

1. The modern concept of set, as developed by Cantor, is (as Mayberry observes) an extension of what the Greeks called an *arithmos* or limited plurality. This concept of discrete finite plurality has an objectivity, transparency and definiteness which Cantor believed would survive the extension of the concept to that of "arbitrary" (i.e., infinite) set. Most mathematicians (but by no means all) came to agree with him.
2. The effort by Dedekind and other 19th century mathematicians to "rigorize" mathematics and to provide proper definitions for hitherto undefined mathematical notions (most notably, those associated with continuity) led to satisfactory formulations of concepts such as real number, and ultimately, of function, in set-theoretical terms. Of particular importance in this respect was the set-theoretical definition of a space of functions, allowing a vast expansion of the realm of mathematical analysis.
3. The contemporaneous emergence, with set theory, of mathematical logic and the resulting extension of the axiomatic method to mathematics as a whole. The *adequacy* of the rules of proof in axiomatic theories is a semantic issue, and the semantic framework for such theories was formulated (by Gödel, Tarski *et al.*) in *set-theoretic* terms. The set concept thus came to be regarded as an indispensable constituent of the *meaning* of the axiomatic method.
4. Finally, the remarkable fact that apparently all "objective" mathematical notions, from natural numbers to Riemannian manifolds and Hilbert spaces, could be supplied with set-theoretical definitions as sets with a structure (morphology).

In this connection ***Gödel***, famously, has made no bones about his realist views concerning sets and classes:

> "Classes and concepts may…be conceived as real objects, namely classes as "pluralities of things" or as structures consisting of a plurality of things and concepts as the properties and relations of things existing independently of our definitions and constructions.
> It seems to me that the assumption of such objects is quite as legitimate as the assumption of physical bodies and there is quite as much reason to believe in their existence. They are in the same sense necessary to obtain a satisfactory system of mathematics as physical bodies are necessary for a satisfactory of our sense perceptions and in both cases it is impossible to interpret the propositions one wants to assert about these entities as propositions about the "data", i.e. in the latter case the actually occurring sense perceptions." [In *Russell's Mathematical Logic", 1944.*]

> "This negative attitude toward Cantor's set theory [held by intuitionists] and toward classical mathematics, of which it is a natural generalization, is by no means a necessary outcome of a closer examination of their foundations, but only the result of a certain

> philosophical conception of the nature of mathematics, which admits mathematical objects only to the extent to which they are interpretable as our own constructions or, at least, can be given completely in mathematical intuition. For someone who considers mathematical objects to exist independently of our constructions and of our having an intuition of them individually, and who requires only that the general mathematical concepts must be sufficiently clear for us to be able to recognize their soundness and the truth of the axioms concerning them, there exists, I believe, a satisfactory foundation for Cantor's set theory in its whole original intent and meaning…"

By this "satisfactory foundation" Gödel has in mind the *iterative concept of set*, "according to which a set is something obtainable from the integers (or some other well-defined objects) by iterated application [if necessary into the transfinite] of the operation "set of", not something obtained by dividing the totality of all existing things into two categories…" As for the primitive concept "set of", Gödel has this to say:

> "The operation "set of *x*'s" cannot be defined satisfactorily (at least not in the present stage of knowledge) but can only be paraphrased by other expressions involving again the concept of set, such as: "multitude of *x*'s", "combination of any number of *x'*s", "part of the totality of *x's*", where a "multitude" ("combination", "part") is conceived of as something which exists in itself no matter whether we can define it in a finite number of words (so that random sets are not excluded [this is what Bernays called the "combinatorial" notion of set]."

It is a consequence of Gödel's realism concerning sets that questions such as the continuum hypothesis must be *objectively true or false*, even though current axiom schemes for set theory do not enable a decision as to their truth or falsity to be made.

> "For if the meaning of the primitive terms of set theory… are accepted as sound, it follows that the set-theoretical concepts and theorems describe **some well-determined reality**, in which [the continuum hypothesis] must be true or false. Hence its undecidability from the axioms being assumed today **can only mean that these axioms do not contain a complete description of that reality."**

Gödel was led by his conviction as to the objectivity of set theory and the independent existence of sets to formulate the idea that new axioms for set theory (axioms of infinity) would be arrived at by reflection on the set concept. These new axioms would, it was hoped lead to a solution to the continuum problem and other outstanding problems of set theory. For Gödel and other set-theoretic realists, the universe of sets represents a kind of Everest which demands to be climbed, like the real Everest, *because it is there.*

*

Let us consider some opposing views.

Saunders Mac Lane's position concerning the ultimate nature of mathematics seems entirely at variance with Mayberry's, to wit,

> "Mathematics is based on human activities and scientific problems."

But Mac Lane recognizes that the set-theoretical foundation for mathematics offers

> "the advantage that every concept can be made absolutely clear and explicit."

Nevertheless he is sceptical concerning the ultimate "existence" of sets:

> "Before axioms are at hand, the (cumulative) hierarchy is a Platonic myth, clearly visible only to those with a sixth sense for sets."

But he is at one with Mayberry in insisting on the necessity of rigorous proof in mathematics: as he says, "proofs are a means of obtaining certainty." One might ask, however, certainty of *what?* Mayberry, as a realist, would respond "certainty of *truth*". But Mac Lane does not espouse realism, and explicitly rejects the traditional notion of mathematical "truth", claiming indeed that mathematical theorems do not assert truths about the world. His answer to our question would necessarily have to be more complicated. In fact he replaces "truth" by the formal notion of "correctness", viz., that the proof proceed in strict accordance with the agreed formal rules. This seems to be a formalist move, and indeed Mac Lane asserts that "Mathematics is concerned not with reality but with rule", that "Mathematics makes no ontological commitments", that "the philosophy of Mathematics need not involve questions about epistemology or ontology." On the other hand the form of mathematics is "chosen to reflect the facts (about reality)".

On the question of foundations of mathematics Mac Lane has this to say:

"Mathematics has access to absolute rigour—because it is about form and about fact. However, there is no single and absolute foundation for Mathematics. Any such fixed foundation would preclude the novelty which might result from the discovery of new form. A form is any development which proceeds by rule rather than by appeal to fact as to meaning."

As for *set theory*, Mac Lane sees it as "strong enough to provide a formulation of most of mathematics but this provision is often artificial… moreover, there is no unique notion of "set". The system of first-order Zermelo-Fraenkel set theory serves as an appropriate formal language for describing mathematical objects and reasoning about them, but since it is incomplete other "foundations", such as elementary toposes, are possible.

*

Mac Lane's views on the nature of mathematics are not incompatible with the claims of the cognitive scientists ***Lakoff*** and ***Nuñez.*** In their recent book *Where Mathematics Comes From : How the Embodied Mind Brings Mathematics into Being*, starting with the observation that the only mathematical ideas that human beings can have are ideas that the human brain allows, they put forward the claim that all abstract ideas, including those of mathematics, arise via *conceptual metaphor*—a mechanism for projecting embodied (that is, sensory-motor) reasoning to abstract reasoning. Briefly, Lakoff and Núñez maintain that mathematics is a product of human beings and is shaped by our brains and conceptual systems, as well as the concerns of human societies and culture. We have evolved so that our cognition fits the world as we know it. Their views on the nature of mathematics form a kind of naturalist's credo:

> "Mathematics is a natural part of being human. It arises from our bodies, our brains, and our everyday experiences in the world. Cultures everywhere have some form of mathematics.
> There is nothing mysterious, mystical, magical, or transcendent about mathematics. It is a consequence of human evolutionary history, neurobiology, cognitive capacities, and culture.
> Mathematics is one of the greatest products of the collective human imagination It is a system of human concepts which makes extraordinary use of the ordinary tools of human cognition. It is special in that it is stable, precise, generalizable, symbolizable, calculable, consistent within each of its subject matters, universally available, and effective for precisely conceptualizing a large number of aspects of the world as we experience it.
> The effectiveness of mathematics in the world is a tribute to evolution and to culture. Evolution has shaped our bodies and brains so that we have inherited neural capacities for the basics of number and for primitive spatial relations. Culture has made it possible for millions of astute observers of nature, through millennia of trial and error, to develop and pass on more and more sophisticated mathematical tools—tools shaped to describe what they have observed. There is no mystery about the effectiveness of mathematics for characterizing the world as we experience it: that effectiveness results from a combination of mathematical knowledge and connectedness to the world. The connection between mathematical ideas and the world as human beings experience it occurs within human minds. It is human beings who have created logarithmic spirals and fractals and who can "see" logarithmic spirals in snails and fractals in palm leaves.
> *Comment: The fact that we do "see" such patterns in nature—indeed the very possibility of doing so—is a feature of reality grasped by human minds but nevertheless objective. The description of the pattern on the snail's shell as a logarithmic spiral is a human construct,*

> *but the relationship between the concept of a logarithmic spiral and the actual configuration on the shell is objective. The number 2 is a human invention, but the relationship between the concept "2"and the poles of a magnet is objective.*
>
> In the minds of those millions who have developed and sustained mathematics, the concepts of mathematics have been devised to fit the world as perceived and conceptualized. This is possible because concepts such as change, proportion, size, rotation, probability, recurrence, iteration, and many others are both everyday ideas and ideas that have been mathematized. The mathematization of ordinary human ideas is an ordinary human enterprise.
>
> Through the development of writing systems over millennia, culture has made possible the notational systems of mathematics. Because human conceptual systems are capable of conceptual precision and symbolization, mathematics has been able to develop systems of precise calculation and proof. Through the use of discretization metaphors, more and more mathematical ideas become precisely symbolizable and calculable. It is the human capacity for conceptual metaphor that makes possible the precise mathematization and even the arithmetization of everyday concepts—concepts such as collections, dimensions, symmetry, causal dependence and independence, and many more.
>
> Everything in mathematics is comprehensible—at least in principle. Since it makes use of general human conceptual capacities, its conceptual structure can be analyzed and taught in meaningful terms.
>
> Mathematics is creative and open-ended. By virtue of the use of conceptual metaphors and conceptual blends, present mathematics can be extended to create new forms by importing structure from one branch to another and by fusing ideas from different branches.
>
> Human conceptual systems are not monolithic. They allow alternative visions of concepts and multiple metaphorical perspectives of many (though by no means all!) important aspects of our lives. Mathematics is every bit as conceptually rich as any other part of the human conceptual system. Moreover, mathematics allows for alternative visions and versions of concepts. There is not one notion of infinity but many, not one formal logic but tens of thousands, not one concept of number but a rich variety of alternatives, not one set theory or geometry or statistics but a wide range of them—all mathematics!
>
> Mathematics is a magnificent example of the beauty, richness, complexity, diversity, and importance of human ideas. It is a marvellous testament to what the ordinary embodied human mind is capable of—when multiplied by the creative efforts of millions over millennia.
>
> The portrait of mathematics has a human face.

*

Some years ago ***Chandler Davis*** put forward some views on mathematics considered from the standpoint of a materialist. According to Davis, being a mathematician and being a materialist are incompatible (despite his own simultaneous status as both!) because the former claims to make verifiable certain statements about abstract mathematical objects which are creations of the human mind, while the latter denies that mental constructs have existence independent of human reality. As a mathematician, Davis regards mathematical activity as being primarily and probably ultimately concerned with abstract *objects*: since these objects evidently have no physical existence, the materialist would be obliged to dismiss them as chimerical. In that case, what is it that the mathematician has certain knowledge *about?*.

Davis goes on to point out that, in direct opposition to what the materialist prescribes, the mathematician pursues theory without relating it to practice. That is, "mathematics is armchair science." But, says Davis, this is defensible, and, indeed, inevitable, because since mathematics "makes possible long chains of reasoning, it tends to lengthen the intervals at which the theoretician, even if willing to appeal to experience, will find it natural to do so." On the other hand, as Davis correctly points out, "by denying the authority of experiment [mathematics will] risk even more than other sciences an escape from reality."

As a materialist, Davis regards the ontological status of mathematical entities as *the* major problem. He observes that mathematical objects are represented syntactically by nouns, and goes on to draw a comparison between the way nouns are used in ordinary language and in mathematics. Since some nouns are not names of objects, he confines attention to "those nouns in common language which are used as existent or at least hypothesized objects." Davis submits that "a mathematical object is whatever can be usefully talked about *in the same way*—logically as well as syntactically—in mathematical reasoning," and that "mathematical existence is usability in the role of name of existent object in mathematical reasoning."

It must be observed that this definition doesn't seem very helpful because it merely replaces the term "exists" by the phrase "naming an existent object." Davis has doen little more than identify correctly the syntactic category in which to assign the names of mathematical objects; the problem of which of these names have real denotations is not resolved.

Davis points out that the problem of existence in mathematics only arises in earnest when working with theories which have no finite models. (This is presumably because finite models are "physically realizable" and so the terms in theories which possess finite models *ipso facto* satisfy the condition of being "names of existent objects".) Moreover, in theories possessing finite models, we can devise a proof of existence of the desired degree of definiteness by reasoning about what would happen if we were to conduct an exhaustive search through the elements of the model. "Reasoning of the same sort," says Davis, "applied to cases where the search is inconceivable, is unsupported." (Of course, this is the constructivist position.)

It should, however, be pointed out that much (infinitistic) mathematics is *not about* infinite search procedures. For example, the proof of the existence of a Lebesgue nonmeasurable set takes the axiom of choice *as given*: once this accepted the proof has nothing to do with "infinite searches". In general, existence statements about infinite structures say nothing about the possibility of actually carrying out a search, and are not intended to.

Davis raises the question as to why mathematicians have tended to ignore foundational issues, and, in particular, why mathematicians have (in Davis's view) simply stuck to Platonism despite its obvious absurdity. He gives two main reasons: the recognition of the destructive nature of the intuitionist critique, and the "irrelevance" of foundational questions for the day-to-day practice of mathematicians. Like Errett Bishop, Davis believes that "classical" mathematics, e.g. Cauchy's residue calculus, would be better off shedding "excess baggage" like the Dedekind construction of the reals, the Heine-Borel theorem and the like. But, unlike the constructivists, Davis doesn't suggest what should replace them.

Davis affirms that (some) mathematics does make statements about objective reality. He deplores the mystification of mathematics and its elevation into a kind of hieratic cult. He quotes Engels approvingly:

> "Pure mathematics has as its objects the spatial forms and quality relations of the real world."

And with equal approval he quotes Dirk Struik:

> "[the] inner truth [of mathematical statements] follows from the fact that they represent objective relations in the material world, which are investigated with regard to their own logic, their own development, and their internal relationships."

However, unlike Davis, who regards mathematical assertions as being primarily about *objects*, both Engels and Struik seem to be saying that these assertions concern *forms* and *relations*, which are *not the same* as objects. (I used to think that it might be possible to justify transfinite mathematics in terms of possible relations among material objects, even though these relations have no physical existence—this is a position I have abandoned.)

Davis puts forward a criterion for "good" mathematics: "A good piece of mathematics is potentially useful in making factual statements about the objective world." (Amusingly, this would be precisely G. H. Hardy's definition of a piece of "bad" mathematics.) Davis admits that this criterion is hard to apply, but asks that it be employed *preferentially*—in such a way as to "prefer one piece of mathematics to another."

I find this realist attitude towards mathematics basically attractive, but I think Davis's view as to what constitutes objective reality is too narrow. In my view, the "real" world contains not just *objects* in the sense that Davis uses the term, but also *relations* and *properties.* Admittedly, the latter are less immediate than material objects or identifiable particulars, but, perhaps, no less real—at least, for us. The extent to which relations and properties can be *reified*, i.e. treated as if they *were objects*, is a central problem in the foundations of mathematics, and one to which Davis gives insufficient attention.

Davis adjures us to reject what he calls "Platonist nonsense" and urges that "arguments which treat infinite sets of distinguishable objects as present and manipulable carry no conviction…" On the other hand, he rejects what he terms "the Kronecker-Brouwer-Bishop doctrine that man-made theories can enter heaven only by proving divine ancestry in the integers." He accepts the mathematical *continuum* , but rejects "the set of all subsets of an uncountably infinite set" on the grounds that it could almost certainly never be used in a "good" piece of mathematics. He thinks that such "monsters" arise as a consequence of performing mental manipulations of the sorting and counting variety where they were not the appropriate ones.

It should be pointed out, however, that this latter criticism loses its force when subsets are regarded as *properties.* Moreover, since set theory has countable models, it may well be the case that the abstract properties involved in set theory have a more concrete embodiment. This would be the case if there are (countably) infinitely many objects in the physical universe—as assumed, for instance, in Newtonian cosmology.

Here is my own position vis-à-vis Davis's materialism. I broadly support the view that mathematics (or 'good" mathematics, at least) says something about objective reality. But the "crudeness" of Davis's materialism leads him to adopt too narrow a view of what constitutes the objective world, and hence what constitutes good mathematics. He is right to criticize Platonist reification as preposterous if taken literally. On the other hand he seems to allow reification of concepts when they are capable of being useful in describing the real world. But he supplies no explanation of this.

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Moshé Machover has employed the idea of reification in his approach to the foundations of mathematics. He says:

> "Mathematical activity is social; but it is social only in a certain specific sense. It is not a kind of social game in which all the players are actually present at the same time and place and engage in a literally joint communal activity. For any mathematician most of the mathematics he comes across takes place first not in his own mind and not even in his own presence, but as it were behind his back, and is later—sometimes much later—presented to him as a *fait accompli* through the various channels of exchange.
> Even though mathematics is basically, or starts as, a constructive activity, it is not generally an activity in which I take part directly. In most cases I am confronted with the end product, which therefore tends to assume the guise of something objective, almost like an object of nature. The temptation to reify is almost irresistible.
> Of course, temptation is not the same as justification. Can a justification be given. I believe that it can. I do not pretend to know *all* the possible justifications, but I believe that there may be more than one, and at different levels.
> Here is *one* way in which it can be done.
> Let us start from a certain type of schematic construction. (I strongly believe that *all* mathematics *starts* from construction, starts as schematic constructive activity.) At this stage every proposition is a statement asserting the feasibility of such-and-such a construction. A proof of such a proposition consists in showing *how* to perform a construction of the kind the proposition claims to be feasible, and showing that the described construction is indeed of the required kind. These propositions become at least partly formalized. (*All* mathematics becomes at least partly formalized, if only in a fragment of natural language. This seems to be one of the rules of the game; perhaps it has to do with the schematic nature of the constructions.)
> The fact that we are still at a constructive stage means that the logic which governs our discourse is constructivist (intuitionist) logic. Though our mathematics is already

formalized—partly or even completely—it is *not* meaningless. On the contrary, formalization is merely a tool of precision. The postulates which we use are by no means arbitrary strings of symbols, neither are they implicit definitions of hypothetical entities. They are postulates in the old traditional sense: self-evident truths about the constructions we are dealing with. Consistency is guaranteed *provided* we have managed to capture correctly in our intuition certain basic facts about these constructions. Of course, we cannot in general be quite sure of this, but experience and careful reflection may correct our mistakes, if any. A proof of consistency—to the extent that we can get one—is at this stage not needed in order to convince us of consistency (indeed, we know that such a proof may in general be less convincing than the direct observation based on the constructive meaning of the postulates) but it may be very illuminating and informative. For example, Gödel's consistency proof for Heyting's arithmetic (the *Dialectica* proof) shows that there is an alternative interpretation of that theory in terms of *primitive-recursive* functionals of finite order (rather than arbitrary constructive functionals).

Now comes the next stage. There are a number of 'faithful translations' from classical to intuitionistic logic such that if Φ is a set of formulas and Φ^* are their translations, then Φ is consistent in classical logic iff Φ^* is consistent in intuitionistic logic. If the members of Φ^* happen to be theorems of our constructive theory, then Φ is classically consistent, and we can take Φ as a set of postulates for a classical theory, in which we use the law of excluded middle, regard a proposition as equivalent to its double negation, identify $\exists x$ with $\neg\forall x\neg$, etc. This means that we can safely interpret our variables as ranging over a complete totality of *objects*—i.e., we can reify. *For, technically speaking, reification amounts to nothing more than the use of classical logic.* The ontology that goes with it is not technically necessary, but is quite harmless (since we are morally certain that the use of classical logic is consistent here) and in practice very useful as an anchor for our intuition.

These posited objects may or may not have a family resemblance or family connection to the constructions with which we started. Superficially, this programme seems to resemble formalism, merely replacing finitary metamathematics by a constructivist metatheory. But there are two vital differences. Firstly, here classical (platonistic) mathematics is no longer largely meaningless, but interpretable in constructive terms. Secondly, this programme is not hit by Gödel's second incompleteness theorem. For this theorem certainly does not preclude a proof (even a finitary proof) of consistency of the classical theory *relative* to the constructive theory. And our faith in the (absolute) consistency of the latter does not hang on a finitary consistency proof.

An obvious example of such a procedure is the derivation of first-order Peano arithmetic from Heyting's arithmetic. A real triumph would be to derive in a similar way one of the usual classical set theories from a sufficiently convincing constructive theory, dealing with a suitable family of constructions which may or may not have some intuitive resemblance to sets. Some progress along these lines has already been made.

Speaking even more speculatively, I would like to suggest that this process may perhaps be iterated. Starting from an already reified ontology, one may consider procedures that are constructive *relative to the objects of that ontology* (e.g., so-called recursive operations on sets, which may themselves be non-constructive). Then one can attempt to reify these constructions.

There may well be other sound ways in which [reification] in mathematics may be legitimized. How much of classical mathematics can be recovered in such a way is a question that cannot be answered in advance. But my own feeling is that any portion of classical mathematics that remains totally resistant to such a reduction is perhaps not worth keeping."

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Kreisel and ***Krivine*** put forward the view that "Foundational studies are concerned with describing and analyzing so-called 'intuitive' or 'informal' mathematics, i.e. mathematics as understood by ordinary working mathematicians." This quasi-sociological view is shored up by the claim that the descriptive part of the subject involves the reformulation of informal mathematics in a *formal language*, e.g. (but not necessarily) set theory.

> "Foundational studies proper are concerned with [questions] which may require considerations quite different in character from those of ordinary mathematics. In particular, in foundations we try to find (a theoretical framework permitting the formulation of) good reasons *for* the basic principles accepted in mathematical practice, while [mathematics itself] is only concerned with derivations *from* these principles."

Accordingly,

"the methods used in foundations will necessarily go beyond those of mathematical practice: the discovery of the new concepts and methods needed may involve distinctly philosophical considerations, and in particular, one's conception of the nature of mathematics. If (1) one holds the view that intuitive mathematics is essentially concerned with certain (abstract) objects, one will be led to a "realist" theory of these basic objects: in such a system of foundations the meaning of intuitive statements is analyzed in terms of this theory and the rules of reasoning are deduced from the laws obeyed by these basic objects. Realist foundations are thus analogous to theoretical physics which explains ordinary physical phenomena in terms of fundamental constituents of the physical world (elementary particles in the current theory). But if, (2), one holds the view that the essence of intuitive mathematics consists in proof, or, more specifically, the various kinds of proof, one will be led to an "idealist" system of foundations, which refers to mathematical activity itself." [Kreisel has also remarked that classical and constructivist mathematics each use the appropriate methods to describe different parts of the same world (respectively, mathematical objects and mathematical evidence).]

As an example of (1) K & K mention *set-theoretic semantic foundations* and as an example of (2) *combinatorial syntactic foundations.* (Another instance of an "idealist" foundation, not in its origin explicitly concerned with proof, is *Brouwerian intuitionism.* Still another, espoused in particular by Hermann Weyl, is *Husserlian phenomenology.*)

As far as set-theoretic foundations are concerned, K & K point out that the concept of set initially involved certain ambiguities, which had to be sorted out before it became acceptable as a foundational tool.

> "The notion of set was introduced as a crude mixture containing at least 3 different elements. Sets were considered:
>
> **1.** as mere analogues of finite collections (a notion which was supposed to be understood) satisfying more or less the same laws [this is the set concept espoused by Mayberry];
> **2.** as arbitrary collections of a *given* collection; this occurs throughout mathematics (sets *of* integers, or sets *of* points; the collection of integers and the collection of points (real numbers) being taken to be well-defined);
> **3.** as an abstraction from the more general notion of *property*, a set being the collection of objects which have a given property. (Since properties defined in different ways may be satisfied by the same objects, the notion of set is here conceived as an invariant of properties [that is, a property treated in extension]). There is little use in mathematics itself of properties for which we have no *a priori* bound on the kind of objects which satisfy them [this is why Cantorian set theory confines attention to *bounded* collections]: but both in logic and in everyday language such properties are used widely. An instance is the property of being non-empty (which, incidentally, applies to itself); or the property of being blue; for, even if it has a bound, we use this property without any clear idea of the class of all blue things (past, present, or future)."

According to K & K, flagrant errors (contradictions) are rare in mathematical uses of the notion of set because in any particular deduction *one* of the notions is tacitly understood.

After the discovery of Russell's paradox, both Russell and Zermelo formulated what K & K identify as the *precise notion of set*, that is, the *type- theoretic notion.* Zermelo's version of this is the *cumulative type structure* (cts). This is defined as follows. We start with some (possibly empty) collection C_0 of individuals, i.e. objects which have no members. Then, for any ordinal $\alpha > 0$,

$C_\alpha = \bigcup_{\beta<\alpha} C_\beta \cup PC_\beta$, the union, for all $\beta < \alpha$, of C_β and all its parts. Besides the basic logical operations on sets there is now present the additional operation *P* and its iteration to transfinite α. [It might be asked whether these two procedures are really "precise"! Because of the non-absoluteness of the power-set operation, we can only require that it be in some sense uniform over the whole hierarchy, that is, that the meaning of the term "all parts of *x*" be independent of *x.* In the case of iteration, the idea of induction must be extended somehow into the transfinite. Given all this, the set-theoretic axioms are then evidently true for the cumulative type structure. Mathematical objects can be identified as sets within this structure, and their properties established by derivation from the axioms.

In combinatorial, or "formalist" foundations, the objects with which one deals are finite combinations of concrete objects such as letters of an alphabet, numerals, symbols of a formal language etc. A combinatorial function is a mechanical rule together with a combinatorial proof of functionality: the rule is applied to an object's description rather than to the object itself. For a proof to be combinatorial it must only involve a finite number of combinatorial functions and the sequence of the basic objects. In a combinatorial foundation reasoning about mathematical objects is studied, not the objects themselves. The truth of mathematical propositions is replaced by demonstrability, and adequacy of the scheme by consistency. This is the essence of *Hilbert's program.*"

K & K sum up the difference between set-theoretic and combinatorial foundations as follows:

> "Both provide an answer to the question: what is mathematics? Set theory formulates a particular "realistic" view, and therefore concentrates on objects, not on the reasoning about them; its answer is that mathematics is the theory of sets, for a suitably precise notion of set. Combinatorialism formulates a particular "idealist" view, regards abstract mathematical objects as figures of speech, and wants to show that our way of using these figures of speech is coherent. What is particular about this view is that … our mathematical reasoning is 'essentially' combinatorial…that the validity of our conclusions which can be combinatorially formulated at all, can also be established by combinatorial methods. This view, if correct, not only asserts a *unity* of mathematical reasoning, but one of a very remarkable kind: since school mathematics is typical of combinatorial mathematics, it presents the whole of mathematics as being of the same kind as school mathematics!"

Elsewhere Kreisel has remarked on realism and set theory:

> "In set theory the emphasis is not on the process of reasoning, but on the results and, in particular, on the objects about which assertions are made In consequence, if formalization is used as the descriptive scheme for reasoning, the justification of the rules consists in showing that the conclusions are *valid.* While the rules themselves are not problematic, in the untutored reasoning about sets they were, and led to contradictions. Closer examination of what objects sets are (cumulative type structure) explains the restrictions on the rules. As a theory, set theory is attractive because it has few primitives, and other mathematical objects are built up naturally: if it was an achievement to build up the physical world from 100-odd atoms, how much more striking to build up the world of mathematics from two primitives. Furthermore the laws for these primitives are elegant and surprisingly clear. [My remark: this might be contrasted with the "achievement" of constructing the English language with an alphabet of 26 letters or Western music with the notes of the chromatic scale.]
>
> More generally, the realist conception is certainly very close to the way a good deal of mathematics presents itself to us, and it explains the objectivity of mathematics, that is, the agreement on results, by its being about external objects with which we are in some kind of contact. As has been pointed out by Gödel, there is considerable similarity in the methods of acquiring knowledge in elementary mathematics and physics. Also, it would be agreed that the realist assumption of external mathematical objects is not more *dubious* than that of physical objects. [My remark: I emphatically disagree here.]

> The *weakness* [of the realist conception] lies elsewhere. .. I do not know a formulation of the realist view for which experience establishes the existence of infinite sets, let alone inaccessible ones. Concerning the relation to physical objects it is certainly unusual in physics to regard the existence of objects as established because we can think of them." [My remark: this seems more characteristic of *conceptualism* rather than realism.]

And on intuitionist and idealist conceptions of mathematics:

> "In the finitist [formulation], constructions are applied only to (concrete, that is, spatio-temporal) configurations, in intuitionist also to abstract objects, such as functions and functionals and, particularly, "logical" operations on so-called undecided properties and proofs.
> Intuitionism is a very narrow version of the idealist conception of mathematics, not unlike solipsism within general idealist philosophy…. As is to be expected from the solipsist tradition, the criticisms of rival positions are extravagant and unconvincing—and possibly intended to cover up the real difficulties in formulating a solipsist position coherently. But it is to be remarked that this position is quite plausible, at least as a first approximation, when one is interested in questions of *evidence* (or in the different question of intelligibility). The solipsist position stresses the *particular* evidence of those ideas which are themselves about other ideas or, more particularly, about mental acts, and not about external objects.
> The idealist conception is probably the most commonplace one at the present time: mathematics is a free mental activity. [My remark: I doubt that this view is the most commonplace among mathematicians. In any case, as a mental activity can hardly be entirely "free", like diatonic music, it is greatly constrained by rules.] As such it does not exclude the existence of mathematical objects external to us; they would have the same structure as the ideas involved, and so mathematics would be about them too. Certainly, at an elementary level, one does not ask oneself whether, for example, simple arithmetic statements are about concrete realizations (finite sets of things), one's ideas of such configurations, or about some abstract entities. Further, even if one admits abstract objects external to us, at *a certain stage* it would be quite in accordance with scientific practice to ignore them if the available information is dubious and confusing. This is done by the realist, too, who certainly (properly) ignores questions about the organization of the brain without denying that this may be involved in a more delicate study of these external objects.
> The *weakness* of the general idealist position is rather that it is not sharp enough. In particular, I know of no formulation which excludes taking the whole of formal set theory as laws of our *ideas* of collections existing independently of us. This suggests the possibility of a *foundation* of the theory of sets in terms of more *general* notions.
> In general, the primitive concepts of an idealist conception would not enter directly into mathematical practice, but would be used as an analysis, that is, for foundations."

Interlude: Variation and Logic

Underlying the evolution of mathematics and philosophy has been the attempt to reconcile a number of interlocked oppositions: the One and the Many, the Finite and the Infinite, the Determinate and the Random, the Discrete and the Continuous, the Constant and the Variable.

It was traditionally assumed that a single overarching system of reasoning, governed by classical logic was applicable *pari passu* to all these oppositions. But does a single logic really suffice?

The world as we perceive it is in a perpetual state of flux. But the objects of mathematics are usually held to be eternal and unchanging. How then, is the phenomenon of variation to be given mathematical expression? Consider, for example, a fundamental and familiar form of variation: *change of position,* or *motion,* a form of variation so basic that the mechanical materialist philosophers of the 18th and 19th centuries held that it subsumes all forms of physical variation. Now motion is itself reducible to a still more fundamental form of variation—*temporal* variation. (It may be noted here that according to Whitehead even this is not the ultimate reduction: cf. his notion of "passage of nature".) But this reduction can only be effected once the idea of *functional dependence* of spatial locations on temporal instants has been grasped. Lacking an adequate formulation of this idea, the mathematicians of Greek antiquity were unable to produce a satisfactory analysis of motion, or more general forms of variation, although they grappled mightily with the problem. (It should be mentioned that the problem of analyzing motion was compounded by Zeno's paradoxes, which were designed to show that motion was impossible, and that in fact the world is a Parmenidean unchanging unity.)

It was not in fact until the 17^{th} century that motion came to be conceived as a functional relation between space and time, as the manifestation of a dependence of variable spatial position on variable time. This enabled the manifold forms of spatial variation to be reduced to the one simple fundamental notion of temporal change, and the concept of motion to be identified as the *spatial representation* of temporal change. (The "static" version of this idea is that space curves are the "spatial representations" of straight lines.)

Now this account of motion (and its central idea, functional dependence) in no way compels one to conceive of either space or time as being further analyzable into static indivisible atoms, or *points.* All that is required is the presence of two domains of variation—in this case, space and time— correlated by a functional relation. True, in order to be able to *establish* the correlation one needs to be able to *localize* within the domains of variation, (e.g. a body is in place x_i at "time" t_i, $i = 0, 1, 2, \ldots$) and it could be held that these domains of variation are just the "ensemble" of all conceivable such "localizations". But even this does not necessitate that the localizations themselves be atomic points—cf. Whitehead's method of "extensive abstraction" and, latterly, the rise of "pointless topology".

The incorporation of variation into mathematics in the 17^{th} century led, as is well known, to the triumphs of the calculus, mathematical physics, and the mathematization of nature. But difficulties arose in the attempt to define the instantaneous rate of change of a varying quantity—the fundamental concept of the differential calculus. Like the ancient Pythagorean effort to reduce the continuous to the discrete, the attempt by 17^{th} century mathematicians to reduce the varying to the static through the use of infinitesimals led to outright *contradictions.*

It was thought, e.g. by Marx and Engels, that the analysis of objective variation would require the creation of a dialectical logic or "logic of contradiction". But what in fact occurred (in the 19^{th} century) was the effective replacement of variation by actual infinity at the hands of Dedekind, Cantor *et al.* Cantor in particular replaces the concept of a varying quantity by that of a completed, static *domain of variation* which may itself be regarded as an ensemble of atomic individuals—thus, like the Pythagoreans at the same time replacing the continuous by the discrete. He also banishes infinitesimals and the idea of geometric objects as being generated by points or lines in motion.

But some mathematicians raised objections to the idea of "discretizing" or "arithmetizing" the linear continuum. Peirce, for example, rejected the idea that a true continuum can be completely analyzed into a collection of discrete points, no matter how many of them there might be. This is reminiscent of the dispute between the atomists and the Stoics over essentially the same issue circa 400 B.C.

It was only with Brouwer at the start of the 20^{th} century that logic becomes genuinely involved in the debate. Rejecting the Cantorian account of the continuum as discrete, Brouwer identifies points on the line as entities "in the process of becoming" in a subjective sense, i.e. as embodying a certain kind of variations. He rejects the law of excluded middle for such objects, a move which leads to a new form of logic, *intuitionistic logic.*

It is a remarkable fact that the intuitionistic logic of Brouwer is compatible with a very general concept of variation, which embraces all forms of (objective) continuous variation, and which in particular allows the use of (continuous) infinitesimals. While its roots lie in the subjective, intuitionistic logic is thus revealed to have an *objective* character. The application of intuitionistic logic to resolve the contradiction engendered by variation shows that it was not in the end necessary—as claimed by "dialectical" philosophy— to reject the *law of noncontradiction* $\neg(A \wedge \neg A)$, but rather its dual the *law of excluded middle* $A \vee \neg A$.

It is a characteristic of intuitionism that, once a property of a mathematical object has been established by means of a construction, the property remains established "for all time"; it is, in a word, *unalterable*. This is reflected in the *persistence property* of the semantics of intuitionistic logic: that a statement, once "forced" be true, remains true. This suggests that intuitionistic logic can, roughly, be regarded as the logic of the *past tense*: a statement of the form "such and such was the case" once true, remains true forever (provided, of course, the universe contains no closed time-like lines). This is a particular case of an association among types of variation, philosophical attitude and logic, as indicated in the following concordance:

Type of variation	**Philosophical attitude**	**Logic**
Static: no variation: eternal present: objective state of affairs independent of our knowledge:	Platonic realism	Classical
Cumulative: no revision of information at later stages: once known, always known	Broad constructivism Kantian idealism	Intuitionistic
Noncumulative: possible revision, falsification, or loss of information at later stages.	Indeterminism Humean scepticism	"Quantum"

Indeed certain general types of variation can be correlated with certain concepts and branches of mathematics:

Type of variation	**Mathematical Correlate**
Temporal	Natural numbers (discrete) Real line (continuous)
Positional (motion)	Real line Differential Calculus Mathematical Analysis
Morphological	Topology Category Theory

Morphological variation—change of form—is, broadly speaking, the subject matter of *category theory*, which we next consider as a possible foundation for mathematics.

Category theory provides a general apparatus for dealing with mathematical structures and their mutual relations and transformations. Invented by Eilenberg and Mac Lane in the 1940s, the discipline originated as a branch of algebra ("homological algebra") by way of topology, but quickly transcended its origins. Category theory may be said to bear the same relation to abstract algebra as the latter does to elementary algebra. For elementary algebra results from the replacement of *constant quantities* (i.e., numbers) by *variables*, leaving the operations on these quantities fixed. Abstract algebra, in tis turn, carries this a stage further by allowing the *operations* to vary while insisting that the ambient mathematical structures maintain a certain prescribed *form* (groups, rings, etc.) Finally, category theory allows even the structure to vary, giving rise to a far-reaching account of mathematical form. The rise of category theory may accordingly be seen as an instance of the dialectical process of replacing the *constant* by the *variable.*

Categories may be considered as frameworks for the analysis of variation. Thus we suppose given

- *domains of variation* or *types A, B, C, …*
- *transformations* or *correlations f: A* → *B* or $A \xrightarrow{f} B$ between such domains: *A* and *B* are said to be *correlated* by *f; A* is the *domain*, *B* the *codomain* of *f.*

As concrete examples we may consider:

Space → *Time*	Analogue clocks
Natural numbers → *Time*	Digital clocks
Time → *Space*	Motions
Space → *Rational/Real numbers*	Thermometers, barometers, speedometers

A correlation *A* → *B* may be thought of as a *B-valued quantity varying over A.* As such, correlations may be *composed*:

$$\frac{A \xrightarrow{f} B \xrightarrow{g} C}{A \xrightarrow{g \circ f} C}$$

E.g. the use of a digital stopwatch amounts to the composite correlation

Natural numbers → *Time* → *Space*

Composition of correlations is *associative* in the evident sense.

Associated with each domain *A* is an *identity* correlation $A \xrightarrow{1_A} A$ satisfying $f \circ 1_A = f$, $g \circ 1_A = g$ for any $A \xrightarrow{f} B$, $C \xrightarrow{g} A$.

These are the basic data of a *category.*

From a philosophical standpoint, a category may be viewed as an explicit presentation of a *mathematical form or concept.* The objects of a category C are the *instances* of the associated form and the morphisms or arrows of C are the transformations between these instances which in some specified sense "preserve" this form. As examples we have:

Category	**Form**	**Transformations**
Sets (Set)	Pure discreteness	Functional correlations
Sets with relations	....	One-many correlations
Groups	Composition/inversion	Homomorphisms
Topological spaces	Continuity	Continuous maps
Differentiable manifolds	Smoothness	Smooth maps

Because the practice of mathematics has, for the past century, been officially founded on set theory, the objects of a category are typically constructed as *sets* of a certain kind, synthesized, as it were, from pure discreteness. As sets, these objects manifest set-theoretic relationships—memberships, inclusions, etc. However, these relationships are irrelevant—and in many cases are actually *undetectable*—when the objects are considered as embodiments of a form, i.e., viewed through the lens of category theory. (For example, in the category of groups the additive group of even integers is isomorphic to, i.e. indistinguishable from, the additive group of all integers.) This fact constitutes one of the "philosophical" reasons why certain category theorists have felt set theory to be an unsatisfactory basis on which to build category theory—and mathematics generally. For categorists, set theory provides a kind of ladder leading from pure discreteness to the category-theoretic depiction of the real mathematical landscape. Categorists are no different from artists in finding the landscape (or its depiction, at least) more interesting than the ladder, which should, following Wittgenstein's advice, be jettisoned after ascent.

The contrasts between set theory and category theory may be summed up in the following table:

Set Theory	**Category Theory**
Analytic	Synthetic
Atomistic	Holistic
Static	Variable
Arithmetic	Geometric
Reduction of mathematical concepts	Direct representation of mathematical concepts

The generality of category theory has enabled it to play an increasingly active role in the foundations of mathematics. Its emergence has had the effect of subtly undermining the prevailing doctrine that all mathematical concepts are to be referred to a fixed absolute universe of sets. Category theory, by contrast, suggests that mathematical concepts should be regarded as possessing meaning only *in relation to* a variety of more or less *local* frameworks. Interpreting a mathematical concept within a category amounts to a kind of refraction or filtering of the concept through the form associated with the category. For example, the interpretation of the concept *group* within the category of topological spaces is *topological group*, within the category of manifolds it is *Lie group*, and within a category of sheaves it is *sheaf of groups*. In this way the group concept, and many other mathematical concepts, acquire a truly protean generality, a further *ambiguity of reference* over and above that already conferred on them through the possession of differing set-theoretic realizations.

In category theory the transformations—*maps, morphisms* or *arrows*—between structures—*objects*—play an autonomous role which is in no way subordinate to that played by the structures themselves. Category theory may thus be compared to a language in which the verbs are on an equal footing with the nouns. In this respect category theory differs crucially from set theory in which the corresponding notion of function is reduced to the concept "set". As a consequence, the notion of transformation in category theory is vastly more general than the set-theoretical notion of function. In particular, the former admits interpretations in which one variable quantity depends functionally on another but where the corresponding "function" is not describable as a set of (ordered pairs of) "points" (for instance, when the functional dependence arises as the phenomenological description of the motion of a body.) The fact that in category theory the concept of transformation is an irreducible basic datum makes it possible to regard arrows in categories as formal embodiments of the idea of *pure variation* or *correlation*, that is, of the idea of *variable quantity* in its original pre-set-theoretic sense. For example, in category theory the variable symbol x with domain of variation X is interpreted as an *identity arrow* (1_X), and this concept is not further analyzable, as it is in set theory, where it is just the set of ordered pairs of the form (x, x). Thus the variable x now suggests the idea of pure variation over a domain, just as intended within the usual functional notation $f(x)$. This latter fact is expressed in category theory by the "trivial" axiomatic condition

$$f \circ 1_X = f,$$

in which the symbol x does not appear: this shows formally that variation is, in a sense, an intrinsic constituent of a category.

There are a number of versions of specifically *category-theoretic foundations* for mathematics. The first was put forward in 1965 by ***Lawvere***. In his *The Category of Categories as a Foundation for Mathematics* he asserts:

> "In the mathematical development of recent decades one sees clearly the rise of the conviction that the relevant properties of mathematical objects are those which can be stated in terms of their abstract structure rather than in terms of the elements which the objects were thought to be made of. The question thus naturally arises whether one can give a foundation for mathematics which expresses wholeheartedly this conviction concerning what mathematics is about, and in particular *in which classes and membership in classes do not play any role.* Here by 'foundation' we mean a single system of first-order axioms in which all usual mathematical objects can be defined and all their usual properties proved. A foundation of the sort we have in mind would seemingly be

> much more natural and readily-usable than the classical one when developing such subjects as algebraic topology, functional analysis, model theory of general algebraic systems, etc. Clearly any such foundation would have to reckon with the Eilenberg-Mac Lane theory of categories and functors. The author believes, in fact, that the most reasonable way to arrive at a foundation meeting these requirements is simply to write down axioms descriptive of properties which the intuitively-conceived category of all categories has until an intuitively adequate list is attained; this is essentially how the theory described below was arrived at. Various metatheorems should then be proved to help justify the feeling of adequacy."

Lawvere is thus, in particular, proposing the use of a certain *first-order theory,* the *elementary theory of the category of categories*, as a foundation for mathematics. Such a proposal runs up against the objection that Mayberry has made to the idea of using any axiomatic theory as a foundation for mathematics. In 1981 I made the following observations on the matter:

> "In what sense could category theory serve as a foundation for mathematics? …First, a strong sense in which *all* mathematical concepts, including those of the current logico-metatheoretic framework for mathematics, are explicable in category-theoretic terms.
> Now it seems to me implausible that category theory is, or could be, foundationally adequate in this strong sense. For consider the metatheoretical framework in which category theory (or any other first-order theory) is embedded. This framework has two basic aspects: the *combinatorial*, which is concerned with the formal, finitely presented properties of the inscriptions of the ambient formal language, and the *semantical*, which is concerned with the interpretation and truth of the expressions of that language. Neither one of these aspects is—at present—reducible to the other. The former deals with *intensional* objects such as proofs and constructions whose actual *presentation* is crucial, while the latter employs extensional objects such as classes whose identity is determined independently of how they may be presented or defined. So if category theory is to furnish a foundation for mathematics in the strong sense, it must provide convincing accounts of *both* these aspects. But a category is defined to be a class of a certain kind, and classes are *extensional*, while combinatorial objects are generally *not*. Since there is no reason to suppose that a satisfactory account of intensional objects can be given solely in terms of extensional ones, it seems to me that category theory as currently formulated in terms of classes must fail to provide a faithful account of the combinatorial aspect at least. (Of course, this 'weakness' is shared by set theory.)
> As far as the *semantical* aspect is concerned, we recall that the interpretation of an expression of a classical first-order language involves a reference to *classes* or *pluralities* in an essential way (as the 'range' of the variables in the expression). In particular, grasping the concept of *logical truth* for sentences of classical first-order languages requires that one has already grasped the concept of class. To put another way, the concept of class is *epistemically prior* to the concept of (classical) logical truth. So if category theory is to serve as an autonomous basis for classical semantics, and in particular give a satisfactory independent account of logical truth, it must be possible to give an explication of class (at least in so far as it is involved in deriving the concept of logical truth) *solely in terms of the notion of category*, and without already having defined the latter notion in terms of classes. But this seems to me highly dubious, for it is surely the case that the unstructured notion of class is epistemically prior to any more highly structured notion such as category: in order to know what a category is, you first have to know what a class is. This applies, *mutatis mutandis,* to the notion of *functor* whose explication involves grasping the idea of *operation.*
> It seems to me that these considerations show that category theory as currently conceived is not capable of serving as a foundation for mathematics in the strong sense. Of course, this is hardly surprising since it is widely recognized that *no* single foundational scheme is at present capable of providing a convincing explication of both combinatorial and set-theoretical objects. What we actually possess is an informal system of 'multiple' foundations, with distinct set-theoretical and combinatorial constituents."

I felt, however, that

> "Category theory is more than just another abstract mathematical theory. Like set theory, it provides a general framework for dealing with mathematical structures, and—again like

set theory—it achieves this by transcending the *particularity* of structures. But set theory and category theory go about doing this in entirely different ways. Set theory strips away structure from the ontology of mathematics leaving pluralities of structureless individuals open to the imposition of new structure. Category theory, on the other hand, transcends particular structure not by doing away with it, but by *generalizing* it, that is, by producing an *axiomatic general theory of structure.* The success of category theory, and its significance to foundations, is due to the *ubiquity of structure* in mathematics.
It may be said that category theory, while still dependent on set theory as the ultimate source of mathematical entities, nevertheless frees mathematics from the particular *form* imposed on it by having to regard these entities as discrete pluralities of elements."

Further work on axiomatizing the category $\mathcal{C}at$ of categories has been carried out by ***Blanc*** and ***Donnadieu***. In their axiomatization, it can be shown that the (meta)category of discrete objects (that is, the counterpart of the category of sets) is a well-pointed topos. Blanc and Donnadieu's arguments are carried out in classical logic and assume the axiom of choice. It would be of interest to determine whether their system can be provided with a suitably constructive formulation in such a way that the category of discrete objects can be shown to be an elementary topos. Suitable constructive axioms for $\mathcal{C}at$ might be found by analyzing its structure within the free topos (see below).

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Another approach to category-theoretic foundations of mathematics is through the notion of (elementary) *topos.* Although topos theory has its origin in sheaf theory, itself arising out of algebraic topology and algebraic geometry, the topos concept may be presented as a generalization of the set concept. Thus we start with the familiar category $\mathcal{S}et$ whose objects are all sets (for precision, in a given model M of set theory) and whose arrows are all mappings (in M) between sets (in M). We observe that $\mathcal{S}et$ has the following properties:

1. There is a 'terminal' object 1 such that, for any object X, there is a unique arrow $X \to 1$ (for 1 we make any one-element set, in particular {0}).
2. Any pair of objects A, B has a Cartesian product $A \times B$.
3. For any pair of objects A, B one can form the 'exponential' object B^A of all mappings $A \to B$.
4. There is an 'object of truth values' Ω such that for each object X there is a natural correspondence between subobjects (subsets) of X and arrows $X \to \Omega$. (For Ω one may take the set $2 = \{0, 1\}$; arrows $X \to \Omega$ are then *characteristic functions* on X, and the exponential object Ω^X corresponds to the *power set* of X.)

All four of these conditions can be formulated in the first-order language of categories: a category satisfying them is called an (elementary) *topos.*

The concept of topos is one of great generality. Not only is $\mathcal{S}et$ a topos, but also, for example, all of the following: (1) the category of Boolean-valued sets and mappings within a Boolean extension of a model of set theory; (2) the category of sheaves or presheaves of sets on a topological space; (3) the category of all diagrams of mappings of sets

$$X_0 \to X_1 \to X_2 \to \ldots$$

The objects of each of these categories may be regarded as sets which are *varying* in some manner: in case (1) over a *Boolean algebra*, in case (2) over a *topological space,* and in case (3) over *discrete time.* (So, in this parlance, the category $\mathcal{S}et$ itself is the category of sets "varying" over the one point set 1.) These examples suggest that a topos may be conceived as a category of *variable sets*: the familiar category $\mathcal{S}et$ is the limiting case in which the variation of the objects has been reduced to zero. For this reason $\mathcal{S}et$ is called a *topos of constant sets.* Thus, like the notion of category itself, the topos concept turns out to be another instance of the dialectical procedure of replacing the *constant* by the *variable.*

In a topos the notion of pure variation is combined with the fundamental principles of construction employed in ordinary mathematics through set theory, viz., forming the *extension of a predicate, Cartesian products*, and *function spaces.*

In set theory one has natural "logical operations" defined on the object 2 of truth values arising as counterparts of "set-theoretic" operations on parts of domains—an idea first made explicit by Boole. The richness of a topos's internal structure enables this correspondence to be carried through there as well. Thus in any topos $\mathscr{E}$ we get natural arrows defined on its object Ω of truth values which may be thought of as internally defined "logical operations" on $\mathscr{E}$. Since these logical operations are defined entirely in terms of the topos's internal mathematical structure, a topos may be regarded as an *apparatus for synthesizing logic from mathematics.* The logical operations arising in this way in general obey the laws of *intuitionistic* logic: usually $\neg \circ \neg \neq \neg$. In a topos Ω is, in general, a *Heyting algebra.* (In the topos of sheaves over a topological space *X,* the points of Ω correspond to the open sets of *X.* So Ω is rarely a Boolean algebra in this case.) The structure of Ω determines the *internal logic* of the ambient topos. Classical bivalent logic requires Ω to have just the two points **true, false,** so that Ω = **2.** This is implied by the condition that any correlation can be reduced to constancy in the following sense: for any $f, g : A \to B$, if for all $1 \xrightarrow{x} A$, $fx = gx$, then *f* = *g*. (Proof: suppose $U \rightarrowtail 1$, $U \neq 0$, and form *U* + *U.* There are then two different arrows $U \to U + U$, and hence an arrow $1 \to U$. It follows that *U* = 1.)

In a topos, as in set theory, every object—and indeed every arrow—can be considered in a certain sense as the extension $\{x: P(x)\}$ of some predicate *P.* The difference between the two situations is that, while in the set-theoretic case the variable *x* here can be construed *substitutionally*, i.e. as ranging over (names for) individuals, in a general topos this is no longer the case: the *"x"* must be considered as a *true variable.* More precisely, while in set theory one has the rule of inference

$$\frac{P(a) \text{ for every individual } a}{\forall x P(x)}$$

in general this rule fails in the internal logic of a topos. In fact, assuming classical set theory as metatheory, the correctness of this rule in the internal logic of a topos forces it to be a model of classical set theory: this result can be suitably reformulated in a constructive setting.

[A recent development of great interest in the relationship between category theory and set theory is the invention by Joyal and Moerdijk [3] of the concept of *Zermelo-Fraenkel algebra.* This is essentially a formulation of set theory based on set *operations,* rather than on properties of the membership relation. The two operations are those of *union* and *singleton*, and Zermelo-Fraenkel algebras are the algebras for operations of these types. (One notes, incidentally, the resemblance of Zermelo-Fraenkel algebras to David Lewis' "megethelogical" formulation of set theory.) Joyal and Moerdijk show that the usual axiom system *ZF* of Zermelo-Fraenkel set theory with foundation is essentially a description (in terms of the membership relation) of the *free* or *initial* Zermelo-Fraenkel algebra, just as the Peano axioms for arithmetic describe the free or initial monoid on one generator. This idea can be extended so as to obtain a charcterization of the class of *von Neumann ordinals* as a free Zermelo-Fraenkel algebra of a certain kind. Thus both well-founded set theory and the theory of ordinals can be characterized category-theoretically in a natural way.]

Topos theory does much more than merely reorganize the mathematical materials furnished by set theory: its function far transcends the purely cosmetic. This is strikingly illustrated by the various topos models (Spaces)—of *synthetic differential geometry* or *smooth infinitesimal analysis*. Here we have an explicit presentation of the form of the smoothly continuous incorporating actual infinitesimals which is simply *inconsistent* with classical set theory: a form of the continuous which, in a word, *cannot* be reduced to discreteness. In these models, *all* transformations are smoothly continuous, realizing Leibniz's dictum *natura non facit saltus* and Weyl's suggestions in *The Ghost of Modality*, and elsewhere. In Spaces all correlations between objects are continuous (thus realizing Leibniz's dictum: *Natura not facit saltus*) and here logic *cannot* be bivalent. (For a connected continuum has no nonconstant continuous maps to **2**, but does possess many nontrivial parts. So Ω cannot coincide with **2.**) Nevertheless, extensions of predicates, and other mathematical constructs, can still be formed in the usual way (subject to intuitionistic logic). Two further arresting features of continuity manifest themselves. First, connected continua are *indecomposable:* no proper nonempty part of a connected continuum has a "proper" complement—cf. Anaxagoras' c. 450 B.C. assertion that the (continuous) world has no parts which can be "cut off by an axe". And secondly, any curve can be regarded as being traced out by the motion of an *infinitesimal tangent vector*—an entity embodying the (classically unrealizable) idea of *pure direction*—thus allowing the direct development of the calculus and differential geometry using nilpotent infinitesimal quantities. These near-miraculous, and yet natural ideas, which *cannot* be dealt with coherently by reduction to the discrete or the notion of "set of distinct individuals" (cf. Russell, who in *The Principles of Mathematics* roundly condemned infinitesimals as "unnecessary, erroneous, and self-contradictory"), can be explicitly formulated in category-theoretic terms and developed using a formalism resembling the traditional one.

Establishing the consistency of smooth infinitesimal analysis through the construction of topos models is, it must be admitted, a somewhat laborious business, considerably more complex than the process of constructing models for the more familiar (discrete) theory of infinitesimals known as *nonstandard analysis*. I think the situation here can be likened to the use of a complicated film projector to produce a simple image (in the case at hand, an image of ideal smoothness), or to the cerebral activity of a brain whose intricate neurochemical structure contrives somehow to present simple images to consciousness. The point is that, although the fashioning of smooth toposes is by no means a simple process, it is designed to realize simple principles. The path to simplicity must, on occasion, pass through the complex.

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One approach to the foundations of mathematics employing topos theory is an idea I put forward some years ago, that of "local mathematics". Here the fundamental idea is

> 'to abandon the unique absolute universe of sets central to the orthodox set-theoretic account of the foundations of mathematics, replacing it by a plurality of local mathematical frameworks—*elementary toposes*—defined in category-theoretic terms. Such frameworks will serve as local surrogates for the classical universe of sets. In particular they will possess sufficiently rich internal structure to enable mathematical concepts and assertions to be interpreted within them. With the relinquishment of the absolute universe of sets, mathematical concepts will in general no longer possess absolute meaning, nor mathematical assertions absolute truth values, but will instead possess such meanings or truth values only *locally*, i.e., *relative* to local frameworks."

I continued:

> "The techniques of Cohen and his successors have led to an enormous proliferation of models of set theory with essentially different mathematical properties, which in turn has engendered a disquieting uncertainty in the minds of set-theorists as to the identity of the "real" universe of sets, or at least as to precisely what mathematical properties it should possess. The upshot is that the set concept—insofar as it is capturable by first-order axioms—has turned out to be *radically underdetermined.*
> What I suggest is that we *accept* the radically underdetermined nature of the set concept and *abandon* the quest for the absolute universe of sets in the form proposed by classical set theory. "

I then took leave of sobriety altogether by proposing that the meaning or reference of set-theoretical concepts be determined *locally*, that is, within elementary toposes (called here "local frameworks"). In this event,

> "an assertion like the continuum hypothesis will no longer be regarded as the possessor of an absolute but unknown truth value, for the unique universe of sets which was presumed to furnish the said truth value will no longer exist. Note, however, that although the concept of *absolute truth* of set-theoretical assertions will have vanished from the scene, there will appear in its place the subtler concept of *invariance*, that is, *validity in all local frameworks.* Thus, e.g., while the theorems of constructive arithmetic will turn out to possess the property of invariance, the axiom of choice or the continuum hypothesis will not, because they will hold true in some local frameworks but not others."

Toposes are precisely the models for theories formulated within a natural typed higher order language based on intuitionistic logic. Each topos $\mathcal{E}$ is associated with such a language whose types match the objects of $\mathcal{E}$ and whose function symbols match the arrows of $\mathcal{E}$. Given a theory T in such a language, (a *local set theory*) a topos $\mathcal{E}_T$ can be constructed whose valid sentences are precisely those of T, and conversely, given a topos $\mathcal{E}$, we can form a theory $T_{\mathcal{E}}$ (the set of sentences "true" in $\mathcal{E}$) whose associated topos $\mathcal{E}_{T_{\mathcal{E}}}$ is categorically equivalent to $\mathcal{E}$.

Any topos may be regarded as a mathematical domain of discourse or "world" in which mathematical concepts can be interpreted and mathematical constructions performed. In this event, the associated local set theory may be treated as a "chart" mapping that world. Just as all the charts in an atlas share a common geometry, so all local set theories share a common logic, the intuitionistic logic of types.

In a topos $\mathcal{E}$, arrows $1 \to \Omega$ are *truth values* of propositions interpreted in $\mathcal{E}$: there are always at least the two truth values $\top, \bot: 1 \to \Omega$ and corresponding to truth and falsehood, but there may be more than just these. A topos in which these are the only truth values is called *bivalent*; this property corresponds to *completeness* of the associated local set theory. Since truth in a topos corresponds to provability in a local set theory, and most local set theories are *incomplete*, we must be prepared to accept the phenomenon of *polyvalence*, that is, the fact that mathematical propositions—formulated in local set theories and interpreted in toposes—will in general possess truth values *different* from truth or falsehood.

A topos and its associated theory may be seen as dual aspects of the same entity—let us call it a *world*—the topos constituting, so to speak, its *substance* and the theory its *form*. Like a work of fiction, the substance of a world is determined entirely by its form: any (meaningful) question which can be asked about the substance is either answerable within the form or, if not (as in the case of an incomplete theory), must be regarded as having no determinate answer within the given world. Just as, for instance, in Kafka's *The Castle*, the remaining letters of its protagonist's *K*'s name must remain forever undetermined since no scrutiny of the text will ever reveal them, so, analogously, the 'truth' or 'falsity' of the continuum hypothesis will always be indeterminate in the world associated with the free classical local set theory.

Topos theory leads to a concordance between *formal* and *substantive* notions:

Formal	**Substantive**
Local set theories	Toposes
Pure local set theory	Free topos
Provability	Truth
Completeness	Bivalence

Confining attention to the left-hand side of this scheme, we get *formalism*; to the right, *realism* (or Platonism). The concordance between the left- and right-hand sides of this scheme suggests the possibility of a rapprochement between these two opposed doctrines, one extending over those parts of mathematics formulable within toposes or local set theories.

Colin McLarty has presented the central point of topos theory with vigour:

> "The point is that toposes describe objective structures. The world around us has a geometric structure that can be idealized in the notion of smooth spaces and maps, as indeed it was in classical analysis in the service of Newton's and later Einstein's physics. The smooth topos **Spaces** formalizes that structure. Another abstraction moves away from geometry to view the world in terms of pure cardinality. This is Cantor's set theory, and is formalized in the topos **Set.** Yet another treats functions as procedures, and so requires them to have algorithms. This is formalized in part of the effective topos **Eff**. These are not competing theories, much less contradictory: they are alternatives suited to different purposes.

> There is no meaningful question of whether all functions from the line to itself are ‘really’ differentiable, or all functions from the natural numbers to themselves are recursive. Rather, we need to study both of these and other idealizations , and the relations between them, whether we model them in sets of toposes or whatever.”

Topos theory is, according to McLarty*,* *objective* and yet at the same time *pluralistic*: toposes describe, in an idealized way, objective aspects of the world, but no unique topos describes the world in its totality. This is somewhat akin to Aristotle’s view of the Forms: grasped by the mind through a process of *abstraction* from sensible objects, but not thereby attaining an autonomous existence detached from these latter.

There are various analogies between topos theory and the theory of relativity. These may be summed up in the following concordance:

Relativity Theory	**Topos Theory**
1.*Geometrization of physics.* Every physical quantity can be represented as a geometric object and every physical law expressed geometrically.	1. *Categorization of mathematics.* Every mathematical concept and every mathematical assertion can be expressed categorically.
2. *Relativity of physical concepts.* In relativity theory physical quantities such as mass, length and energy are measured with respect to a local coordinate system or reference frame (usually inertial). Thus the measured values of these quantities are not absolute, but are determined only relative to the coordinate system with respect to which the measurement is effected. Of course, in the case of mass, for example, one can introduce the concept of rest mass, i.e. mass measured in a coordinate system with respect to which the body is at rest. This is an invariant quantity, but still involves a reference to a coordinate system so that even it cannot be regarded as the measure of an attribute entirely intrinsic to the body. Rest mass could only be considered an absolute quantity if there existed an absolute coordinate system (absolute space)with respect to which the body was at rest. But in relativity theory there is no absolute space, only spacetime.	2. *Relativity of mathematical concepts.* In topos theory mathematical concepts and assertions are interpreted in local mathematical frameworks, i.e., toposes. This gives rise to a relativity of mathematical concepts which is well illustrated by the phenomenon of *cardinal collapse* (Skolem’s paradox). Suppose we are given a set I in a topos $\mathscr{S}$ of constsnt sets and I has uncountable cardinality in $\mathscr{S}$. Then we can shift (via an admissible transformation—see below) to a new local framework $\mathscr{S}'$ (a Boolean extension of $\mathscr{S}$) in which the cardinality of I is countable. Accordingly the cardinality of an infinite set is not absolute but is determined relative to the local framework w.r.t. which the cardinality is “measured”. One can, to be sure, reintroduce a degree of invariance to the concept by defining the “true” cardinality of I to be its cardinality with respect to the initial framework $\mathscr{S}$ in which I is situated. But this definition again involves a reference to a framework and so cannot be regarded as defining an attribute intrinsic to *I.* The only circumstance in which the cardinality of I could be regarded as absolute would arise when the framework $\mathscr{S}$ is absolute or universal (*the* “universe of sets). But category theory suggests that no such framework exists.
3. *Admissible Changes of Local Coordinate System.* In special relativity these are the Lorentz transformations; in general relativity the smooth maps.	3. *Admissible transformations between Local Frameworks.* In topos theory these are the continuous maps—*geometric morphisms.* Just as in astronomy one effects a change of coordinate system to simplify the description of, e.g. a planet, so it also becomes possible to simplify the formulation of a mathematical concept by effecting a shift of mathematical framework. Consider, for example, the concept “real-valued continuous function on a topological space *X*.” Any such function may be regarded as a real number (or quantity) varying continuously over *X.* Now consider the topos

	$\mathscr{S}h(X)$ of sheaves on X. Here *everything* is varying continuously over X, so shifting from $\mathscr{S}$ to $\mathscr{S}h(X)$ amounts to placing oneself in a framework which is, as it were, itself "comoving" with the variation over X of any given variable real number. This causes its variation not to be "noticed" in $\mathscr{S}h(X)$; it is accordingly regarded as a *constant* real number. In this way the concept "real-valued continuous function on X" is transformed into the concept "real number" when interpreted in $\mathscr{S}h(X)$. Putting it the other way around, the concept "real number", interpreted in $\mathscr{S}h(X)$, corresponds to the concept "real valued continuous function on X" interpreted in $\mathscr{S}$. In topos theory, therefore, a mathematical concept may possess a fixed *sense*, but a variable *reference*. The *sense* of the concept "real number" may be taken as fixed by its definition within a local set theory, but its reference varies with the framework of interpretation.
4. *Invariant physical laws.* These are the statements of mathematical physics (e.g. Maxwell's equations) which, suitably formulated, hold in every local coordinate system.	4. *Invariant mathematical laws.* These are the mathematical assertions which hold in every local framework, viz., the theorems of higher-order intuitionistic logic. Thus the invariant mathematical laws are those that are provable *constructively.* A theorem of classical logic which is not a theorem of intuitionistic logic (e.g. the law of excluded middle) will not hold universally until it has been transformed into its intuitionistic correlate via, e.g., the Gödel translation. The procedure of translating classical into intuitionistic logic is the mathematical counterpart of casting physical laws into invariant (or covariant) form.
5. *Inertial coordinate systems.* The validity of *Newton's first law of motion* singles out the inertial coordinate systems among all possible ones. (An inertial coordinate system is one in which a body initially at rest remains at rest provided it is not subject to impressed forces.) These are the "classical" coordinate systems acting as local surrogates for Newtonian absolute space.	5. *Classical local frameworks.* The truth of the *axiom of choice* in a local framework $\mathscr{E}$ entails that the internal logic of $\mathscr{E}$ is classical. A local framework satisfying the axiom of choice may then be regarded as possessing the essential properties of a classical model of set theory, i.e. as being a local surrogate for the absolute universe of sets. Accordingly the axiom of choice corresponds to Newton's first law and classical local frameworks to inertial coordinate systems.
6. *Non-inertial coordinate systems.* In these bodies undergo spontaneous changes of velocity.	6. *Non-classical toposes.* In these objects undergo internal "variation".
7. *Local introducibility of inertial coordinate system at each point in spacetime.*	7. *Barr's theorem: every (Grothendieck) topos is the surjective image of a classical topos.*
8. *Special Principle of Relativity.* Any pair of inertial local coordinate systems are equivalent under Lorentz transformations. This equivalence asserts that the laws of physics are equivalent in both systems, i.e., they cannot be distinguished by *physical* means.	8. *Principle of Equivalence for Classical Frameworks.* This is the theorem (Lawvere) that any pair of classical local frameworks linked by an admissible transformation are categorically equivalent. Accordingly they cannot be distinguished by *mathematical* means.
9. *Newtonian absolute space* was presumed to provide a universal inertial frame to which all physical phenomena are to be referred.	9. *The classical universe of sets* was presumed to provide a universal framework to which all mathematical assertions and concepts are to be

	referred.
10. *Global Riemannian spacetime/Contrast between global and local physics.* In general relativity, global Riemannian spacetime constitutes the universe or domain in which all physical phenomena actually occur: as such it replaces Newtonian absolute space, and forms a kind of objective metatheory for local physics. Local physics is the study of physical phenomena referred to local coordinate systems; global physics is the study of global Riemannian spacetime.	10. *Global metacategory $\mathcal{T}op$ of local frameworks/Contrast between global and local mathematics.* In topos theory the global metacategory $\mathcal{T}op$ of local frameworks and admissible transformations is the universe in which all the frameworks "live": as such it replaces the classical universe of sets, and forms a metatheory for local mathematics. Local mathematics is the study of mathematical concepts referred to local frameworks; global mathematics (as opposed to absolute mathematics) may be identified as the study of $\mathcal{T}op$. However, an axiomatization of this metacategory is lacking.

The idea of admissible transformation between toposes gives rise, I claimed, to another "dialectical" phenomenon, one that I termed *the negation of constancy.* Suppose that we are given a topos $\mathcal{S}$ of constant sets (i.e. one satisfying the axiom of choice) and an admissible transformation from $\mathcal{S}$ to a topos $\mathcal{E}$: we shall say that $\mathcal{E}$ is *defined* over $\mathcal{S}$. By "set" we shall mean "object of $\mathcal{S}$". We may regard $\mathcal{E}$ as a framework which results when the objects of $\mathcal{S}$ are allowed to *vary* in some fashion: for example, when $\mathcal{E}$ is $\mathcal{S}h(X)$, the objects of $\mathcal{E}$ are objects of $\mathcal{S}$ "varying" continuously over the opens of X. In passing from $\mathcal{S}$ to $\mathcal{E}$ we thus dialectically *negate* the "constancy" of the objects in $\mathcal{E}$, and in so doing inject "variation" or "change" into the new objects of $\mathcal{E}$. In the passage we are, in short, *negating constancy.*

Now in certain important cases we can proceed in turn to dialectically negate the "variation" in $\mathcal{E}$ to obtain a new classical framework $\mathcal{S}^*$ in which constancy again prevails: $\mathcal{S}^*$ may be seen as arising from $\mathcal{S}$ by the dialectical process of *negating negation.* In general, $\mathcal{S}^*$ is not equivalent to $\mathcal{S}$, and so, according to a well-known result of topos theory, the passage from $\mathcal{E}$ to $\mathcal{S}^*$—the second "negation"—cannot be an admissible transformation (but it is a "logical" functor). Thus the act of negating negation in this sense *transcends* admissibility: this is the price exacted for reinstating constancy in passing finally to $\mathcal{S}^*$. I regard the process as a nice example of Hegelian *Aufhebung*—"sublation" or "synthesis".

This particular kind of sublation may be envisioned as underlying two key developments in the foundations of mathematics—*Robinson's non-standard analysis and Cohen's independence proofs in set theory.*

Given a set I, each element $i \in I$ may be identified with the principal ultrafilter $U_i = \{X \subseteq I: i \in X\}$ over I. This identification suggests that we think of *arbitrary* ultrafilters over I as *generalized points* of I. The collection of generalized points of I forms a new set βI (which topologists call the Stone-Čech compactification of I). Identifying I as a subset of βI, we call elements of I *standard* points of I, and elements of $\beta I - I$ *ideal* points of I. If I is infinite, it always has ideal points.

Now consider the topos $\mathcal{S}^I$ of sets varying over I. Objects of $\mathcal{S}^I$—which we shall call *variable sets*—are I-indexed families of sets $X = \langle X_i: i \in I\rangle$. An 'element' of a variable set X is an I–indexed family $x = \langle x_i: i \in I\rangle$ such that $x_i \in X_i$ for all $i \in I$, i.e. a choice function on X. Thus the Cartesian product $\prod_{i \in I} X_i$ is the set of 'elements' of the variable set X.

Each (constant) set A is associated with the variable set $\hat{A}$ given by the function on I with constant value A. The set of 'elements' of $\hat{A}$ is A^I. The maps $A \mapsto \hat{A}$, $X \mapsto \prod_{i \in I} X_i$ define an admissible transformation from $\mathcal{S}$ to $\mathcal{S}^I$.
Given an element $i_0 \in I$, we can arrest the variation of any variable set X by *evaluating* at i_0, *i.e,* by considering X_{i_0} . If we apply this in particular to the set A^I of 'elements' of the variable set $\hat{A}$, i.e.,

if we evaluate each such 'element' at i_0, we merely retrieve *A.* So, in this case, if we negate the constancy of (the elements of) *A* by passing to the set A^I of (variable) 'elements" of *A*, and then negate the variation of these latter by evaluating at a *standard* point of *I*, we come full circle. The situation is decidedly otherwise, however, when the evaluation is made at an *ideal* point of *I.*

Given an ideal point *U* of *I*, i.e., a nonprincipal ultrafilter over *I*, how shall we 'evaluate' functions in A^I at *U*? To this end, observe that the result of evaluating at a standard point i_0 of *I* is essentially the same as *identifying* functions in A^I when their values at i_0 coincide, i.e. stipulating that, for $f, g \in A^I$,

$$f \approx_{i_0} g \ \text{ iff } \ f(i_0) = g(i_0) \ \text{ iff } \ \{i \in I : f(i) = g(i)\} \in U_{i_0} .$$

This last equivalence is used as the basis for evaluating functions in A^I at an ideal point *U* of *I.* That is, we define

$$f \approx_U g \ \text{ iff } \ \{i \in I : f(i) = g(i)\} \in U \ .$$

Then the result of 'evaluating' all the functions in A^I at *U* is the set A^* of equivalence classes of A^I modulo $\approx_U$. It is well known that if *I* is infinite (and *U* an ideal point of *I*), then A^* cannot coincide with *A.* In particular if, for example, *A* is the real line $\mathbb{R}$, then $\mathbb{R}^*$ will have the same elementary properties as $\mathbb{R}$ but will in addition have new 'infinite' and 'infinitesimal' elements. That is, $\mathbb{R}^*$ will be a *nonstandard model of the reals.* This, in essence, is the basis of Robinson's non-standard analysis.

In sum, we get Robinson's infinitesimals by 'sublation': first negating the constancy of the classical real line, and then negating the resulting variation by arresting it an ideal point.

Now if we arrest the variation of all the objects of $\mathscr{S}^I$ *simultaneously* at an ideal point of *I*, we obtain a new constant topos $\mathscr{S}^*$ (an ultrapower or enlargement of $\mathscr{S}$) which has the same elementary properties as $\mathscr{S}$. So this instance of 'sublation" leads to a constant topos which, although not identical with $\mathscr{S}$, is nevertheless *internally indistinguishable* from it. By contrast, the whole purpose of *Cohen's* techniques in set theory is to fashion new constant toposes which are internally *distinguishable* from $\mathscr{S}$.

Let *P* be a partially ordered set: think of the elements of *P* as *stages of knowledge* and $p \le q$ as meaning that *q* is a deeper (or, at any rate, later) stage of knowledge than *p.* A *set varying over P* is a map *X* which assigns to each $p \in P$ a set $X(p)$ and to each pair $p, q \in P$ such that $p \le q$ a map $X_{pq}: X(p) \to X(q)$ such that $X_{pr} = X_{qr} \circ X_{pq}$ whenever $p \le q \le r$. Let $\mathscr{E}$ be the topos—defined over $\mathscr{S}$— whose objects are all sets varying over *P* (and in which an arrow $f: X \to Y$ is a collection of maps $F_p: X(p) \to Y(p)$ such that $f_q \circ X_{pq} = Y_{pq} \circ f_p$ for $p \le q$). The objects of $\mathscr{E}$ may be thought of as sets varying over the stages of knowledge assembled in *P.*

Within $\mathscr{E}$ we consider objects *X* for which $X(p) \subseteq X(q)$ and X_{pq} is the insertion map for $p \le q$. Such an object will be called a set *varying steadily* over *P.* If we think of *X(p)* as the collection of elements of the variable set *X secured* at stage *p*, then the 'steadiness' condition on *X* means that no secured elements are ever lost. For $p \in P$ and sets *X, Y* varying steadily over *P*, we write

$$p \Vdash X \subseteq Y$$

for

$$\forall q \ge p \forall x \in X \exists r \ge q [x \in Y(r)],$$

that is, given stage *p*, *X eventually coincides* with *Y.*

Two elements *p, q* $\in$ *P* are *compatible* if $\exists r \in P[p \le r \;\&\; q \le r]$. A set of compatible elements of *P* is called a *body of knowledge.* A body of knowledge is *complete* if whenever $p \in P$ is compatible with every member of *K,* then *p* belongs to *K.*

Given a complete body of knowledge *K*, define the equivalence relation $\sim_K$ on the collection of sets varying steadily over *P* by

$$X \sim_K Y \Leftrightarrow \exists p \in K\, [\, p \Vdash X \subseteq Y].$$

Thus $X \sim_K Y$ means that our body of knowledge *K* tells us that *X* and *Y* eventually coincide. The collection of equivalence classes modulo $\sim_K$ of steadily varying sets constitute the objects of a *new* constant topos $\mathscr{S}^*$, which is, in general internally *distinguishable* from $\mathscr{S}$ in the sense of not sharing all the elementary properties of $\mathscr{S}$: $\mathscr{S}^*$ is in fact a (possibly non-standard) *Cohen extension* of $\mathscr{S}$.

In sum, the topos $\mathscr{E}$ was obtained from $\mathscr{S}$ by negating constancy in allowing variation, or 'growth' over stages of knowledge, and the Cohen extension $\mathscr{S}^*$ obtained from (the steadily varying objects in)$\mathscr{E}$ by invoking a complete body of knowledge to determine the 'eventual' identities between the variable sets, that is, to 'negate' their variation. Here passing from $\mathscr{S}$ to $\mathscr{S}^*$ —'sublation'— preserves some of the principles associated with constancy of sets (e.g., axiom of choice, classical logic), but as Cohen famously showed, the partially ordered set *P* may be chosen in such a way—now familiar to every set theorist—as to ensure that other such principles (e.g., axiom of constructibility, continuum hypothesis) are *violated* in this passage. In passing from $\mathscr{S}$ to $\mathscr{E}$ (negation of constancy), the classical bivalent logic of $\mathscr{S}$ is replaced by the intuitionistic polyvalent logic of $\mathscr{E}$. And passage from $\mathscr{E}$ to $\mathscr{S}^*$ ('negation of negation') restores classical logic and some, but not all, principles associated with constancy.

Now we could have *refrained* from performing the return passage to constancy (i.e., the second 'negation') and instead remained in the topos $\mathscr{E}$ of variable sets. The set-theoretic independence proofs can be obtained by scrutinizing the internal properties of $\mathscr{E}$ (more precisely, by employing the Scott-Solovay method of replacing $\mathscr{E}$ by its associated Boolean topos of double-negation sheaves). If we agree more generally to abstain from returning to constancy then some truly startling possibilities begin to emerge. For example, in certain more radical choices of the topos $\mathscr{E}$ of variable sets (in which the sets vary over a certain category of rings in a natural way) the part $\Delta = \{x\colon x^{1/2} = 0\}$ of the real line consisting of nilsquare infinitesimals is nondegenerate, moreover every map on the real line is infinitesimally affine on Δ, and hence smooth: $\mathscr{E}$ is thus a model of *smooth infinitesimal analysis.* This is one of the rewards of remaining within a framework of variable sets, resolutely adopting a local viewpoint in which the grip of constancy and classical logic has been loosened.

To sum up: The replacement of absolute by local mathematics results, it seems to me, in a considerable gain in *flexibility of application* of mathematical ideas, and so offers the possibility of providing a convincing explanation of their "unreasonable effectiveness". For now, instead of being obliged to force an intuitively given concept onto the Procrustean bed of absolute mathematics, with the attendant distortion of meaning, we are at liberty to *choose* the local mathematics naturally fitted to express and develop the concept. To the extent that the given concept embodies aspects of (our experience of) the real world, so also will the associated local mathematics; the 'effectiveness' of the latter, i.e., its conformability with objective reality, thus loses its 'unreasonableness' and instead is shown to be a product of design.

So the local interpretation of mathematics implicit in category theory accords closely with the unspoken belief of many mathematicians that their science is ultimately concerned, not with abstract sets, but with the structure of the real world.

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Lambek and ***Scott*** advocate the adoption of the *free topos*—the elementary topos generated by pure type theory—as an appropriate foundation for mathematics. Here by a *type theory* is meant any system of intuitionistic higher-order logic which includes a type $\boldsymbol{N}$ for the natural numbers and in which Peano's axioms for arithmetic hold. *Pure type theory* contains no types, terms and assumptions other than those it has to contain by virtue of being a type theory. Lambek and Scott call a *model* any topos $\mathcal{E}$ that resembles the usual category of sets in satisfying the following conditions:

1. no contradiction holds in $\mathcal{E}$
2. if $p \vee q$ holds in $\mathcal{E}$, then either p holds in $\mathcal{E}$ or q does
3. if $\exists x{\in}A\varphi(x)$ holds in $\mathcal{E}$, then there is an entity of type A in $\mathcal{E}$ such that $\varphi(a)$ holds in $\mathcal{E}$.

Lambek and Scott show that the free topos is a model in this sense. They see models as possible mathematical worlds acceptable to what they term "moderate" intuitionists. They claim that among these models the free topos stands out as a kind of ideal world in the Platonic sense. Noting that the construction of the free topos is linguistic (its objects being terms of the language of pure type theory), they claim that within it three competing traditional philosophies are reconciled:

- intuitionism, according to which only knowable statements are true
- Platonism (or realism), which asserts that mathematical expressions refer to entities whose existence is independent of the knowledge we have of them
- Formalism, whose principal concern is with expressions in the formal language of mathematics.

While it is not unreasonable to see the free topos as reconciling intuitionism and formalism, the claim that it also embodies Platonism or realism seems dubious. For an essential constituent of realism is classical logic, and the logic of the free topos is not classical. The free Boolean topos $\mathcal{B}_0$ would seem to be the "ideal world" for the Platonist, but this is not a model in Lambek and Scott's sense. And while $\mathcal{B}_0$ might be considered to "reconcile" Platonism and formalism, at least in a weak sense, it can hardly be claimed also to reconcile intuitionism, since its internal logic is classical.

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I come finally to the idea of *constructive type theory* (CTT) as providing a foundation for mathematics. The earliest type theory, that of **Russell,** was intended to provide a foundation for mathematics, but he was, so to speak, "forced" into adopting a type-theoretic stance in order to avoid paradox. (Frege before him, unaware of the paradoxes, had assumed that typing was unnecessary, at least, as far as the universe of objects was concerned.) The idea of typing as a natural fact of logical life was grasped by **Hermann Weyl** in *Das Kontinuum* (1919), which begins:

> "A *judgment* affirms a *state of affairs.* If this state of affairs obtains, then the judgment is *true*; otherwise, it is untrue. States of affairs involving *properties* are particularly important… A judgment involving properties asserts that a certain object possesses a certain property, as in the example 'This leaf (given to me in the present act of perception) has this definite green color (given to me in this very perception).' **A property is always affiliated with a definite category of object in such a way that the *proposition* '*a* has that property' is *meaningful*— i.e., expresses a judgment and thereby affirms a state of affairs—only if *a* is an object of that category.** For example, the property 'green' is affiliated with the category 'visible thing'. So the proposition that, say, an ethical value is green is neither true nor false but meaningless. A judgment corresponds only to a *meaningful* proposition, a state of affairs only to a *true* judgment; a state of affairs, however, *obtains*—purely and simply. Perhaps meaningless propositions can appear only in thought about language, never in thought about things."

Weyl had come to believe that, the work of Cauchy, Weierstrass, Dedekind and Cantor notwithstanding, mathematical analysis at the beginning of the 20th century would not bear logical scrutiny, for its essential concepts and procedures involved vicious circles to such an extent that, as he says, "every cell (so to speak) of this mighty organism is permeated by contradiction." In *Das Kontinuum* he tries to overcome this by providing analysis with a *predicative* formulation—not, as Russell and Whitehead had attempted, by introducing a hierarchy of logically ramified types, which Weyl seems to have regarded as too complicated—but rather by confining the comprehension principle to formulas whose bound variables range over just the initial given entities (numbers). Thus he restricted analysis to what can be done in terms of natural numbers with the aid of three basic logical operations, together with the operation of substitution and the process of "iteration", i.e., primitive recursion. Weyl recognized that the effect of this restriction would be to render unprovable many of the central results of classical analysis—e.g., Dirichlet's principle that any bounded set of real numbers has a least upper bound—but he was prepared to accept this as part of the price that must be paid for the security of mathematics.

Like Russell and Whitehead's ramified types, Weyl's system combined type theory and predicativity. But neither was completely "constructive" in the strictest sense of the word, since the common underlying logic of both was, of course, classical rather than intuitionistic. The first truly constructive theory of types, in the sense of being both predicative and based on intuitionistic logic, to undergo systematic development, was that of **Per Martin-Löf**. In introducing it his purpose was to provide, as he put it, "a full scale system for formalizing intuitionistic mathematics as developed, for example, in the book by Bishop." The chief advantage of Martin-Löf's system over traditional intuitionistic systems was in allowing proofs to be constituents of propositions, so enabling propositions to express properties of proofs, and not merely individuals, as in first-order predicate logic. Indeed, Martin-Löf's system provides a complete embodiment of the "propositions-as-types (or sets)" idea originally suggested by Curry, Feys, and Howard. At the root of the "propositions-as-types" conception lies the idealist notion, which may be traced back to Kant, that the meaning of a proposition does not derive from an absolute standard of truth external to the human mind, but resides rather in the evidence for its assertability in the form of a mental construction or proof. In the "propositions-as-types" interpretation, and more generally, in constructive type theories, each proposition is the type, or set, of its proofs. A major consequence is that *under this interpretation these are the only sets, or types.* In other words, a set *is a set of proofs*, or more generally, constructions. It is this feature that has made constructive type theory particularly suitable for developing the theory of computer programming. (Here the somewhat hazy idea of "mental constructions" has been replaced by the precise notion of a computer program.)

Here is Martin-Löf himself on the matter.

> "Every mathematical object is of a certain kind or *type*. Better, a mathematical object is always given together with its type, that is it is not just an object: it is an object of a certain type. … A type is defined by prescribing what we have to do in order to construct an object of that type… Put differently, a type is well-defined if we understand…what it means to be an object of that type. … Note that it is required, neither that we should be able to generate somehow all the objects of a given type, nor that we should so to say know all of them individually. It is only a question of understanding what it means to be an *arbitrary* object of the type in question.
> A *proposition* is defined by prescribing how we are allowed to prove it, and a proposition *holds* or is *true* intuitionistically if there is a proof of it. … It will not be necessary, however, to the notion of proposition as a separate notion because of that proposition.
> Conversely, each type determines a proposition, namely, the proposition that the type in question is nonempty. This is the proposition which we prove by exhibiting an object of the type in question. On this analysis, there appears to be no fundamental difference between propositions and types. Rather, the difference is one of point of view: in the case of a proposition, we are not so much interested in what its proofs are as in whether it has a proof, that is, whether it is true or false, whereas, in the case of a type, we are of course interested in what its objects are and not only in whether it is empty or nonempty.

A key element in Martin-Löf's formulation of type theory is the distinction, which goes back to Frege, between *propositions* and *judgments.* Propositions (which, as we have seen, in Martin-Löf's systems are identified with types) are syntactical objects on which mathematical operations can be performed and which bear certain formal relationships to other syntactical objects called proofs. Propositions and proofs are, so to speak, *objective* constituents of the system. Judgments, on the other hand, typically involve the *idealist* notion of "understanding" or "grasping the meaning of". Thus, for example, while $2 + 2 = 4$ is a proposition, "$2 + 2 = 4$ is a proposition" and "$2 + 2 = 4$ is a true proposition" are judgments.

Martin-Löf also follows Frege in taking the rules of inference of logic to concern judgments rather than propositions. Thus, for example, the correct form of the rule of $\rightarrow$ -elimination is not

$$\frac{A \quad A \rightarrow B}{B}$$

but

$$\frac{A \text{ true} \quad A \rightarrow B \text{ true}}{B \text{ true}}.$$

That is, the rule does not say that the proposition B follows from the propositions A and $A \rightarrow B$, but that the *truth* of the proposition B follows from the *truth* of the proposition A conjoined with that of $A \rightarrow B$. In general, judgments may be characterized as expressions which appear at the conclusions of rules of inference.

Another important respect in which Martin-Löf follows Frege is in his insistence that judgments and formal rules must be accompanied by full explanations of their *meaning.* (This is to be contrasted with the usual model-theoretic semantics which is really nothing more than a translation of one object-language into another.) In particular, the judgment *A is a proposition* may be made only when one knows what a (canonical) proof of *A* is, and the judgment *A is a true proposition* only when one knows how to find such a proof. Judgments, and the notion of truth, are thus seen to be mind-dependent.

Martin-Löf's various systems abound in subtle distinctions. For example, in addition to the distinction between proposition and judgment, there is a parallel distinction between *type* (or set) and *category* (or species). In order to be able to judge that *A* is a category one must be able to tell what kind of objects fall under it, and when they are equal. To be in a position to make the further judgment that a category is a type, or set, one must be able to specify what its "canonical" or typical, elements are. In judging something to be a set, one must possess sufficient information concerning the its elements to enable quantification over it to make sense. Thus, for example, the natural numbers form a set $\mathbb{N}$, with canonical elements given by: 0 is a canonical element of $\mathbb{N}$, and if n is a canonical element of $\mathbb{N}$, then $n + 1$ is a canonical element of $\mathbb{N}$. On the other hand the collection of subsets of $\mathbb{N}$ forms a category, but not a set.

The "propositions-as-types" interpretation has been subjected to a searching analysis by ***Bill Tait.*** In his "The Law of Excluded Middle and the Axiom of Choice" of 1990, he remarks:

> "There are really two parts to the constructivist conception of mathematical reasoning. One part concerns the basic role of the notions of constructing an object of given type…and of constructing a proof of a proposition. The second part concerns the "effective" or "computable" character of these objects or proofs.
> I truly wish that the term "constructive" had been reserved for just the first part, since it seems most appropriately applied to the view that the basic notion of mathematics is that of construction [*cf. Machover*], without further specification of what kinds of construction are to be permitted. But the term has been pre-empted for the narrower conception. So I will call the conception I want to present the "construction-theoretic" conception. As you will see immediately, I could also have called it the "proof-theoretic" conception.
> There are, at first blush, two kinds of conception involved: constructions of proofs of some proposition and construction of objects of some type. But I will argue that, from the point of view of foundations of mathematics, there is no difference between the two notions. A proposition may be regarded as a type of object, namely the type of its proofs. Conversely, a type *A* may be regarded as a proposition, namely the proposition whose proofs are the objects of type *A.* So a proposition is true just in case there is an object of type *A.*

> The identification of propositions with types has two sides. The first is technical and unproblematic. If we simply translate the relation of proposition to proof as the relation of type to object… the rules of proof correspond exactly to rules of construction of objects of the corresponding type. So every proposition that we recognize as such may, from the point of view of proof theory, be treated as a type. The other side of the identification needs argument. For it implies that a proposition is determined when it is determined proof-theoretically, that is, when it is determined *qua* type. *On this view, truth can only mean provability.*

It would seem that the italicized passage here that points up the *idealist,* or at least *formalist,* character of the "propositions-as-types" interpretation. But Tait goes on to explain:

> "When I say that truth can only mean provability, I do not mean that "A is true' is defined to mean "*A* is provable" for each type *A. "A* is true" means, simply, that *A* [*this an instance of the so-called "redundancy theory of truth"*]. So the force of the identification of truth with provability is simply that the only warrant for asserting *A* is a proof of *A*."

It is interesting to contrast this with the topos-theoretic account of truth. Here one has an "objective" semantic notion of truth à la Tarski in a local framework (topos) $\mathcal{E}$, together with what one might call a "subjective" (although perfectly rigorous) notion of provability from a local set theory *T*. The two are defined quite differently, but they can always be made to coincide in the sense that *truth in* $\mathcal{E}$ *coincides with provability from the theory* $Th(\mathcal{E})$ *determined by* $\mathcal{E}$*, and conversely provability from T coincides with truth in the linguistic topos* $\mathcal{E}_T$ *determined by T.* So, even though truth can be shown to coincide with provability, it is not *defined* so as to bring the two concepts into coincidence. In this respect the topos-theoretic account of truth is, of course, no different from the "propositions-as-types" account. The two notions of truth are distinguished, rather, by the fact that, in the "propositions-as-types" framework, as Tait says, *the only warrant for asserting a proposition A is a proof of A*, while in the topos-theoretic framework one has in addition to the "subjective" warrant of a proof, *the "objective" warrant of truth in a local framework.* It is this latter feature that, I think, lends the topos-theoretic account its "realist" character.

Tait recognizes that the formalist identification of truth with provability, on which the whole "propositions-as-types" conception rests can be questioned.

> "For [as he says] there are many who would argue that truth is a prior notion which cannot be captured by the notion of proof. For one who holds this opinion, the identification of propositions as types would be unsatisfactory. For it would be inadequate simply to specify how to construct objects of the various types *A*. We would also have to explain the conditions under which *A* is true (and explain in these terms why objects of type *A* should count as proofs of *A*)."

The "propositions-as-types" conception (which for convenience we abbreviate to PAT) gives rise to a correspondence between logical operators and operations on types (or sets). To begin with, consider two propositions/types *A* and *B*. What should be required of a proof *f* of the implication $A \to B$? Just that, given any proof *x* of *A, f* should yield a proof of *B,* that is, *f* should be a function from *A* to *B.* In other words, the proposition $A \to B$ is just the type of functions from *A* to *B*:

$$A \to B = B^A$$

Similarly, all that should be required of a proof *c* of the conjunction $A \wedge B$ is that it should yield proofs *x* and *y* of *A* and *B,* respectively. From the construction-theoretic point of view $A \wedge B$ is accordingly just the type $A \times B$ of all pairs (x, y), with *x* of type *A* (we write this as x: A) and y: B.

A proof of the disjunction $A \vee B$ is either a proof of *A* or a proof of *B* together with the information as to which of *A* or *B* it is a proof. That is, if we introduce the type 2 with the two distinct elements 0 and 1, a proof of $A \vee B$ may be identified as a pair (c, n) in which either *c* is a proof of *A* and *n* is 0, or *c* is a proof of *B* and *n* is 1. This means that, from the construction-theoretic point of view, $A \vee B$ is the disjoint union $A + B$ of *A* and *B.*

The true proposition $\top$ may be identified with the one element type $1 = \{0\}$: 0 thus counts as the unique proof of $\top$. The false proposition $\bot$ is taken to be a proposition which lacks a proof altogether: accordingly $\bot$ is identified with the empty set $\emptyset$. The negation $\neg A$ of a proposition A is defined as $A \to \bot$, which therefore becomes identified with the set $A^{\emptyset}$.

As we have already said, a proposition A is deemed to be true if it (i.e, the associated type) has an element, that is, if there is a function $1 \to A$. Accordingly the *law of excluded middle* for a proposition A becomes the assertion that there is a function $1 \to A + \emptyset^A$.

If a and b are objects of type A, we introduce the *identity proposition* or *type* $a =_A b$ expressing that a and b are identical objects of type A. This proposition is true, that is, the associated type has an element, if and only if a and b are identical. In that case id(a) will denote an object of type a.

In PAT one must distinguish sharply between *propositions,* which have proofs, and *judgements,* which do not. For example $0 =_2 0$ is a proposition, while "0 is of type 2" is a judgement. Rather than being true or false, a judgement is either assertable, or nonsensical.

While 2^A does not have a very natural interpretation as a proposition, it may be considered the type of all *decidable sets of objects* of type A. For given f: 2^A and x: A, if we define elementhood by

$$x \in_A f \quad \text{iff } fx = 1,$$

then it is easy to see that $x \in_A f \vee \neg\, x \in_A f$.

In order to deal with the quantifiers we require operations defined on families of types, that is, types $\varphi(x)$ depending on objects x of some type A. By analogy with the case $A \to B$, a proof f of the proposition $\forall x{:}A\ \varphi(x)$, that is, an object of type $\forall x{:}A\ \varphi(x)$, should associate with each x: A a proof of $\varphi(x)$. So f is just a function with domain A such that, for each x: A, fx is of type $\varphi(x)$. That is, $\forall x{:}A\ \varphi(x)$ is the *Cartesian product* $\Pi x{:}A\ \varphi(x)$ of the $\varphi(x)$'s. We use the λ-notation in writing f as $\lambda x f x$.

A proof of the proposition $\exists x{:}A\ \varphi(x)$, that is, an object of type $\exists x{:}A\ \varphi(x)$, should determine an object x: A and a proof y of $\varphi(x)$, and *vice-versa.* So a proof of this proposition is just a pair (x, y) with x: A and y: $\varphi(x)$. Therefore $\exists x{:}A\ \varphi(x)$ is the *disjoint union* $\coprod x : A\varphi(x)$ of which φ is the constant function with value B.

We introduce the functions σ, π, π' of types $\forall x{:}A(\varphi(x) \to \exists x{:}A\ \varphi(x))$, $\exists x{:}A\ \varphi(x) \to A$, and $\forall y$: $(\exists x\varphi(x))$. $\varphi(\pi(y))$ as follows. If b: A and c: $\varphi(b)$, then σbc is (b, c). If d: $\exists x{:}A\ \varphi(x)$, then d is of the form (b, c) and in that case $\pi(d) = b$ and $\pi'(d) = c$. These yield the equations

$$\pi(\sigma bc) = b \qquad \pi'(\sigma bc) = c \qquad \sigma(\pi d)(\pi' d) = d.$$

The *axiom of choice* (AC) is the proposition

$$\forall x{:}A \exists y{:}B\ \varphi(x, y)) \to \exists x{:}B^A \forall x{:}A\ \varphi(x, fx)).$$

AC is true in PAT, as the following argument shows. Let u be a proof of the antecedent $\forall x{:}A\exists y{:}B\ \varphi(x, y))$. Then, for any x: A, $\pi(ux)$ is of type B and $\pi'(ux)$ is a proof of $\varphi(x, \pi ux)$. So $s(u) = \lambda x.\pi(ux)$ is of type B^A and $t(u) = \lambda x.\ \pi'(ux)$ is a proof of $\forall x{:}A\ \varphi(x,\ s(u)x)$. Accordingly $\lambda u.\sigma s(u) t(u)$ is a proof of $\forall x{:}A\exists y{:}B\ \varphi(x, y)) \to \exists x{:}B^A\ \forall x{:}A \varphi(x, fx))$.

In ordinary set theory this argument establishes the *isomorphism* of the sets $\Pi x{:}A \coprod y{:}B\ \varphi(x, y))$ and $\coprod f{:}B^A\ \Pi x{:}A\ \varphi(x, fx))$, but not the validity of the axiom of choice. In set theory AC is not is not represented by this isomorphism, but is rather (equivalent to) the *equality* in which Π is replaced by $\bigcap$ and $\coprod$ by $\bigcup$:

$$\bigcap_{x\in A}\bigcup_{y\in B}\phi(x,y) = \bigcup_{f\in B^A}\bigcap_{x\in A}\varphi(x, fx).$$

While in PAT AC has no "untoward" logical consequences, in intuitionistic set theory (IST) this is far from being the case, for there AC implies the law of excluded middle. It is worth rehearsing the argument, in its original form due to Diaconescu:

Suppose given a choice function f on the power set of the set 2 = {0, 1}. Let α be any proposition, and define

$$U = \{x\in 2: x = 0 \vee \alpha\} \quad V = \{x\in 2: x = 1 \vee \alpha\}.$$

Writing $a = fU$, $b = fV$, we have $a \in U$, $b \in V$, i.e.,

(#) $$[a = 0 \vee \alpha] \wedge [b = 1 \vee \alpha].$$

It follows that

$$[a = 0 \wedge b = 1] \vee \alpha,$$

whence

(*) $$a \neq b \ \vee \alpha,$$

Now clearly

$$\alpha \Rightarrow U = V = 2 \ \Rightarrow a = b,$$

whence

$$a \neq b \Rightarrow \neg\alpha.$$

But this and (*) together imply $\neg\alpha \vee \alpha$.

[Remark: In fact, we need only assume the choice function to be defined on the set {U, V}. This form of the argument can be reproduced in the intuitionistic ε-calculus with extensional ε-terms, thus showing that it is in fact classical.]

Given that AC holds in PAT, it is of interest to ask why these arguments cannot be reproduced there. Now the first argument seems to hinge on two assumptions, first, that the sets U and V are well defined and satisfy the usual "eliminability" conditions leading to the assertability of (#) above. And secondly, that the choice function f satisfies extensionality in the sense that, if U and V are extensionally identical, then $fU = fV$. It seems to be the case that when subset types are added to PAT (in Martin-Löf's system), the 'eliminability" condition

(!) $$a \in \{x: \varphi(x)\} \rightarrow \varphi(a)$$

fails. Concerning the second argument, this seems to fail essentially because in PAT the value of a function defined on a (sub)set X depends not only on the variable member x of X but also on the *proof* that x is in fact in X. Thus suppose given types A, B and a subset $X = \{x: \beta(x)\}$ of A. Write $p \vdash \alpha$ for "p is a proof of α". Then in PAT from $\forall x{:}A[\beta(x) \rightarrow \exists y{:}B\varphi(x, y)]$ we can infer the existence of a function $f: \{(x, p): p \vdash \beta(x)\} \rightarrow B$ for which $\forall x\forall p[p \vdash \beta(x) \rightarrow \varphi(x, f(x,p))]$. Now return to Diaconescu's argument. Here A is P2, the power set of 2 (supposing that to be present), $\beta(x)$ is $\exists x.\ x \in X$ (X a variable of type P2), B is 2 and $\varphi(X, y)$ is $y \in X$. Now, given a proposition α, define the subsets U and V as before. Constructively, the only proof of $\exists x.\ x \in U$ to be had is by exhibiting a member of U, and , since α is not known to be true, the only exhibitable member of U is 0. Similarly, the only exhibitable member of V is 1. Accordingly, writing $a = f(U, 0)$ and $b =$

$f(V, 1)$, we derive (*) above as before (assuming subset eliminability). But now while $\alpha \rightarrow U = V$, we cannot infer that $U = V \rightarrow a = b$, so blocking the derivation of $\alpha \rightarrow a = b$.

So if extensional power sets are suitably added to PAT, logic becomes classical there.

In IST even AC *for singletons* has an untoward logical consequence, namely, the nonconstructive $(\alpha \rightarrow \beta) \vee (\beta \rightarrow \alpha)$. Here by AC for singletons we mean that for any set U there is a function f with domain $\{U\}$ such that, if $\exists x\, x \in U$, then $f(U) \in U$ – such an f will be called a *choice function* on $\{U\}$.

Now let α and β be any propositions and define

$$U = \{x\in 2\colon (x = 0 \wedge \alpha) \vee (x = 1 \wedge \beta)\}.$$

Let f be a choice function on $\{U\}$, and write $a = f(U)$. Clearly $\exists x\, x \in U \leftrightarrow \alpha \vee \beta$, so that

$$(*) \qquad \alpha \vee \beta \rightarrow a \in U \rightarrow (a = 0 \wedge \alpha) \vee (a = 1 \wedge \beta).$$

Since $a \in 2$, we have $a = 0 \vee a = 1$, which assertion may be conjoined with (*) to give

$$(a = 0 \vee a = 1) \wedge [\alpha \vee \beta \rightarrow (a = 0 \wedge \alpha) \vee (a = 1 \wedge \beta)],$$

whence

$$[a = 0 \wedge [\alpha \vee \beta \rightarrow (a = 0 \wedge \alpha) \vee (a = 1 \wedge \beta)] \vee$$
$$[a = 1 \wedge [\alpha \vee \beta \rightarrow (a = 0 \wedge \alpha) \vee (a = 1 \wedge \beta)],$$

which implies, using $0 \neq 1$,

$$[\alpha \vee \beta \rightarrow \alpha) \vee [\alpha \vee \beta \rightarrow \beta],$$

whence

$$[\beta \rightarrow \alpha) \vee [\alpha \rightarrow \beta],$$

as claimed.

This argument fails in PAT with subsets because of the noneliminability of subset terms, that is, the failure of (!) above.

[Remark: this last argument can be reproduced in the intuitionistic ε-calculus, thus showing that $[\alpha \rightarrow \beta) \vee [\beta \rightarrow \alpha]$ is derivable there.]

What are we to make of PAT, or constructive type theory, as a "foundation" for mathematics? First, one must note its radically "internal" character, in Gödel's words (originally stated in connection with Russell's "no-class" theory) "the tendency to eliminate assumptions about the existence of objects outside the 'data' and to replace them by constructions on the basis of these data." By "data" here Gödel meant logic without the assumption of the existence of classes and objects. In the case of CTT or PAT the "data" include *expressions, rules, judgements, propositions, proofs/constructions, sets/types, species/categories.* And in place of logical operations one has operations on *types*: disjoint union, Cartesian product. These are *formal* notions.

In CTT impredicativity is avoided by strict adherence to the principle that universally quantified variables should range only over previously defined sets. While this has positive features, for example, in enabling the axiom of choice to become derivable, it also imposes severe constraints on possible extensions of the system. For instance, if one attempts to add to CTT power sets, or even the power set of 1, the law of excluded middle becomes derivable. And treating the species of propositions as a set, in other words, allowing second-order quantification, leads to outright inconsistency.

Set theory and constructive type theory have one thing in common which distinguishes them from topos theory: they are, at least in intention, *monistic*. The one purports to crystallize the truths holding in the unique universe of sets; the other gives expression to the, presumably unique, corpus of principles underlying the intuitionistic conception of mathematics. (The fact that both frameworks admit extensions (e.g. set theory by the addition of large cardinal axioms and CTT by the addition of subset principles and the like – cf. the extension of Brouwerian intuitionism by the inclusion of the "creative subject"— does not belie their essential monisticity.) In their monism both have an affinity to number theory. By contrast, topos theory resembles algebra in that its central concept was never intended to have a unique reference. Indeed the concept of topos was designed to capture the common features of a wide spectrum of categories arising in topology and algebraic geometry—the sheaf categories.

It has been said that CTT is not a theory for everyday practical use, but one for understanding the foundations of constructive mathematics. In fact much of the driving force behind the development of CTT has come from computing science: indeed CTT is itself a functional programming language. The subtlety and complexity of CTT has rendered difficult the development of "real" mathematics within it: in the words of **G. Sambin** and **S. Valentini,**

> "Our experience in developing pieces of actual mathematics within type theory has brought us to believe that 'orthodox' type theory is not suitable because its control of information is too strict for this purpose. In fact, the fully analytic character of type theory becomes a burden when dealing with the synthetic methods of mathematics, which 'forget' or [take] for granted most of the details. This, in our opinion, could be the reason why type theory has [attracted], up to now, more interest among logicians and computer scientists as a foundational theory…"

They go on to show how, by "liberalizing" type theory somewhat, it becomes possible to develop general topology there in a natural way. The underlying motivation for such a development is still, in their words, "to bridge the gap between ordinary mathematics and computer languages."

Another way of making type theory more accessible to practicing mathematicians is by assimilating it to *set* theory. This has led to the development, largely through the efforts of Peter Aczel, of a predicative form of intuitionistic set theory called *constructive set theory.* Constructive set theory shows real promise as a system combining the precision of type theory with the "user-friendly" character of set theory.