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Cohesiveness

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ABSTRACT: It is characteristic of a continuum that it be "all of one piece", in the sense of being inseparable into two (or more) disjoint nonempty parts. By taking "part" to mean open (or closed) subset of the space, one obtains the usual topological concept of *connectedness*. Thus a space *S* is defined to be connected if it cannot be partitioned into two disjoint nonempty open (or closed) subsets – or equivalently, given any partition of *S* into two open (or closed) subsets, one of the members of the partition must be empty. This holds, for example, for the space **R** of real numbers and for all of its open or closed intervals. Now a truly radical condition results from taking the idea of being "all of one piece" literally, that is, if it is taken to mean inseparability into any disjoint nonempty parts, or subsets, *whatsoever*. A space *S* satisfying this condition is called *cohesive* or *indecomposable.* While the law of excluded middle of classical logic reduces indecomposable spaces to the trivial empty space and one-point spaces, the use of intuitionistic logic makes it possible not only for nontrivial cohesive spaces to exist, but for every connected space to be cohesive.In this paper I describe the philosophical background to cohesiveness as well as some of the ways in which the idea is modelled in contemporary mathematics.

Key Words. Continuum, cohesiveness, connectedness, intuitionistic set theory, topos.

RÉSUMÉ : Le continu cohésif. Un continuum est « d'une seule pièce », au sens où il ne peut être divisé en deux (ou plusieurs) parties non vides disjointes. Si la « partie » désigne un ouvert (ou un fermé) de l'espace, on est conduit au concept topologique classique de connexité. Ainsi un espace *S* est-il connexe s'il est impossible de le partitionner en deux sous-ensembles ouverts (ou fermés) non vides et disjoints – ou de façon équivalente, si, pour toute partition par deux ouverts (ou deux fermés) de *S*, l'un d'eux est vide. Tel est le cas, par exemple, de l'espace **R** des réels et de tous ses intervalles ouverts et fermés. Un tournant radical s'opère si l'on prend l'expression « d'une seule pièce » au sens litttéral, c'est-à-dire au sens de l'inséparabilité de l'espace en deux parties, ou sous-ensembles, quelconques. Un espace *S* satisfaisant une telle condition est dit *cohésif* ou indécomposable. En logique classique, en raison de la validité du tiers exclu, les espaces cohésifs se réduisent à des espaces vides ou à des singletons ; en revanche, la logique intuitioniste, non seulement, garantit l'existence d'espaces cohésifs non triviaux, mais, de plus, fait de tout espace connexe un espace cohésif. Dans cet article, nous présentons le substrat philosophique de l'indécomposabilité ainsi que les diverses modélisations de cette notion dans les mathématiques contemporaines.

Mots clés : Continuum, indécomposabilité, connexité, théorie intuitioniste des ensembles, topos

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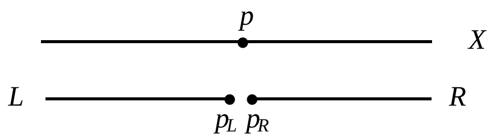
1. What is cohesiveness?

It is characteristic of a continuum[1] that it is "gapless" or "all of one piece", in the sense of not being *actually separated* into two or more disjoint proper parts. On the other hand it has been taken for granted from antiquity that continua are *limitlessly divisible,* or *separable* into parts in the sense that any part of a continuum can be "divided", or "separated" into two or more disjoint proper parts. Now there is a traditional conceptual difficulty in seeing just how the parts of a continuum obtained by separation – assumed disjoint – "fit together" exactly so as to reconstitute the original continuum. This difficulty is simply illustrated by considering the case in which a straight line X is divided into two segments L, R by cutting it at a point p. What happens to p when the cut is



made? On the face of it, there are four possibilities (not all mutually exclusive): (*i*) p is neither in L nor in R; (*ii*) p may be identified as the right-hand endpoint p_L of L: (*iii*) p may be identified as the left-hand endpoint p_R of R; (*iv*) p may be identified as *both* the right-hand endpoint of L *and* the left-hand endpoint of R. Considerations of symmetry suggest that there is nothing to choose between (*ii*) and (*iii*), so that if either of the two holds, then so does the other. Accordingly we are reduced to possibilities (*i*) and (*iv*). In case (*i*), L and R are disjoint, but since neither contains p, they together fail to cover X; while in case (*iv*), L and R together cover X, but since each contains p, they are not disjoint. This strongly suggests that a (linear) continuum *cannot* be separated, or decomposed, into two disjoint proper parts *which together cover it*.[2] Herein lies the germ of the idea of cohesiveness.

Of course, this analysis is quite at variance with the standard set-theoretic (Cantor-Dedekind) account of the linear continuum as a discrete linearly ordered set **R** of real numbers. "Cutting" **R** (or any interval thereof) at a point p amounts to partitioning it into the pairs of subsets $(\{x : x \leq p\}, \{x : p < x\})$ or $(\{x : x < p\}, \{x : p \leq x\})$: the first and second of these correspond, respectively, to cases (*ii*) and (*iii*) above. Now in the discrete case, one cannot appeal to symmetry as before: consider, for instance, the partitions of the set of natural numbers into the pairs of subsets $(\{n : n \leq 1\}, \{n : 1 < n\})$ and $(\{n : n < 1\}, \{n : 1 \leq n\})$. The first of these is $(\{0, 1\}, \{2, 3, ...\})$ and the second $(\{0\},$

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[1] For an extended analysis and account of the development of the continuum concept, see Bell (2005). A briefer presentation is given in Bell (online).

[2] That is, in the words of Weyl [1925], "A continuum cannot be put together out of parts".

{1, 2, ...}). Here it is manifest that the symmetry naturally arising in the continuous case does not apply: in the first partition 1 is evidently a member of its first component and in the second partition, of its second. In sum, when a discrete linearly ordered set X is "cut", no ambiguity arises as to which segment of the resulting partition the cut point is to be assigned, so that the segments of the partition can be considered disjoint while their union still constitutes the whole of X [3].

Acknowledging the fact that the set-theoretic continuum, as a discrete entity, *can* be separated into disjoint parts, set theory proceeds to capture the characteristic "gaplessness" of a continuum by restricting the *nature* of the parts into which it can be so separated. In set-theoretic topology this is done by confining "parts" to *open* (or *closed*) subsets, leading to the standard topological concept of *connectedness*. Thus a space S is defined to be connected if it cannot be partitioned into two disjoint nonempty open (or closed) subsets[4] – or equivalently, given any partition of S into two open (or closed) subsets, one of the members of the partition must be empty. It is a standard topological theorem that the space **R** of real numbers and all of its intervals are connected in this sense.

But now let us return to our original analysis. This led to the idea that a continuum cannot be decomposed into disjoint parts. Let us take the bull by the horns and attempt to turn this idea into a definition. We shall call a space S *cohesive* or *indecomposable*[5], or a (genuine) *continuum* if, for *any* parts, or subsets U and V of S, whenever $U \cup V = S$ and $U \cap V = \varnothing$, then one of U, V must be empty, or, equivalently, one of U, V must coincide with S. Put succinctly, a space is cohesive if it cannot be partitioned into two nonvoid parts. Another form of cohesiveness, slightly weaker than the version just stated, is: for any subsets U, V each of which contains at least one point, if $U \cup V = S$, then $U \cap V$ cannot be empty.

Cohesiveness may also be phrased in the following way. Call a subset U of S *detachable* if a "complementary" subset V of S exists satisfying $U \cup V = S$

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[3] Even so, as Michael White remarks (White 1992, p. 20), "if we take a (supposedly continuous) physical object and cut it into two pieces, it would seem strange to say that one piece contains its limit (at the place of bisection) but that the other piece does not contain such a limit – that the end where it has been cut, although obviously limited, does not *contain* its terminus in the way that the other piece does." While certainly not inconsistent, this violation of intuition is part of the price that must be paid for treating the continuous as discrete.

[4] We note that the partitions obtained above by cutting **R** at p each consist of an open set and a closed set.

[5] The term *unsplittable* is also used. We shall use the term *decomposable* for "not cohesive". It should be noted that in topos theory the term "indecomposability" sometimes receives the considerably stronger meaning of *irreducibility* (see, e.g. Bell (1988) or Lambek and Scott (1986)). A space S is said to be *irreducible* if, for any subsets U and V of S for which $U \cup V = S$, we have $U = S$ or $V = S$, even when U and V *are not assumed to be disjoint.* (Notice that **R** can never be irreducible.) Clearly S is irreducible if the trivial filter $\{S\}$ over S is prime. The "logical" condition for irreducibility is :

$$\forall x \in S\, [P(x) \vee Q(x)] \rightarrow [\forall x \in S\, P(x) \vee \forall x \in S\, Q(x)].$$

for arbitrary properties P and Q.

and $U \cap V = \varnothing$. Then S is cohesive precisely when its only detachable subsets are $\varnothing$ and S itself.

Cohesiveness can be furnished with various "logical" formulations. Namely, S is cohesive in the first, stronger sense, if and only if, for any property P defined on S, the following implication holds:

$$(*)\ \forall x \in S\, [P(x) \vee \neg P(x)] \rightarrow [\forall x \in S\, P(x) \vee \forall x \in S\, \neg P(x)].$$

And S is cohesive in the second, weaker sense if and only if, for any properties P, Q defined on S:

$$[\forall x \in S\, [P(x) \vee Q(x)] \wedge \exists\, x \in S\, P(x) \wedge \exists\, x \in S\, Q(x)] \rightarrow [\neg \forall x \in S\, \neg [P(x) \wedge Q(x)].$$[6]

We observe that the law of excluded middle of classical logic confines cohesive spaces to the trivial empty space and one-point spaces. For nontrivial cohesive spaces to become admissible, therefore, it is necessary to *abandon the law of excluded middle.*[7] In fact the existence of nontrivial cohesive spaces is compatible with *intuitionistic* logic. Indeed much more can be said: it is compatible with intuitionistic logic that *every space which is connected in the usual topological sense is cohesive.* How does this come about? To get a clue, let us reformulate our definitions in terms of maps, rather than parts. If we denote by **2** the two-element discrete space, then connectedness of a space S is equivalent to the condition that any *continuous* map $S \rightarrow \mathbf{2}$ is constant, and cohesiveness of S to the condition that any map $S \rightarrow \mathbf{2}$ *whatsoever* is constant. Supposing S to be connected and to possess more than one point, then from the law of excluded middle it follows that there exist nonconstant – and hence discontinuous – maps $S \rightarrow \mathbf{2}$. But the situation would be decidedly otherwise if *all* maps defined on S were continuous, for then, clearly, the connectedness of S would immediately yield its cohesiveness. So if S could be conceived as inhabiting a universe **U** in which all maps defined on S are continuous, then, within **U**, S would be both nontrivial *and* cohesive. Such universes **U** can in fact be constructed within category theory as *toposes*. Their underlying logic is intuitionistic, and within them the law of excluded middle fails in just the way necessary to allow for the presence of nontrivial cohesive spaces. In certain toposes, every connected space is cohesive. We return to this below.

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[6] Cohesiveness in this sense admits the following homely illustration (see also the remarks on Veronese below). Suppose one attempts to paint a board (S) exactly half-black and half-white. For any point x on the board write $P(x)$, $Q(x)$ for "x is painted black, or white, respectively". Then cohesiveness of S entails that it *cannot* be the case that there is *no* point on the board which is painted *both* black and white. In painting it seems to be difficult, if not impossible in practice to prevent "leakage at the borders"; the revelation that this may also be impossible in principle will perhaps prove consoling to those of us who, like myself, are constantly frustrated in our efforts to produce neat paint jobs.

[7] It is easy to see that the law of excluded middle in the form $\forall x\ \forall y\ [x = y \vee x \neq y]$ must fail for any cohesive space S with at least two elements. For if that law held in S, then, for $a \in S$, the sets $\{a\}$, $S \setminus \{a\}$ would constitute a partition of S into two nonvoid parts.

2. TRACING THE IDEA OF COHESIVENESS: ARISTOTLE, VERONESE, BRENTANO

While cohesiveness and connectedness as we have defined the terms are modern mathematical concepts, related ideas in regard to continuous entities can be traced back to antiquity. Anaxagoras, for example, asserts around 450 B.C. that

> The things in the one world-order are not separated one from the other nor cut off with a hatchet, neither the hot from the cold nor the cold from the hot.

Here the "one world-order" is the homogeneous continuum supposed by Anaxagoras to constitute the world.

It was Aristotle who first undertook the systematic analysis of continuity and discreteness. Aristotle's theory of the continuum rests upon the assumption that all change is continuous and that continuous variation of quality, of quantity and of position are inherent features of perception and intuition. Aristotle considered it self-evident that a continuum cannot consist of points. Any pair of unextended points, he observes, are such that they either touch or are totally separated: in the first case, they yield just a single unextended point, in the second, there is a definite gap between the points. Aristotle held that any continuum – a continuous path, say, or a temporal duration, or a motion – may be divided *ad infinitum* into other continua but not into what might be called "discreta" – parts that cannot themselves be further subdivided. Accordingly, paths may be divided into shorter paths, but not into unextended points; durations into briefer durations but not into unextended instants; motions into smaller motions but not into unextended "stations". Nevertheless, this does not prevent a continuous line from being divided at a point constituting the common border of the line segments it divides. But such points are, according to Aristotle, just *boundaries,* and not to be regarded as actual *parts* of the continuum from which they spring. If two continua have a common boundary, that common border unites them into a single continuum. Such boundaries exist only *potentially*, since they come into being when they are, so to speak, marked out as connecting parts of a continuum; and the parts in their turn are similarly dependent as parts upon the existence of the continuum.

Aristotle identifies continuity (and discreteness) as attributes applying to the category of Quantity[8]. As examples of continuous quantities he offers lines, planes, solids (i.e., solid bodies), extensions, movement, time and space; among discrete quantities he includes number[9] and speech[10]. He lays down the following definition of continuity:

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[8] In Book VI of the *Categories.* Quantity (*ποσον*) is introduced by Aristotle as the category associated with *how much.* In addition to exhibiting continuity and discreteness, quantities are, according to Aristotle, distinguished by the feature of being *equal* or *unequal.*

[9] Here it must be noted that for Aristotle, as for ancient Greek thinkers generally, the term "number" – *arithmos* – means just "plurality".

> I mean by one thing being continuous with another that those extremities of the two things in virtue of which they are in contact with each other become one and the same thing and (as the very name indicates) are "held together", which can only be if the two limits do not remain two but become one and the same. [11]

In effect, Aristotle here defines continuity as a *relation* between entities rather than as an *attribute* appertaining to a single entity; that is to say, he does not provide an explicit definition of the concept of *continuum.* He indicates that a single continuous whole can be brought into existence by "gluing together" two things which have been brought into contact, which suggests that the continuity of a whole should derive from the way its *parts* "join up". That this is indeed the case is revealed by turning to the account of the difference between continuous and discrete quantities offered in the *Categories:*

> Discrete are number and language; continuous are lines, surfaces, bodies, and also, besides these, time and space. For the parts of a number have no common boundary at which they join together. For example, ten consists of two fives, however these do not join together at any common boundary but are separate; nor do the constituent parts three and seven join together at any common boundary. Nor could you ever in the case of number find a common boundary of its parts, but they are always separate. Hence number is one of the discrete quantities…. A line, on the other hand, is a continuous quantity. For it is possible to find a common boundary at which its parts join together – a point. And for a surface – a line; for the parts of a plane join together at some common boundary. Similarly in the case of a body one would find a common boundary – a line or a surface – at which the parts of the body join together. Time also and space are of this kind. For present time joins on to both past time and future time. Space again is one of the continuous quantities. For the parts of a body occupy some space, and they join together at a common boundary. So the parts of the space occupied by various parts of the body themselves join together at the same boundary as the parts of the body do. Thus space is also a continuous quantity, since its parts join together at one common boundary.[12]

Accordingly for Aristotle quantities such as lines and planes, space and time are continuous by virtue of the fact that their constituent parts "join together at some common boundary".

Let us attempt to turn Aristotle's idea of continuity into a mathematical definition. Suppose then that we are given "quantities" *A, B, C,…, U, V, X, Y, Z.* We suppose also that we have an inclusion relation $\subseteq$ between quantities: thus $U \subseteq A$ is understood to mean that *U* is a *subquantity* or *part* of *A*, or that *U* is *included* in *A.* We assume that for any quantity *A* there is a void subquantity

[10] Aristotle points out that (spoken) words are analyzable into syllables or phonemes, linguistic "atoms" themselves irreducible to simpler linguistic elements.

[11] Physics, V, 3.

[12] Categories, VI.

θ with the property that $\theta \subseteq U$ for all subquantities U of A. Given subquantities U, V of a quantity A, we suppose that there are subquantities $U \oplus V$, $U \otimes V$ of A – the *join* and *meet*, respectively, of U and V, with the property that, for any subquantity X of A, $U \oplus V \subseteq X$ if and only if $U \subseteq X$ and $V \subseteq X$, and $X \subseteq U \otimes V$ if and only if $X \subseteq U$ and $X \subseteq V$. So $U \oplus V$ is the "least" subquantity including both U and V, and $U \otimes V$ is the "greatest" subquantity included in both U and V – the *boundary* of U and V. The assertion $U \otimes V \neq \theta$ may be understood as "U and V have a common (i.e., nonvoid) boundary".

For Aristotle, constituent parts of continuous quantities "always join together at a common boundary". This suggests that we call a quantity A *continuous in the Aristotelian sense* or an *Aristotelian continuum*, if any pair of (nonvoid) "constituent parts" of A have a "common boundary", that is, whenever $U, V \subseteq A$ are such that $U \neq \varnothing$ and $V \neq \varnothing$ and $U \oplus V = A$, then $U \otimes V \neq \varnothing$. This corresponds (essentially) to the weaker version of cohesiveness formulated above.

Of particular relevance to our discussion is an observation Aristotle makes (in the Metaphysics) in connection with the joining and division of bodies:

> But points, lines and planes, although they exist at one time and at another do not, cannot be in the process of being either generated or destroyed; for whenever bodies are joined or divided, at one time, when they are joined, one surface is instantaneously produced, and at another, when they are divided, two. Thus, when the bodies are combined the surface does not exist, but has perished; and when they are divided, surfaces exist which did not exist before. (The indivisible point is of course never divided into two.)[13]

Aristotle's view is accordingly that the actual division of a body produces bounding surfaces, and so, analogously, in the words of the contemporary scholar Michael White, that the "actual bisection of an interval results in two distinct points, a limit or terminus of each sub-interval, where there was formerly one 'position' "[14]. Moreover, when the subintervals are rejoined, the two distinct endpoints "become" one. This brings to mind case (iv) of the analysis with which we started and from which the cohesiveness (as we have defined it) of a linear continuum was inferred.

It would of course be grossly anachronistic to infer from all this that Aristotle conceived of continua as being cohesive in the exact technical sense in which it has been defined here. Nevertheless, the quotations do suggest that

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[13] *Metaphysics,* III. Compare also the following passage in *Physics,* VIII, in which Aristotle comments, in connection with Zeno's paradoxes of motion, on the division of space and movement: "For whoever divides the continuous into two halves thereby confers a double function upon the point of division, for he makes it both a beginning and an end. And that is just what the counting man, or the dividing man whose half-sections he counts, is doing; and by the very act of division both the line and the movement cease to be continuous".

[14] White (1992), 20. In fact, according to White, Aristotle "does not seem to recognize" the "open (or half open) intervals of magnitude", that is, intervals lacking at least one of their end points, generated by cutting the set-theoretic linear continuum.

Aristotle saw something like cohesiveness as the feature distinguishing continuous from discrete quantity[15].

We now jump forward a couple of millenia to consider the views on the continuum of the geometer Giuseppe Veronese (1854-1917). Of particular importance for Veronese's analysis of the continuum is his account of the nature of points. For him points are nothing more than signs indicating "positions of the uniting of two parts" of a (linear) continuum. To illustrate this Veronese offers two thought experiments. In the first of these it is supposed that

> ...the part *a* of the rectilinear object is painted red, the remaining part □*'* white, and suppose further that there is no other colour between the white and the red. That which separates the white from the red can be coloured neither white nor red, and therefore cannot be a part of the object, since by assumption all its parts are white or red. And this sign of separation of uniting can be considered as belonging either to the white or to the red, if one considers them independently of one another. If we now abstract from the colours, we can assume that the sign of separation between the parts *a* and □*'* belongs to the object itself.[16]

In the second thought experiment, Veronese proposes to

> cut a very fine thread at the place indicated by *X* with the blade of an extremely sharp knife, *[so that]* the two parts *a* and *a'* separate *(Fig.1)* and we assume that one can put the thread back together *(Fig. 2)* without seeing where the cut was, in other words, without a particle of the thread being lost. One produces this, apparently, if one looks at the thread from a certain distance. If one now considers the part a from right to left as the arrow above a indicates, then what one sees of the cut is surely not part of the thread, just as what one sees from a body is not part of the body itself. It happens analogously if one looks at the part *a'* from left to right. If the sign of separation *X* of the parts *a* and *a'*, which by assumption belongs to the thread itself, were part of the thread, then looking at a from right to left, one would not see all of this part, since that which separates the part a from *a'* is only that which one sees in the way indicated above when one supposes the thread put back together. [17]

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[15] White's "principle of non-supervenience of continuity" (*ibid.*, p. 29), which he attributes to Aristotle, in fact amounts essentially to cohesiveness. This principle reads:"(*) Each partition of a continuous magnitude into proper parts yields parts each of which is pairwise continuous with at least one other part".

Here the notion of a part being "continuous with" another part is taken in the Aristotelian sense of "the limits of both parts at which they touch are the same", in other words, part of their boundaries coincide. In particular if a continuous magnitude is divided into two parts, (*) asserts that these parts cannot be disjoint. That is, (*) asserts that a continuous magnitude is cohesive in the weaker sense.

[16] Quoted in Fisher (1994), p. 139

[17] *Ibid.* pp. 139-40.

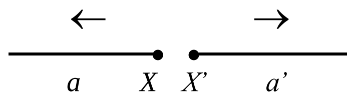

Figure 1

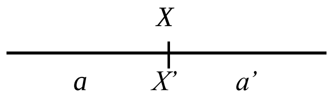

Figure 2

The premise of Veronese's first thought experiment is that a continuum can be coloured completely by two different colours, which is tantamount to claiming that it is decomposable. He then attempts to get round the difficulty of which segment of the resulting partition of the continuum the boundary ("that which separates") is to be assigned by arguing that it is not actually part of the continuum – it is, in fact, no more than a "sign of separation" which can be considered as belonging to either segment. But he goes on, using an appeal to "abstraction", to assert that the sign of separation, i.e. the boundary between the segments, can be regarded as being part of the continuum itself. Unfortunately, this move reintroduces precisely the difficulty we noted at the beginning, namely, to which segment is the boundary, now conceived as being an actual part of the continuum, to be assigned? As long as the "sign of separation" remains a potentiality, that is, as long as the continuum is not actually separated, this "boundary issue" does not arise. But conceiving of the continuum as being actually separated into two segments inevitably rekindles the boundary issue[18].

Veronese's second thought experiment postulates that a thread, or more generally a linear continuum, can be cut into two segments which, upon being rejoined, reconstitute the continuum in its entirety – thus once again making the assumption that the continuum is decomposable. From this he infers that the point, or sign, of separation cannot be part of the continuum. His argument is essentially that since, on intuitive grounds, the "cut" itself cannot be part of either segment, it cannot be part of the whole continuum either.

Veronese's purpose is to establish that a point cannot be a part of a continuum, and to do this he makes what seems to him the natural assumption that any continuum is decomposable. He would not, perhaps, have taken very seriously the contrapositive version of his argument to the effect that if points can be considered parts of continua, then the latter must be cohesive. But presumably he would have regarded it as valid – if unsound.

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[18] In fact, Veronese's initial assumption that the given rectilinear continuum is coloured in the way he specifies *already entails* that it is actually separated into segments.

The Austrian philosopher Franz Brentano (1838-1917) devoted considerable attention to the nature of the continuum. Brentano held that the idea of the continuous is derived from primitive sensible intuition, and is not reducible to anything else. Our grasp of the concept of continuity, according to Brentano, emerges in three stages. First, sensation presents us with objects having parts that coincide. From such objects the concept of *boundary* is abstracted in turn, and then one grasps that these objects actually *contain* coincident boundaries. Finally one sees that this is all that is required in order to have grasped the concept of continuity, and so also of a continuum.

Like Aristotle, Brentano considered it self-evident that a continuum cannot consist of points. Points are just boundaries, and not to be regarded as actual parts of the continuum from which they spring. If two continua have a common boundary, that common border unites them into a single continuum. Such boundaries exist only potentially, since they come into being when they are, so to speak, marked out as connecting parts of a continuum; and the parts in their turn are similarly dependent as parts upon the existence of the continuum. In this connection he writes:

> We must ask those who say that the continuum ultimately consists of points what they mean by a point. Many reply that a point is a cut which divides the continuum into two parts. The answer to this is that a cut cannot be called a thing and therefore cannot be a presentation in the strict and proper sense at all. We have, rather, only presentations of contiguous parts. … The spatial point cannot exist or be conceived of in isolation. It is just as necessary for it to belong to a spatial continuum as for the moment of time to belong to a temporal continuum.[19]

For Brentano the essential feature of a continuum is its inherent capacity to engender boundaries, and the fact that such boundaries can be grasped as coincident. Brentano ascribes particular importance to the fact that points in a continuum can *coincide.* On this matter he writes:

> Various other thorough studies could be made [*on the continuum concept*] such as a study of the impossibility of adjacent points and the possibility of coincident points, which have, despite their coincidence, distinctness and full relative independence. [*This*] has been and is misunderstood in many ways. It is commonly believed that if four different-coloured quadrants of a circular area touch each other at its centre, the centre belongs to only one of the coloured surfaces and must be that colour only. Galileo's judgment on the matter was more correct; he expressed his interpretation by saying paradoxically that the centre of the circle has as many parts as its periphery. Here we will only give some indication of these studies by commenting that everything which arises in this connection follows from the point's relativity as involves a continuum and the fact that it is essential for it to belong to a continuum. Just as the possibility of the coincidence of different

[19] Brentano (1974), p. 354.

> points is connected with that fact, so is the existence of a point in diverse or more or less perfect plerosis[20]. All of this is overlooked even today by those who understand the continuum to be an actual infinite multiplicity and who believe that we get the concept not by abstraction from spatial and temporal intuitions but from the combination of fractions between numbers, such as between 0 and 1.[21]

Brentano took a dim view of the efforts of mathematicians to construct the continuum from numbers, or points. His attitude varied from rejecting such attempts as inadequate to according them the status of "fictions"[22]. For example, in a discussion of Dedekind's construction of the real numbers we read:

> Dedekind believes that either the number 1/2 forms the beginning of the series 1/2 to 1, so that the series 0 to 1/2 would thereby be spared a final member, i.e. an end point which would belong to it, or conversely. But this is not how things are in the case of a true continuum. Much rather it is the case that, when one divides a line, every part has a starting point, but in half plerosis.[23]

> That one has... postulated something completely absurd is seen immediately if one splits the supposedly continuous series of fractions between 0 and 1 into two parts at some arbitrary position. One of the two parts will then end with some fraction *f*, the second however could now start only if there were some fraction in the series which was the immediate neighbour of *f*, which is however not the case .... We should apparently have something that began but without having any beginning.[24]

And again:

> Geometry teaches that a line that is halved is halved in a single point. The line *a b c* in the point *b*. And further, that one is able to lay the one half over the other, for example in such a way that *cb* would come down on *ba*, the point *c* coinciding with the point *b*, the other end coinciding with the point *a*. According to the doctrine here considered [i.e., the Cantor-

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[20] The concept of *plerosis* (from Greek "fullness") plays an important role in Brentano's account of the continuum. Plerosis is a quality possessed by boundaries and may be regarded as is a measure of the number of directions in which a given boundary actually bounds. Thus, for example, within a temporal continuum the endpoint of a past episode or the starting point of a future one bounds in a single direction, while the point marking the end of one episode and the beginning of another may be said to bound doubly. In the case of a spatial continuum there are numerous additional possibilities: here a boundary may bound in all the directions of which it is capable of bounding, or it may bound in only some of these directions. In the former case, the boundary is said to exist in *full plerosis*; in the latter, in *partial plerosis.* Brentano believed that the concept of plerosis enabled sense to be made of the idea that a boundary possesses "parts", even when the boundary lacks dimensions altogether, as in the case of a point.

[21] Brentano (1974), p. 357.

[22] In a letter to Husserl drafted in 1905, Brentano asserts that "I regard it as absurd to interpret a continuum as a set of points."

[23] Brentano (1988), pp. 40 – 41.

[24] *Ibid.*, p. 4.

> Dedekind construction of the continuum], the divisions of the line would not occur in points, but in some absurd way behind a point and before all others, of which none, however, would stand closest to the cut. One of the two lines into which the line would be split upon division would therefore have an end point, but the other no beginning point. This inference has quite correctly been drawn by Bolzano, who was led thereby to his monstrous doctrine that there would exist bodies with and without surfaces.[25.]

From these quotations it becomes clear that Brentano rejected the idea of "splitting" a continuum into two parts, one of which lacks a boundary. His view is that the boundary is common to both parts, but with a difference in "plerosis" depending on which part the boundary is considered as bounding. Thus, for example, when one divides a closed interval $[a, b]$ at an intermediate point c, one necessarily obtains the closed intervals $[a, c]$, $[c, b]$, with the common point c. Brentano would maintain that the "plerosis" of the point c is different in its two manifestations: as a right-hand endpoint of the first interval, it is in half-plerosis to the left; in the second, analogously, in half-plerosis to the right. But this does not affect the fact that the point c is common to both intervals. That being the case, Brentano would probably have regarded a continuous line as indecomposable, into disjoint intervals at least.

3. THE COHESIVENESS OF THE INTUITIONISTIC CONTINUUM: BROUWER AND WEYL

While none of the above thinkers can be claimed to have asserted explicitly that the continuum is cohesive, cohesiveness was a cornerstone of Brouwer's view of the continuum – the intuitionistic continuum. In his early thinking Brouwer held that that the continuum is presented to intuition as a whole, and that it is impossible to construct all its points as individuals. But in his mature thought, he radically transformed the concept of "point", endowing points with sufficient fluidity to enable them to serve as generators of a "true" continuum. This fluidity was achieved by admitting as "points", not only fully defined discrete numbers such as $\sqrt{2}$, π, e, and the like – which have, so to speak, already achieved "being" – but also "numbers" which are in a perpetual state of "becoming" in that their the entries in their decimal (or dyadic) expansions are the result of free acts of choice by a subject operating throughout an indefinitely extended time. The resulting choice sequences cannot be conceived as finished, completed objects: at any moment only an initial segment is known. Thus Brouwer obtained the mathematical continuum in a manner compatible with his belief in the primordial intuition of time – that is, as an unfinished, in fact unfinishable entity in a perpetual state of growth, a "medium of free development". In Brouwer's vision, the mathematical continuum is indeed "constructed", not, however, by initially shattering, as did Cantor and Dedekind, an intuitive continuum into isolated points, but rather by assembling it from a complex of continually changing overlapping parts.

The mathematical continuum as conceived by Brouwer presents a number of features rendering it bizarre to the classical eye. For example, in the Brouwerian continuum the usual law of comparability, namely that for any real numbers a, b either $a < b$ or $a = b$ or $a > b$, fails. Even more fundamental is the failure of the law of excluded middle in the form that for any real numbers a, b, either $a = b$ or $a \neq b$.

While the intuitionistic continuum may possess a number of negative features from the standpoint of the classical mathematician, it has the merit of corresponding more closely to the continuum of intuition than does its classical counterpart. Hermann Weyl, who in the early 1920s was closely associated with Brouwer, pointed out a number of respects in which this is so:

> In accordance with intuition, Brouwer sees the essential character of the continuum, not in the relation between element and set, but in that between part and whole. The continuum falls under the notion of the 'extensive whole', which Husserl characterizes as that "which permits a dismemberment of such a kind that the pieces are by their very nature of the same lowest species as is determined by the undivided whole. [26]

Far from being bizarre, the failure of the law of excluded middle for points in the intuitionistic continuum was seen by Weyl as "fitting in well with the character of the intuitive continuum":

> For there the separateness of two places, upon moving them toward each other, slowly and in vague gradations passes over into indiscernibility. In a continuum, according to Brouwer, there can be only continuous functions. The continuum is not composed of parts. [27]

For Brouwer had indeed shown, in 1924, that every function defined on a closed interval of the continuum as he conceived of it is continuous, in fact uniformly continuous[28]. As a consequence, the intuitionistic continuum is cohesive, a fact which Weyl found thoroughly agreeable. Here is what Weyl had to say on the issue in 1921:

> ...if we pick out a specific point, say, $x = 0$, on the number line C (i.e., on the variable range of a real variable x), then one cannot, under any circumstance, claim that every point either coincides with it or is disjoint from it. The point $x = 0$ thus does not at all split the continuum C into two parts C^-: $x < 0$ and C^+: $x > 0$, in the sense that C would consist of the union of C^-, C^+ and the one point 0 ... If this appears offensive to present-day mathematicians with their atomistic thought habits, it was in earlier times a self-evident view held by everyone: Within a continuum, one can very well

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[26] Weyl (1949), p. 52.

[27] *Ibid.,* p. 54.

[28] One might be inclined to regard this claim as impossible: is not a counterexample provided by, for example, the function f given by $f(0) = 0$, $f(x) = |x|/x$ otherwise? No, because from the intuitionistic standpoint this function is not everywhere defined on the interval [–1, 1], being undefined at those arguments x for which it is unknown whether $x = 0$ or $x \neq 0$.

> generate subcontinua by introducing boundaries; yet it is irrational to claim that the total continuum is made up of the boundaries and the subcontinua. The point is, a genuine continuum is something connected in itself, and it cannot be divided into separate fragments; this conflicts with its nature. [29]

The uniform continuity of functions defined on a closed interval of the intuitionistic continuum – hence the indecomposability of any closed interval, as well as the whole continuum itself – was shown by Brouwer to follow from certain intuitionistically plausible principles he held choice sequences should satisfy. One such principle is the *Continuity Principle*: given a relation $R(\alpha, n)$ between choice sequences α and numbers n, if for each α a number n may be determined for which $R(\alpha, n)$ holds, then n can already be determined on the basis of the knowledge of a finite number of terms of α[30] From this it can be shown that every function from **R** to **R** is continuous[31]. Another such principle is *Bar Induction*, a certain form of induction for well-founded sets of finite sequences[32]. Brouwer used Bar Induction and the Continuity Principle in proving that every real-valued function defined on a closed interval is uniformly continuous.

Brouwer gave the intuitionistic conception of mathematics an explicitly subjective twist by introducing the *creative subject*. The creative subject was conceived as a kind of idealized mathematician for whom time is divided into discrete sequential stages, during each of which he may test various propositions, attempt to construct proofs, and so on. In particular, it can always be determined whether or not at stage n the creative subject has a proof of a particular mathematical proposition p. While the theory of the creative subject remains controversial, its purely mathematical consequences can be obtained by a simple postulate which is entirely free of subjective and temporal elements.

The creative subject allows us to define, for a given proposition p, a binary sequence $\langle a_n \rangle$ by

$a_n = 1$ if the creative subject has a proof of p at stage n; $a_n = 0$ otherwise.

Now if the construction of these sequences is the only use made of the creative subject, then references to the latter may be avoided by postulating the principle known as *Kripke's Scheme:*

> For each proposition p there exists an increasing binary sequence $\langle a_n \rangle$ such that p holds if and only if $a_n = 1$ for some n.

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[29] Weyl 1921, 111.

[30] The plausibility of this assertion emerges when one considers that according to Brouwer the construction of a choice sequence is incompletable; at any given moment one can know nothing about it outside the identities of a finite number of its entries. Brouwer's principle amounts to the assertion that every function from $\mathbf{N}^{\mathbf{N}}$ to **N** is continuous.

[31] Bridges and Richman (1987), p. 109.

[32] For an explicit statement of the principle of Bar Induction, see Ch. 3 of Dummett (1977), or Ch. 5 of Bridges and Richman (1987).

Taken together, these principles have been shown[33] to have remarkable consequences for the cohesiveness of subsets of the continuum. Not only is the intuitionistic continuum cohesive, but, assuming the Continuity Principle and Kripke's Scheme, it remains cohesive even if one pricks it with a pin at a point.[34] So "the [intuitionistic] continuum has, as it were, a syrupy nature, one cannot simply take away one point."[35] If in addition Bar Induction is assumed, then, even more surprisingly, cohesiveness is maintained even when all the rational points are removed from the continuum.

4. COHESIVENESS OF SPACES IN MODELS OF INTUITIONISTIC SET THEORY

We have observed that in classical set theory the only cohesive spaces are trivial. It is a remarkable fact, however, that the existence of a whole range of nontrivial cohesive spaces, including the real line and all of its intervals, is consistent with *intuitionistic* set theory **IST**[36]. This is established by constructing models of **IST** in which the existence of such spaces can be demonstrated. These models are category-theoretic in nature: to be precise, each is a certain type of category known as a *topos*. By definition, a topos is a category $\mathcal{E}$ which resembles the familiar category $\mathcal{SET}$ of sets (whose objects are all sets and whose maps are all functions) in that it satisfies the following conditions:

- $\mathcal{E}$ has a terminal object 1 such that, for any object X, there is a unique map $X \to 1$. (Maps $1 \to X$ correspond to elements of X.)
- Each pair of objects A, B of $\mathcal{E}$ has a product $A \times B$.
- Each object A of $\mathcal{E}$ has a power object PA whose elements correspond to subobjects (subsets) of A.

It can be shown that any topos $\mathcal{E}$ has the two further properties:

- Each pair of objects A, B of $\mathcal{E}$ has a *coproduct* $A + B$, the categorical counterpart of the disjoint union of A and B
- Each pair of objects A, B of $\mathcal{E}$ has an *exponential* B^A, the categorical counterpart of the set of all maps $A \to B$.

There are several ways of characterizing cohesive objects in a topos in terms of coproducts and exponentials. If we write 2 for 1 + 1, and $\cong$ for "is isomorphic to", then the following equivalent conditions on an object A of a topos expresses the cohesiveness of A:

- $2^A \cong 2$
- for any object X, $(X + X)^A \cong X^A + X^A$

[33] van Dalen (1997)

[34] More exactly, for any real number a, the complement $\mathbf{R} \setminus \{a\}$ of $\{a\}$ is cohesive.

[35] *Ibid.* There the classical continuum is described as the "frozen intuitionistic continuum".

[36] By "intuitionistic set theory" we mean the theory in intuitionistic first-order logic whose axioms are the "usual" axioms of Zermelo set theory (without the axiom of choice), namely: Extensionality, Pairing, Union, Power set, Infinity and Separation.

- for any objects X, Y, $(X + Y)^A \cong X^A + Y^A$

Toposes can arise in a variety of ways, for example[37]:

(i) as categories of "sets undergoing variation", with maps respecting the particular form of variation;

(ii) as categories of "sets with a generalized equality relation", with maps preserving that relation in an appropriate sense;

(iii) as the category-theoretic embodiment of an intuitionistic higher-order theory.

There is a natural interpretation of the usual language of set theory in any topos under which all the axioms of **IST** are satisfied. In this sense toposes may be considered a model of **IST**, and we shall treat them as such.

There are a number of toposes which contain nontrivial cohesive objects. We consider five of these.

1. *The topos T of sheaves on the site of topological spaces equipped with the open cover topology*[38]

This topos is of sort (i). Here the "variation" takes place over a suitable small category of topological spaces containing the usual real line **R**. It can shown that, in *T*, every map from the space $\mathbf{R_D}$ of Dedekind real numbers to itself is continuous (with respect to the usual open-interval topology). Now Stout (1976) has shown that, in **IST**, $\mathbf{R_D}$ is connected in the following sense:

(*) For all subsets U, V of $\mathbf{R_D}$ open in the usual open-interval topology,

$$[\mathbf{R_D} = U \cup V \;\&\; \exists x.\, x \in U \;\&\; \exists x.\, x \in V] \Rightarrow U \cap V \neq \varnothing.$$

It follows that, in *T*, $\mathbf{R_D}$ is cohesive in the weaker sense mentioned at the beginning. For suppose U, V are (arbitrary) subsets of $\mathbf{R_D}$ for which $\mathbf{R_D} = U \cup V$, $U \cap V \neq \varnothing$.and $\exists x.\, x \in U$ & $\exists x.\, x \in V$. Define the map $f : \mathbf{R_D} \to \mathbf{2}$ by $f(x) = 0$ if $x \in U$, $f(x) = 1$ if $x \in V$. Then, in *T*, f is continuous, so U and V are open. We infer from (*) that $U \cap V \neq \varnothing$. The weak cohesiveness of $\mathbf{R_D}$ follows.

2. *The topos SHV*(R) *of sheaves over the real line* R[39]

This topos is also of sort (i). Here the "variation" takes place over the category of open subsets of **R.** It can be shown that in *SHV*(**R)** every map from a closed interval of $\mathbf{R_D}$ to $\mathbf{R_D}$ is *uniformly* continuous, and it follows easily from this that $\mathbf{R_D}$ and all of its intervals are cohesive.

3. *The free topos F*

This topos is of sort (iii). In fact *F* is the category-theoretic embodiment of the *free* or *minimal* higher-order intuitionistic theory **F** (with an object of natural numbers): **F** is minimal in the sense that it is the common part of all such theories. It has been shown by Joyal that, in *F*, all maps $\mathbf{R_D} \to \mathbf{R_D}$ are continuous. It follows, as above, that $\mathbf{R_D}$ is cohesive in the weaker sense.

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[37] For an extensive list, see Johnstone (2002).

[38] This example receives a detailed discussion in Mac Lane and Moerdijk (1992).

[39] See Scott (1970), Hyland (1979).

In ***F*** power objects, or sets, exhibit an exotic property we shall call *extreme cohesiveness*[40]. A space S is *extremely cohesive* if whenever $\{X_n : n \in \mathbf{N}\}$ is a (not necessarily disjoint) covering of S indexed by the set **N** of natural numbers, then $S = X_n$ for some n. Obviously an extremely cohesive space is cohesive. Notice that the real line, while it can be cohesive, is never extremely cohesive because it can be expressed as the union of the family of closed intervals $[-n, n]$ for $n \in \mathbf{N}$.

Extreme cohesiveness of S is equivalent to the *uniformity rule* for S, namely,

$$(*)\ \forall x \in S\, \exists n \in \mathbf{N}\ \phi(x, n) \Rightarrow \exists n \in \mathbf{N}\ \forall x \in S\ \phi(x, n).$$

Extreme cohesiveness can also be expressed within infinitary logic: S is extremely cohesive if, for any countable collection $\{P_n : n \in \mathbf{N}\}$ of properties of S, the following implication is valid:

$$\forall x \in S \bigvee_{n\in\mathbf{N}} P_n(x) \Rightarrow \bigvee_{n\in\mathbf{N}} \forall x \in S\ P_n(x).$$

Clearly, if S is extremely cohesive, any map $S \to \mathbf{N}$ is constant.

Extreme cohesiveness may be construed in a variety of ways:

(a) As an extreme version of the *pigeonhole principle.* The pigeonhole principle (in a general form) states that, given two sets S and I, if the cardinality of M exceeds that of I, then any I-indexed covering of S has a non-singleton member. The more the cardinality of S exceeds that of I, the larger must one of these non-singleton members be: in particular, it might be S itself. Let us call *S incomparably bigger* than I if any I-indexed covering of S *always* contains S as a member. Clearly, if S is incomparably bigger than I, then any map $S \to I$ is constant. Any set with more than one element is incomparably bigger than any singleton. Extreme cohesiveness is the same as being incomparably bigger than **N**.

(b) As a kind of *measurability* in the set-theoretic sense[41]. This follows if one observes that S is incomparably bigger than I exactly when the trivial filter $\{S\}$ on S is I-complete.[42] So S is incomparably bigger than **N**, i.e. S is extremely cohesive, if and only if the trivial filter $\{S\}$ on S is countably complete. Compare this with the definition of a measurable cardinal in the set-theoretic sense: a cardinal K is measurable if it supports a countably complete ultrafilter not generated by a singleton.

(c) As an extreme version of countable compactness with arbitrary countable covers replacing open covers.

(d) As an extreme kind of *amorphousness*. Let us call a space *D discrete* if it satisfies $\forall x \in D\ \forall y \in D\ (x = y \vee x \neq y)$ In any topos the set **N** of natural numbers is discrete. It is natural to call a space which is not discrete

[40] See Lambek and Scott (1986).

[41] I owe this observation to Jean Petitot.

[42] Recall that a filter F on a set M is *I-complete* if whenever the union of any I-indexed family S of subsets of M is a member of F, then at least one member of S is also a member of F.

amorphous: elements of an amorphous space may be thought of as being only partially distinguishable. It is easy to see that a cohesive – and *a fortiori* an extremely cohesive – space is amorphous. Note also that in any non-Boolean topos (i.e., one in which the law of excluded middle fails) the power set of a set with at least one element is always amorphous.

Recall now the uniformity rule (*) for S. Its premise may be construed as an attempt to coordinate each element of S via ϕ with a natural number, that is, to give a function f: $S \to \mathbf{N}$ for which $\forall x \in S\ \phi(x, fx)$. If S is sufficiently amorphous, there is no way of distinguishing elements x and y of S so as to make the elements fx, fy of the discrete set **N** distinct. This means that f has to be constant, and we infer the consequent of (*).

It can be shown that in ***F*** the power set of any set with at least one element is extremely cohesive, hence incomparably bigger than **N**.

4. *The effective topos EFF* [43]

This topos is of type (ii). The "generalized equality relation" here on a set X is a P**N**-valued predicate on $X \times X$ satisfying formal versions of symmetry and transitivity formulated in terms of the notion of *recursive realizability*. A map in *EFF* between two sets X, Y equipped with generalized equality relations in this sense is a P**N**-valued predicate R on $X \times Y$ satisfying the corresponding formal version of the condition "R is a single-valued relation with domain X and codomain Y".

In *EFF* maps between objects constructed from the natural numbers correspond to recursive functions between them. In particular maps from **N** to **N** may be considered as being (total) recursive functions on **N**. Hence, in *EFF*, *Church's thesis* holds in the strong sense that *every* function $\mathbf{N} \to \mathbf{N}$ is recursive. It follows from this that in *EFF* the domain of Cauchy real numbers $\mathbf{R_C}$ corresponds to the *recursive reals*, that is, the real numbers arising as limits of recursive Cauchy sequences of rationals. (This means that "real analysis" in *EFF* coincides with recursive analysis.) Using the fact from classical recursion theory that recursive maps on the recursive reals are continuous, it follows that, in *EFF*, every map $\mathbf{R_C} \to \mathbf{R_C}$ is continuous. So, as above, we infer that $\mathbf{R_C}$ is cohesive in the weaker sense[44].

Finally, it has been shown that, in *EFF*, P**N** is extremely cohesive.

5. *Smooth toposes*

There are various examples of so-called *smooth* toposes, each of which may be considered to be an enlargement of the category *MAN* of manifolds (or spaces) and smooth maps to a topos *E* which contains no new maps between spaces (so that all such maps in *E* are still smooth, and so *a fortiori* continuous), but does contain – unlike *MAN* – certain "infinitesimal" spaces as described below. The fact that all maps between spaces are continuous in a

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[43] See Hyland (1982).

[44] Hence $\mathbf{R_D}$ is also cohesive since in *EFF* $\mathbf{R_D}$ can be shown to be isomorphic to $\mathbf{R_C}$.

smooth topos E guarantees that, in E every connected space, in particular the real line, is cohesive.

Each smooth topos is a model of a theory extending **IST** called *smooth infinitesimal analysis* (**SIA**). Here are the basic axioms of the theory[45].

Axioms for the continuum, or smooth real line R. These include the usual axioms for a commutative ring with unit expressed in terms of two operations + and •, (we usually write xy for x•y) and two distinguished elements $0 \neq 1$. In addition we stipulate that **R** is an *intuitionistic field,* i.e., satisfies the following axiom:

$$x \neq 0 \text{ implies } \exists y\, xy = 1.$$

Axioms for the strict order relation < on **R**. These are:

O1. $a < b$ and $b < c$ implies $a < c$.

O2. $\neg(a < a)$

O3. $a < b$ implies $a + c < b + c$ for any c.

O4. $a < b$ and $0 < c$ implies $ac < bc$

O5. either $0 < a$ or $a < 1$.

O6 $a \neq b$ [46] implies $a < b$ or $b < a$.

O7. $0 < x$ implies $\exists y\, x = y^2$.

We write Δ for the subset $\{x : x^2 = 0\}$ of **R** consisting of (nilsquare) *infinitesimals* or *microquantities;* use the letter ε as a variable ranging over Δ. Δ is subject to the

Microaffineness Axiom. For any map $g : \Delta \to \mathbf{R}$ there exist unique $a, b \in \mathbf{R}$ such that, for all ε, we have

$$g(\varepsilon) = a + b\varepsilon.$$

We define the relation $\leq$ on **R** is defined by $a \leq b$ iff $\neg\,(b < a)$. The open interval (a, b) and closed interval $[a, b]$ are defined as usual, viz. $(a, b) = \{x : a < x < b\}$ and $[a, b] = \{x : a \leq x \leq b\}$; similarly for half-open, half-closed, and unbounded intervals.

From these axioms the following can be deduced:

- Δ is *nondegenerate,* i.e. $\Delta \neq \{0\}$.[47]

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[45] Moerdijk and Reyes (1991) or Bell (2008). Lambek (forthcoming) provides a nice account of infinitesimals from an algebraic point of view.

[46] Here $a \neq b$ stands for $\neg\, a = b$. It should be pointed out that axiom 6 is omitted in some presentations of **SIA**, e.g. those in Kock (1981) and McLarty (1992).

[47] It should be noted that, while Δ does not reduce to $\{0\}$, nevertheless 0 is the sole *element* of Δ in the (weak) sense that the assertion "there exists an element of Δ which is $\neq 0$" is refutable. Figuratively speaking, Δ is the "atom" 0 encased in an infinitesimal carapace.

- Call x, $y \in$ **R** *indiscriminable* (resp. *indistinguishable*) and write $x \approx$ □ (resp. $x \sim y$) if $x - y \in \Delta$ (resp. $\neg x \neq y$). Then $x \approx y$ implies $x \sim y$ (but not vice-versa).
- any closed interval is closed under indiscriminability.
- Any $f : [a, b] \to$ **R** is *indiscriminably continuous* in the sense that, for $x, y \in [a, b]$, $x \approx y$ implies $fx \approx fy$, and hence also $fx \sim fy$. (Note that it follows trivially from $x \sim y$ that $fx \sim fy$.)

In **SIA** one also assumes the

Constancy Principle. If A is any closed interval on **R,** or **R** itself, and f : $A \to$ **R** is *locally constant* in the sense that $x \approx y$ implies $fx = fy$ for all $x, y \in A$, then f is constant.

Now call a subset D of **R** *discrete* if it satisfies

$$\forall x \in D\ \forall y \in D\ [x = y \vee x \neq y].$$

Notice that if D is discrete, then, for $x, y \in D$, $x \sim y$ implies $x = y$.

It follows quickly from the Constancy Principle that, *if A is any closed interval on* **R** *or* **R** *itself, then any map on A to a discrete subset of* **R** *is constant.*[48] (To see this, let f be a map of A to a discrete set D. Then from $x \approx y$ in A we deduce $fx \sim fy$, and hence $fx = fy$, in D. So f is locally constant, and hence constant.) And from this it follows in turn that **R** and all of its closed intervals are cohesive. To see this, let A be **R** or any closed interval, and suppose that $A = U \cup V$ with $U \cap V = \varnothing$. Let **2** be the discrete subset {0, 1} of **R,** and define $f : A \to$ **2** by $f(x) = 1$ if $x \in U$, $f(x) = 0$ if $x \in V$. Then f is constant, that is, constantly 1 or 0. In the first case $V = \varnothing$, and in the second $U = \varnothing$.

From the cohesiveness of closed intervals it can be inferred[49] that in **SIA** *all intervals in* **R** *are cohesive.*

In **SIA** cohesive subsets of **R** correspond, *grosso modo*, to connected subsets of **R** in classical analysis, that is, to intervals. This is borne out by the fact that any puncturing of **R** is *decomposable*, for it follows immediately from Axiom O6 that

$$\mathbf{R} \setminus \{a\} = \{x : x > a\} \cup \{x : x < a\}.$$

The set **Q** of *rational numbers* is defined as usual to be the set of all fractions of the form m/n with $m, n \in$ **N**, $n \neq 0$. The fact that **N** is cofinal in **R** ensures that **Q** is dense in **R**.

The set **R** \ **Q** of *irrational* numbers is decomposable as

$$\mathbf{R} \setminus \mathbf{Q} = [\{x : x > 0\} \setminus \mathbf{Q}] \cup [\{x : x < 0\} \setminus \mathbf{Q}\}.$$

This is in sharp contrast with the situation in intuitionistic analysis (augmented by Kripke's scheme, the continuity principle, and bar induction). For we have observed that in intuitionistic analysis not only is any puncturing of **R**

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[48] This property may be considered another strong form of cohesiveness.
[49] Bell (2001).

cohesive, but that this is even the case for the irrational numbers. This would seem to indicate that in some sense the continuum in smooth infinitesimal analysis is considerably less "syrupy" [50] than its counterpart in intuitionistic analysis.

Finally, consider the set of *infinitesimals:*

$$\mathbf{I} = \{x : \neg\, x \neq 0\}.$$

I is an *ideal*, in fact a *maximal* ideal in the ring **R.** That being the case, we may construct the quotient ring **R/I.** In certain smooth toposes **R/I** can be shown to be isomorphic to the ring $\mathbf{R_D}$ of Dedekind real numbers, from which the cohesiveness of $\mathbf{R_D}$ follows immediately from that of **R.**

5. CONCLUDING REMARKS

There is an old philosophical argument, going back to the pre-Socratics, to the effect that, if continuous extended magnitudes are limitlessly divisible, then they must be composed of indivisible atoms. For suppose one starts with such a magnitude, a line, say, then proceeds to divide it in two, then divides each half in two, indefinitely. Imagine this process to be carried out completely. The result is a multitude of parts that cannot be further divided, that is, atoms. But now, the argument continues, since at no stage in the process of division is any part of the original line "lost", that is, at any stage the original line remains the sum of the parts obtained by division, it follows that this must still be the case when the process of division is completed and the parts into which the line has been divided have become atoms. Conclusion: any line is the sum of atoms. Hermann Weyl put the matter thus:

> The old principle that "one cannot separate that which is not already separated (Gassendi) here again comes into its own. Indeed, Democritus argues with good reason that if I can break a stick, then it was from the outset not a whole. Strictest atomism is the inescapable conclusion of this.[51]

Now let us grant for the purposes of argument that the process of division can be "completed". Then drawing the conclusion depends crucially on the assumption – call it "*A*" – that *at no stage in the process of division is any part "lost".* (Without *A* one is left at the completion of the division with a multitude of atoms whose sum does not necessarily coincide with the given line.) Now to posit *A* is tantamount to asserting that *lines (or continua generally) are decomposable.* Consequently (granting the completability of the division process), *if* continua are decomposable, they are sums of atoms, hence discrete. This conclusion is, of course, the central claim of atomism. Those who (like myself) grant the limitless divisibility of continua, are happy to accept the imaginative possibility of "completing" the infinite process of dividing them, and grant as well that atoms are the result – but who nevertheless find

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[50] It should be emphasized that this phenomenon is a consequence of axiom O6: it cannot necessarily be affirmed in versions of **SIA** not including this axiom.

[51] Weyl (1925), 135

unpalatable the conclusion that continua are the sums of atoms, and so ultimately discrete – can avoid that conclusion by the simple expedient of denying that continua are decomposable, that is, by asserting that *continua are cohesive*. Cohesiveness forces us to recognize that, in separating a continuum into parts, something *is* lost thereby – call it the "glue" uniting the parts into the original whole. Before the whole is separated into parts, those parts are implicit, and so the potential existence of the "glue" uniting them lies unrecognized. The loss of that "glue", indeed its very presence within the whole, only becomes apparent once the whole is separated. When the process of dividing a continuum is "completed", one is left with a multitude of separated atoms, but the "glue" uniting them within the whole has vanished without trace. Without that "glue" to hold them together, the atoms fail to sum to the original whole.

It has to be admitted that, even if one accepts the argument just presented, cohesiveness, as defined here, may still seem a bizarre notion. For *of course* a stick can be cut into two pieces, and *of course* a board can be painted half black and half white. But if such objects were truly cohesive, wouldn't it then become impossible to carry out such routine procedures? Perhaps only those few (necessarily philosophers) bent on avoiding atomism would be forced to countenance such a curious idea! But it is not really necessary to invoke the spectre of atomism, or even to be a philosopher, to acknowledge the concept of cohesiveness. For a moment's thought shows that cohesiveness is not such an unreasonable notion after all. In fact the cohesiveness principle does not make the cutting of a stick in two or the painting of a board half black and half white impossible *per se*; it asserts nothing more than that it is impossible to do such things with *complete exactitude*. In the first case, the two half sticks cannot exactly reconstitute the original stick, and in the second the board cannot be painted with sufficient precision so as to cover it and at the same time avoid an overlapping, however small, of the painted areas. From an empirical standpoint, these facts are quite commonplace. The concept of cohesiveness may thus be seen as the result of elevating certain limitations *in practice* in the handling of continuous objects into a limitation *in principle*. These practical limitations not only make the idea of cohesiveness less offensive to intuition, they actually serve to distinguish continuous from discrete objects in their everyday handling. Thus, while dividing a stick leads to subtleties, none arise in separating a dozen eggs into two half-dozens!

Given universal discreteness, or classical logic, as in classical set theory, cohesiveness collapses into the trivial property of having no nonvoid proper parts. Consequently the counterpart of cohesiveness in set-theoretic topology – connectedness – is not and cannot be an intrinsic property of a space; it is rather a property of the topology imposed on the space: with a different topology, e.g. the discrete topology, a connected space can become highly disconnected. The property of cohesiveness, on the other hand, is intrinsic to a space, marking it as a genuine continuum in itself. A space is guaranteed to be cohesive if all maps on it are continuous. It is remarkable that the use of intuitionistic logic is compatible with the pervasive continuity of maps, hence also with the existence of cohesive spaces and genuine continua, so allowing the latter to assume their rightful place in mathematics and philosophy.

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