

The Axiom of Choice

John L. Bell

Dedicated to the memory of

My Dear Wife Mimi

1947-2009

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Volume 20
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Dov Gabbay

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ERNST ZERMELD
1871 - 1953

1947-2009

FELIX HAUSDORFF
1868 - 1942

MAX ZORN
1906 - 1993

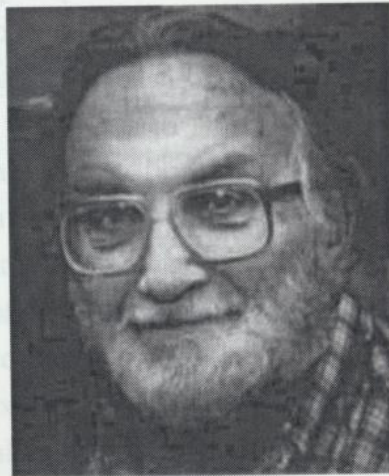
Preface



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Preface

The Axiom of Choice has fascinated me from my student days. As a young aspiring mathematician, I was struck by the fact that through the areas of mathematics to which I was most attracted – abstract algebra, general topology, functional analysis – there ran a common thread: the Axiom of Choice and the “working mathematician’s” surrogate for it known as Zorn’s Lemma. Teasing out that thread is the purpose of the present book.

The book provides an overview of the development of the Axiom of Choice since its introduction by Zermelo at the beginning of the last century. In it the Axiom of Choice is surveyed from three perspectives. The first, or *mathematical* perspective, is that of the abovementioned “working mathematician”. This perspective brings into view the manifold applications of the Axiom of Choice—usually in the guise of Zorn’s Lemma—in a great variety of areas of mathematics. The second, *foundational*, perspective, is that of the logician or constructive mathematician concerned with the foundational status of the Axiom of Choice. The third, *topos-theoretical*, perspective is that taken by the mathematician or logician investigating the role of the Axiom of Choice in topos theory.

In this book certain topics—for instance mathematical applications of the Axiom, and its relationship with logic—are discussed in considerable detail. Others—notably the consistency and independence of the Axiom of the usual systems of set theory—are given no more than summary treatment, the justification being that these topics have been given full expositions elsewhere. The book contains seven chapters and two appendices. Chapter I describes the origins of the Axiom of Choice and its status within set theory. Chapter II introduces

maximal principles, in particular, Zorn's Lemma, and discusses their development and their relationship to the Axiom of Choice. Chapter III provides a detailed account of the many applications of the Axiom of Choice and Zorn's Lemma within classical mathematics. Chapter IV offers a compressed account of how the Axiom of Choice is proved consistent with, and independent of, Zermelo-Fraenkel set theory. Chapter V describes the relationship between the Axiom of Choice and logic, chiefly focussing attention on intuitionistic logic. Chapter VI begins with a discussion of the Axiom of Choice in a categorical setting, and continues with what amounts to a crash course in topos theory and the associated local set theory, leading up to an account of the role played by the Axiom of Choice therein. Chapter VII is devoted to a discussion of the role of the Axiom of Choice in Constructive Type Theory. The book concludes with two Appendices, the first outlining intuitionistic logic and the second charting the basic concepts of category theory.

*

I wish to thank my old friends and fellow-logicians Peter Aczel and John Mayberry for their helpful comments on the first draft of the book.

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