Zeitschr. f. math. Logik und Grundlagen d. Math.
Bd. 22, S. 307–310 (1976)

A NOTE ON GENERIC ULTRAFILTERS

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Let $\mathfrak{M} = \langle M, \in \rangle$ be a transitive $\in$-model of ZFC (ZERMELO-FRAENKEL set theory with the axiom of choice) and let $B \in M$ be a complete BOOLEAN algebra in the sense of $\mathfrak{M}$. Thus, if $X \in P^{(\mathfrak{M})}(B) = P(B) \cap M$, then $\bigvee X$ and $\bigwedge X$, the join and meet of X, exist and are in M. (Here $P(B)$, $P^{(\mathfrak{M})}(B)$ are, respectively, the power set of B and the power set of B computed in $\mathfrak{M}$.) An ultrafilter U in B (U need not be in M!) is said to *respect* the join $\bigvee X$, for $X \in P^{(\mathfrak{M})}(B)$, if

$$\bigvee X \in U \Rightarrow X \cap U \neq \emptyset$$

(the reverse implication holding trivially). If $\mathscr{S} \subseteq P^{(\mathfrak{M})}(B)$, then U is said to be $\mathscr{S}$-*complete* if U respects the join $\bigvee X$ for every $X \in \mathscr{S}$. A $P^{(\mathfrak{M})}(B)$-complete ultrafilter is said to be $\mathfrak{M}$-*generic*. In this note we show that several apparently weak conditions on an ultrafilter in B are actually *sufficient* for $\mathfrak{M}$-genericity.

We construct the *B-extension* $M^{(B)}$ of $\mathfrak{M}$ in the usual way, following JECH [1]. We write $\|\sigma\|$ for the B-value of any sentence σ of the language of set theory (which may contain names for elements of $M^{(B)}$). Well-known is the fact that $\|\sigma\| = \mathbf{1}$[1]) whenever σ is a theorem of ZFC. We also recall that there is a natural map $x \mapsto \hat{x}$ of $\mathfrak{M}$ into $M^{(B)}$.

A subset $\{b_i \colon i \in I\}$ of B is called an $\mathfrak{M}$-*partition* of B if (1) $b_i \neq \mathbf{0}$ for all $i \in I$, (2) $\langle b_i \colon i \in I \rangle \in M$, (3) $\bigvee_{i \in I} b_i = \mathbf{1}$ and (4) $b_i \wedge b_j = \mathbf{0}$ whenever $i, j \in I$ and $i \neq j$. We have the following well-known facts (the first of which is a special case of the so-called „Mixing Lemma", see [1], Lemma 49):

Lemma 1. *Let $\{b_i \colon i \in I\}$ be an $\mathfrak{M}$-partition of B and let $\{x_i \colon i \in I\} \in M$. Then there is $x \in M^{(B)}$ such that $b_i = \|x = \hat{x}_i\|$ for all $i \in I$.*

Lemma 2. *Let U be an ultrafilter in B. Then U is generic iff whenever $\{b_i \colon i \in I\}$ is an $\mathfrak{M}$-partition of B, there is $i \in I$ such that $b_i \in U$.*

Now let U be an ultrafilter in B which we assume fixed once and for all. There is a natural way of factoring $M^{(B)}$ by U; for $x, y \in M^{(B)}$ we put $x \sim_U y \Leftrightarrow \|x = y\| \in U$. It is easily verified that $\sim_U$ is an equivalence relation on $M^{(B)}$. We write x^U for the $\sim_U$-class of x and define the relation $\in_U$ on the set $M^{(B)}/U$ of all $\sim_U$-classes by

$$x^U \in_U y^U \Leftrightarrow \|x \in y\| \in U.$$

We define the quotient of $M^{(B)}$ by U to the structure $\mathfrak{M}^{(B)}/U = \langle M^{(B)}/U, \in_U \rangle$. We have the following well-known version of Łoś' theorem for $\mathfrak{M}^{(B)}/U$:

Lemma 3. *For any formula $\varphi(v_0, \ldots, v_n)$ and any $x_0, \ldots, x_n \in M^{(B)}$,*

$$\mathfrak{M}^{(B)}/U \models \varphi[x_0^U, \ldots, x_n^U] \Leftrightarrow \|\varphi(x_0, \ldots, x_n)\| \in U.$$

It follows immediately from this lemma that $\mathfrak{M}^{(B)}/U$ is a model of ZFC.

Now let $\boldsymbol{Ord}(x)$ be the formula expressing that x is an ordinal. An element $a \in M^{(B)}/U$ is called an *ordinal in* $\mathfrak{M}^{(B)}/U$ if $\mathfrak{M}^{(B)}/U \models \boldsymbol{Ord}[a]$. Let $\boldsymbol{OR}^{(\mathfrak{M})}$ be the set of ordinals of $\mathfrak{M}$; then for each $\alpha \in \boldsymbol{OR}^{(\mathfrak{M})}$, $\hat{\alpha}^U$ is an ordinal in $\mathfrak{M}^{(B)}/U$ called a *standard* ordinal in $\mathfrak{M}^{(B)}/U$.

Let $\mathscr{S}_1$ be the subfamily of $P^{(\mathfrak{M})}(B)$ consisting of all subsets of B of the form $\{\|x = \hat{\alpha}\| : \alpha \in \boldsymbol{OR}^{(\mathfrak{M})}\}$ for $x \in M^{(B)}$. It is well-known that, for any $x \in M^{(B)}$, we have

(1) $$\|\boldsymbol{Ord}(x)\| = \bigvee_{\alpha \in \boldsymbol{OR}^{(\mathfrak{M})}} \|x = \hat{\alpha}\|.$$

[1]) **1**, **0** denote the unit and zero element of B, respectively.

We are now ready to state and prove our first result.

Theorem 1. *The following conditions are equivalent:*

(i) *Each ordinal in* $\mathfrak{M}^{(B)}/U$ *is standard;*

(ii) U *is* $\mathscr{S}_1$*-complete;*

(iii) U *is* $\mathfrak{M}$*-generic.*

Proof. (i) $\Rightarrow$ (ii). Assume (i); then if $\bigvee_{\alpha \in \boldsymbol{OR}^{(\mathfrak{M})}} \|x = \hat{\alpha}\| \in U$, we have $\|\boldsymbol{Ord}(x)\| \in U$ by (1), so $\mathfrak{M}^{(B)}/U \models \boldsymbol{Ord}[x^U]$ by Lemma 3. Hence (i) gives $x^U = \hat{\alpha}^U$ for some $\alpha \in \boldsymbol{OR}^{(\mathfrak{M})}$, whence $\|x = \alpha\| \in U$, and (ii) follows.

(ii) $\Rightarrow$ (iii). Assume (ii), let $\alpha \in \boldsymbol{OR}^{(\mathfrak{M})}$ and let $\{b_\xi : \xi < \alpha\}$ be an $\mathfrak{M}$-partition of B with $\alpha \in \boldsymbol{OR}^{(\mathfrak{M})}$. By Lemma 1 we can find $x \in M^{(B)}$ such that $b_\xi = \|x = \hat{\xi}\|$ for all $\xi < \alpha$. We have

$$\bigvee_{\xi \in \boldsymbol{OR}^{(\mathfrak{M})}} \|x = \hat{\xi}\| \geqq \bigvee_{\xi < \alpha} \|x = \hat{\xi}\| = \mathbf{1} \in U,$$

so by (ii) there is $\xi_0 \in \boldsymbol{OR}^{(\mathfrak{M})}$ such that $\|x = \hat{\xi}_0\| \in U$. If $\xi \geqq \alpha$, $\xi \in \boldsymbol{OR}^{(\mathfrak{M})}$, it is clear that $\|x = \hat{\xi}\| = \mathbf{0}$, so we must have $\xi_0 < \alpha$. Hence $b_{\xi_0} = \|x = \hat{\xi}_0\| \in U$ and (iii) now follows from Lemma 2.

(iii) $\Rightarrow$ (i). Assume (iii); then we have, using (1) and Lemma 3,

$$\begin{aligned} \mathfrak{M}^{(B)}/U \models \boldsymbol{Ord}[x^U] &\Leftrightarrow \|\boldsymbol{Ord}(x)\| \in U \\ &\Leftrightarrow \bigvee_{\alpha \in \boldsymbol{OR}^{(\mathfrak{M})}} \|x = \hat{\alpha}\| \in U \\ &\Leftrightarrow \|x = \hat{\alpha}\| \in U \text{ for some } \alpha \in \boldsymbol{OR}^{(\mathfrak{M})} \\ &\Leftrightarrow x^U = \hat{\alpha}^U \text{ for some } \alpha \in \boldsymbol{OR}^{(\mathfrak{M})}, \end{aligned}$$

and (i) follows. □

Corollary. *If* U *is* $\mathscr{S}_1$*-complete, then* $\in_U$ *is a well-founded relation.*

Proof. If U is $\mathscr{S}_1$-complete, then Theorem 1 shows that the map $\alpha \mapsto \hat{\alpha}^U$ sends the well-ordered set $\boldsymbol{OR}^{(\mathfrak{M})}$ onto the set of ordinals of $\mathfrak{M}^{(B)}/U$. This map is clearly order-preserving, and it follows that the ordinals of $\mathfrak{M}^{(B)}/U$ are well-ordered under $\in_U$. The usual rank argument now yields the well-foundedness of $\in_U$. □

It is worth observing that, assuming the existence of a measurable cardinal, the converse of this corollary fails. This can be seen as follows. Take $M = V$, the universe of all sets, let $\varkappa$ be a measurable cardinal and let D be a non-principal $\varkappa$-complete ultrafilter in the complete Boolean algebra $P\varkappa$. Then by standard results $V^{(P\varkappa)}/D$ is isomorphic to the ultrapower $V^\varkappa/D$ and is therefore well-founded by Scott [2]. On the other hand D cannot be $\mathscr{S}_1$-complete, because if it were it would be V-generic by Theorem 1, and hence principal, contrary to assumption. (This argument is due to Dana Scott.)

Suppose now that $\in_U$ *is* a well-founded relation. Then $\mathfrak{M}^{(B)}/U$ can be collapsed uniquely to a transitive $\in$-structure $\mathfrak{M}[U] = \langle M[U], \in \rangle$ via the map h defined recursively on $\in_U$ by

$$h(x^U) = \{h(y^U) : y^U \in x^U\} = \{h(y^U) : \|y \in x\| \in U\}.$$

Thus $h : \mathfrak{M}^{(B)}/U \to \mathfrak{M}[U]$ is a bijection satisfying $x^U \in_U y^U \Leftrightarrow h(x^U) \in h(y^U)$. Under these conditions we can define a surjection $i : M^{(B)} \to \mathfrak{M}[U]$ by putting $i(x) = h(x^U)$ for $x \in M^{(B)}$. Clearly we have, for $x \in M^{(B)}$,

$$i(x) = \{i(y) : \|y \in x\| \in U\}. \tag{2}$$

Our next lemma follows immediately from Lemma 3 and the fact that h is an isomorphism of $\mathfrak{M}^{(B)}/U$ onto $\mathfrak{M}[U]$.

Lemma 4. *For any formula* $\varphi(v_0, \ldots, v_n)$ *and any* $x_0, \ldots, x_n \in M^{(B)}$,

$$\mathfrak{M}[U] \models \varphi[i(x_0), \ldots, i(x_n)] \Leftrightarrow \|\varphi(x_0, \ldots, x_n)\| \in U.$$

Now define $j: \mathfrak{M} \to \mathfrak{M}[U]$ by putting $j(x) = i(\hat{x})$ for $x \in M$. Then j is an $\in$-monomorphism. Also, let $\mathscr{S}_2$ be the subfamily of $P^{(\mathfrak{M})}(B)$ consisting of all subsets of B of the form $\{\|x = \hat{z}\| : z \in y\}$ for $x \in M^{(B)}$, $y \in M$. We note the following well known fact

$$\|x \in \hat{y}\| = \bigvee_{z \in y} \|x = \hat{z}\|. \tag{3}$$

Theorem 2. *The following conditions are equivalent:*

(i) U *is* $\mathscr{S}_2$*-complete;*
(ii) $\in_U$ *is well-founded and* j *is the identity on* $\mathfrak{M}$;
(iii) $\in_U$ *is well-founded and* $j[\mathfrak{M}]$ *is transitive;*
(iv) U *is* $\mathfrak{M}$*-generic.*

Proof. (i) $\Rightarrow$ (iv) Assume (i), and let $\{b_i : i \in I\} = y$ be an $\mathfrak{M}$-partition of B. Then by Lemma 1 we can find $x \in M^{(B)}$ such that $b_i = \|x = \hat{b}_i\|$ for all $i \in I$. Hence

$$\bigvee_{z \in y} \|x = \hat{z}\| = \bigvee_{i \in I} \|x = \hat{b}_i\| = \mathbf{1} \in U,$$

so by (i) there is $z \in y$ such that $\|x = \hat{z}\| \in U$. Since $z = b_i$ for some $i \in I$, it follows that $b_i = \|x = \hat{b}_i\| \in U$, and (iv) now follows from Lemma 2.

(iv) $\Rightarrow$ (iii). Assume (iv). Then $\in_U$ is well-founded by the Corollary to Theorem 1. Also, if $y \in M$ and $x \in j(y)$ then, since $\mathfrak{M}[U]$ is transitive, there is $x' \in M^{(B)}$ such that $x = i(x')$. Thus $i(x') \in j(y) = i(\hat{y})$, so that, by (2), $\|x' \in \hat{y}\| \in U$. It follows from (iv) and (3) that there is $z \in y$, hence $z \in M$ such that $\|x' = \hat{z}\| \in U$, whence $x = j(z) \in j[\mathfrak{M}]$. This proves (iii).

(iii) $\Rightarrow$ (i). Assume (iii). If $x \in M^{(B)}$, $y \in M$, then

$$\begin{aligned} \bigvee_{z \in y} \|x = \hat{z}\| \in U &\Rightarrow \|x \in \hat{y}\| \in U \qquad \text{(by (3))} \\ &\Rightarrow x^U \in_U \hat{y}^U \\ &\Rightarrow h(x^U) \in h(\hat{y}^U) \\ &\Rightarrow i(x) \in i(\hat{y}) = j(y) \\ &\Rightarrow i(x) = j(z) \text{ for some } z \in M \qquad \text{(by (iii))} \\ &\Rightarrow z \in y. \end{aligned}$$

Hence $i(x) = j(z) = i(\hat{z})$, so, by Lemma 4, $\|x = \hat{z}\| \in U$. (i) follows.

Finally, the equivalence of (ii) and (iii) follows immediately from the transitivity of $\mathfrak{M}$ and the fact that j is an $\in$-isomorphism of $\mathfrak{M}$ onto $j[\mathfrak{M}]$. □

Remark. The equivalence of (i) and (iv) was suggested to me by B. Balcar.

Now define $U_* \in M^{(B)}$ by domain$(U_*) = \{\hat{x} : x \in B\}$ and $U_*(\hat{x}) = x$ for $x \in B$. U_* is called the *canonical generic ultrafilter* in $\mathfrak{M}^{(B)}$, since, as is well-known, we have $\|U_*$ is a $P^{(\mathfrak{M})}(B)^\wedge$-complete ultrafilter in $\hat{B}\| = 1$. We observe that, for $x \in M^{(B)}$,

$$\|x \in U_*\| = \bigvee_{y \in B} y \wedge \|x = \hat{y}\|. \tag{4}$$

Let $\mathscr{S}_3$ be the subfamily of $P^{(\mathfrak{M})}(B)$ consisting of all subsets of B of the form $\{y \wedge \|x = \hat{y}\| : y \in B\}$ for $x \in M^{(B)}$.

Our final result in this note is the following

Theorem 3. *The following conditions are equivalent:*

(i) *U is $\mathscr{S}_3$-complete;*
(ii) *$\in_U$ is well-founded, j is the identity on U, and $i(U_*) = U$;*
(iii) *U is $\mathfrak{M}$-generic.*

Proof. (i) $\Rightarrow$ (iii). Assume (i). Let $\{b_i : i \in I\}$ be an $\mathfrak{M}$-partition of B; then by Lemma 1 we can find $x \in M^{(B)}$ such that $b_i = \|x = \hat{b}_i\|$ for $i \in I$. We have

$$\bigvee_{y \in B} y \wedge \|x = \hat{y}\| \geqq \bigvee_{i \in I} b_i \wedge \|x = \hat{b}_i\| = \bigvee_{i \in I} b_i = \mathbf{1}.$$

Hence, by (i), there is $y \in B$ such that $y \wedge \|x = \hat{y}\| \in U$. If $y \notin \{b_i : i \in I\}$, then

$$\|x = \hat{y}\| = \|x = \hat{y}\| \wedge \mathbf{1} = \|x = \hat{y}\| \wedge \bigvee_{i \in I} \|x = \hat{b}_i\| \leqq \bigvee_{i \in I} \|\hat{y} = \hat{b}_i\| = \mathbf{0},$$

so that $\|x = \hat{y}\| \notin U$. It follows that $y = b_i$ for some $i \in I$, whence $b_i \wedge \|x = \hat{b}_i\| \in U$, and so $b_i \in U$. (iii) now follows from Lemma 2.

(iii) $\Rightarrow$ (ii). Assume (iii). Then $\in_U$ is well-founded and j is the identity on U by Theorem 2. We also claim that for $x \in M^{(B)}$, $\{i(x) : \|x \in U_*\| \in U\} = \{i(\hat{y}) : y \in U\}$. In fact this follows from the chain of equivalences:

$$\begin{aligned} \|x \in U_*\| \in U &\Leftrightarrow \bigvee_{y \in B} y \wedge \|x = \hat{y}\| \in U \quad \text{(by (4))} \\ &\Leftrightarrow \exists y \in B[y \wedge \|x = \hat{y}\| \in U] \quad \text{(by (iii))} \\ &\Leftrightarrow \exists y \in U[\|x = \hat{y}\| \in U]. \\ &\Leftrightarrow \exists y \in U[i(x) = i(\hat{y})]. \end{aligned}$$

Now we have $i(U_*) = \{i(x) : \|x \in U_*\| \in U\} = \{i(\hat{y}) : y \in U\} = \{j(y) : y \in U\} = U$ and so (ii) follows.

(ii) $\Rightarrow$ (i). Assuming (ii), we have

$$U = i(U_*) = \{i(x) : \|x \in U_*\| \in U\}. \tag{5}$$

Hence

$$\begin{aligned} \bigvee_{y \in B} y \wedge \|x = \hat{y}\| \in U &\Rightarrow \|x \in U_*\| \in U \quad \text{(by (4))} \\ &\Rightarrow i(x) \in U \quad \text{(by (5))} \\ &\Rightarrow i(x) = y = j(y) \text{ for some } y \in U \text{ (since } j \text{ is the identity on } U) \\ &\Rightarrow i(x) = i(\hat{y}) \\ &\Rightarrow \|x = \hat{y}\| \in U \\ &\Rightarrow y \wedge \|x = \hat{y}\| \in U, \end{aligned}$$

and (i) follows. □

Acknowledgements. I wish to express my gratitude to Bohuslav Balcar and Dana Scott for helpful observations and suggestions.

References

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(Eingegangen am 28. November 1974 und 9. Oktober 1975)