

ISOMORPHISM OF STRUCTURES IN S-TOPOSES

J. L. BELL

It is a well-known fact that two structures are $\infty\omega$ -equivalent if and only if they are isomorphic in some Boolean extension of the universe of sets (cf. [4]; an early allusion to this result appears in [8]). My principal object here is to show that arbitrary toposes defined over the category of sets may be used instead. Thus $\infty\omega$ -equivalence means isomorphism in the extremely general context of some universe of "variable" sets in which not only is much of the usual set-theoretic machinery unavailable but the underlying logic is not even classical. This provides further support for the view that $\infty\omega$ -equivalence is a relation between structures of fundamental importance.

§1. We recall that a *topos* is a category E satisfying the following conditions (for notation, see [6] or [7]):

- E has a terminal object 1 .
- E has finite products.
- There is an object Ω in E , the *subobject classifier* or *truth-value object*, and an arrow $1 \xrightarrow{\text{true}} \Omega$ such that for each monic $A \xrightarrow{m} B$ there is a unique $B \xrightarrow{\chi_m} \Omega$, the *characteristic arrow* of m (or A) such that

$$\begin{array}{ccc} A & \xrightarrow{\quad} & 1 \\ m \downarrow & & \downarrow \text{true} \\ B & \xrightarrow{\chi_m} & \Omega \end{array}$$

is a pullback, and conversely every arrow to Ω arises in this way. Where there is no possibility of confusion, we write χ_A for χ_m .

- For each object A there is an object Ω^A , the *exponential* of A , and an arrow $A \times \Omega^A \xrightarrow{\text{ev}} \Omega$ such that, for each arrow $A \times B \xrightarrow{f} \Omega$ there is a unique arrow $B \xrightarrow{g} \Omega^A$ such that

$$\begin{array}{ccc} A \times B & & \\ \text{id}_A \times g \downarrow & \searrow f & \\ A \times \Omega^A & \xrightarrow{\text{ev}} & \Omega \end{array}$$

commutes. (Here id_A is the identity arrow on A .)

It is well known that any topos has an initial object 0 , arbitrary finite coproducts,

Received November 18, 1978.

and pullbacks. We shall always assume that a topos has small hom-sets, and that it is nondegenerate, i.e. $0 \not\cong 1$.

Let S be the category of sets. A topos E is said to be *defined over* S or to be simply an S -topos if it has arbitrary set-indexed copowers of its objects. It is easy to see that for this to be the case it is enough to be able to form arbitrary set-indexed copowers of 1 in E . Moreover, it is well known (see [6, Chapter 4]) that in any S -topos one can form arbitrary coproducts of subobjects of any given fixed object, in particular, of subobjects of 1 . If $\{X_i : i \in I\}$ is any family of objects, we write $\coprod_{i \in I} X_i$ or $\coprod X_i$ for the coproduct, assuming it exists.

We shall continually use the standard fact that, in a topos, if for each $i \in I$ the diagram

$$\begin{array}{ccc} X_i & \longrightarrow & Y_i \\ \downarrow & & \downarrow \\ A & \longrightarrow & B \end{array}$$

is a pullback, then so is the diagram

$$\begin{array}{ccc} \coprod X_i & \longrightarrow & \coprod Y_i \\ \downarrow & & \downarrow \\ A & \longrightarrow & B \end{array}$$

with the natural arrows.

Given an S -topos E , there is a natural functor $\hat{} : S \rightarrow E$ (left adjoint to the hom-functor $E(1, \cdot) : E \rightarrow S$) defined as follows: for each set A we put $\hat{A} = \coprod_A 1$ and for $f : A \rightarrow B$ we define $\hat{f} : \hat{A} \rightarrow \hat{B}$ to be the unique arrow in E making the diagrams

$$\begin{array}{ccc} 1 & \xrightarrow{\hat{a}} & \hat{A} \\ & \searrow f(a)^\wedge & \downarrow \hat{f} \\ & & \hat{B} \end{array}$$

commute for all $a \in A$, where $\hat{a}, \hat{b}$ are the canonical injections of 1 into $\hat{A}, \hat{B}$ respectively. It is easy to verify that $\hat{}$ is faithful and preserves products (in fact, all finite limits).

If $B \subseteq A$ and $a \in A$, it is not hard to see that¹

$$a \in B \Leftrightarrow \chi_{\hat{B}} \hat{a} = \text{true};$$

$$a \notin B \Leftrightarrow \chi_{\hat{B}} \hat{a} = \text{false},$$

where $1 \xrightarrow{\text{false}} 0$ is the characteristic arrow of $0 \rightarrow 1$.

Let $\text{Sub}(E)$ be the set of all subobjects of 1 in an S -topos E , where we agree to

¹Composition of arrows is denoted, as usual, in the "set-theoretic" manner, i.e. by juxtaposition in the order opposite to that in which they appear in a diagram.

identify isomorphic subobjects. The relation $\leq$ on $\text{Sub}(E)$ defined by $U \leq V$ iff there is a (necessarily unique) monic $U \rightarrow V$ is then a partial ordering on $\text{Sub}(E)$ and, as is well known, $\langle \text{Sub}(E), \leq \rangle$ is a complete Heyting algebra in which the meet $U \wedge V$ of two elements U, V coincides with their product $U \times V$ and, for any set $\{U_i : i \in I\}$, the join $\bigvee_{i \in I} U_i$ is the unique member of $\text{Sub}(E)$ such that

$$\begin{array}{ccc} \prod_{i \in I} U_i & \xrightarrow{\quad} & 1 \\ & \searrow k \quad \swarrow m & \\ & \bigvee_{i \in I} U_i & \end{array}$$

commutes with k epic and m monic. (We recall that in a topos any arrow has a "unique" epic-monic factorization.)

For each object $A \in E$ we define a (partial) *section* of A to be an arrow of the form $\text{dom}(u) \xrightarrow{s} A$, where $\text{dom}(u)$ is a subobject of 1. Let $\tilde{A}$ be the set of sections of A , and let $u, v \in \tilde{A}$. We define $\llbracket u = v \rrbracket \in \text{Sub}(E)$ to be the subobject of 1 such that

$$\begin{array}{ccc} \llbracket u = v \rrbracket & \xrightarrow{\quad} & \text{dom}(u) \\ \downarrow & & \downarrow \\ \text{dom}(v) & \xrightarrow{\quad} & A \end{array}$$

is a pullback. Notice that we have, for $a, a' \in A$, $\llbracket \hat{a} = \hat{a}' \rrbracket = 1$ and $\llbracket \hat{a} = \hat{a}' \rrbracket = 0$ whenever $a \neq a'$.

Let H be a complete Heyting algebra. Following [5], we define an H -set to be a pair $\langle X, \delta \rangle$ consisting of a set X and a function $\delta : X \times X \rightarrow H$ satisfying

$$\delta(x, y) = \delta(y, x), \quad \delta(x, y) \wedge \delta(y, z) \leq \delta(x, z).$$

(δ may be thought of as an " H -valued equality relation" on X .) The category S_H of H -sets is defined as follows. Its objects are all H -sets; an arrow between two objects $\langle X, \delta \rangle$ and $\langle Y, \varepsilon \rangle$ is a function $f : X \times Y \rightarrow H$ such that, for all $x, x' \in X$, $y, y' \in Y$,

$$\begin{aligned} \delta(x, x') \wedge f(x, y) &\leq f(x', y), \\ f(x, y) \wedge \varepsilon(y, y') &\leq f(x, y'), \\ f(x, y) \wedge f(x, y') &\leq \varepsilon(y, y'), \\ \bigvee_y f(x, y) &= \delta(x, x). \end{aligned}$$

(Such a function may be thought of as an H -valued relation which is functional and defined on X .) Given two arrows $\langle X, \delta \rangle \xrightarrow{f} \langle Y, \varepsilon \rangle$ and $\langle Y, \varepsilon \rangle \xrightarrow{g} \langle Z, \eta \rangle$, the composition $\langle X, \delta \rangle \xrightarrow{gf} \langle Z, \eta \rangle$ is defined by $(gf)(x, z) = \bigvee_{y \in Y} f(x, y) \wedge g(y, z)$; the identity arrow on $\langle X, \delta \rangle$ is just δ . It is well known that S_H is an S -topos in which $\text{Sub}(S_H) \cong H$ and, for any set A , $\hat{A}$ is the H -set $\langle A, \delta \rangle$ where δ is defined by $\delta(a, a) = 1$, $\delta(a, a') = 0$ for any $a, a' \in A$ such that $a \neq a'$.

We observe that, for any S -topos E , and any object $A \in E$, the pair $\langle \hat{A}, [\cdot = \cdot] \rangle$ is a $\text{Sub}(E)$ -set.

If B is a complete Boolean algebra, the *Boolean extension* $V^{(B)}$ of the universe of sets may be regarded as a category in the following way. First, we identify elements of $V^{(B)}$ which are equal with probability 1. The objects of the category $V^{(B)}$ are then the (thus identified) elements of $V^{(B)}$ and the arrows are those elements of $V^{(B)}$ which are functions with probability 1. It is easy to see that the resulting category is a topos, and indeed an S -topos in which the natural functor $S \hookrightarrow V^{(B)}$ defined above is just the usual injection of V into $V^{(B)}$ (see [1]). It is known [5] that $V^{(B)}$ and S_B are equivalent as categories.

Let E be a topos. A (binary) E -structure² is a pair $\mathfrak{A} = \langle A, R \rangle$ consisting of an object A of E and a subobject R of $A \times A$. If $\mathfrak{A} = \langle A, R \rangle$ and $\mathfrak{B} = \langle B, S \rangle$ are two E -structures, an arrow $f: A \rightarrow B$ is said to be *compatible* with R and S if

$$\chi_R = \chi_S(f \times f).$$

An E -isomorphism between A and B is an isomorphism f (in the usual categorical sense in E) which is compatible with R and S . Under these conditions we write $f: \mathfrak{A} \cong \mathfrak{B}$. If such an isomorphism exists in E we write $\mathfrak{A} \cong_E \mathfrak{B}$ and say that $\mathfrak{A}$ and $\mathfrak{B}$ are E -isomorphic.

If $\mathfrak{A} = \langle A, R \rangle$ is a (binary) structure in the usual sense, and E is an S -topos, we denote by $\hat{\mathfrak{A}}$ the E -structure³ $\langle \hat{A}, \hat{R} \rangle$. If $\mathfrak{B}$ is a structure in the usual sense, we say that $\mathfrak{A}$ and $\mathfrak{B}$ are E -equivalent, written $\mathfrak{A} \equiv_E \mathfrak{B}$, if $\hat{\mathfrak{A}} \cong_E \hat{\mathfrak{B}}$. We shall call $\mathfrak{A}$ and $\mathfrak{B}$ *topos-equivalent*, and write $\mathfrak{A} \equiv_T \mathfrak{B}$, if $\mathfrak{A} \equiv_E \mathfrak{B}$ for some S -topos E .

Given two (usual) structures $\mathfrak{A} = \langle A, R \rangle$ and $\mathfrak{B} = \langle B, S \rangle$, a *partial isomorphism* between $\mathfrak{A}$ and $\mathfrak{B}$ is a nonempty family P of functions such that:

- for each $f \in P$, we have $\text{dom}(f) \subseteq A$, $\text{ran}(f) \subseteq B$, and f is an isomorphism of $\mathfrak{A} \upharpoonright \text{dom}(f)$ to $\mathfrak{B} \upharpoonright \text{ran}(f)$;
- if $f \in P$, $a \in A$, $b \in B$, then there exist $g, h \in P$ both extending f such that $a \in \text{dom}(g)$, $b \in \text{ran}(h)$ ("back and forth" property).

If these conditions are satisfied we write $P: \mathfrak{A} \cong_P \mathfrak{B}$; if there is such a P we write $\mathfrak{A} \cong_P \mathfrak{B}$ and say that $\mathfrak{A}$ and $\mathfrak{B}$ are *partially isomorphic*.

It is well known (see [3]) that *two structures are $\omega\omega$ -equivalent if and only if they are partially isomorphic*; it is this latter relation we shall employ in the sequel.

§2. We first want to formulate necessary and sufficient conditions on an S -topos E for two given structures to be E -equivalent. Since we are ultimately going to show that E -equivalent structures are partially isomorphic, our strategy will be to start with two partially isomorphic structures, use the partial isomorphism to construct an S -topos in which they turn out to be equivalent, and then 'read off' the appropriate conditions on the topos from the corresponding properties of the partial isomorphism.

²There is really no necessity to confine attention to binary structures; we do so solely for the sake of notational simplicity. By complicating the symbolism the results given here go through for structures with any number of finitary relations.

³In order for $\langle \hat{A}, \hat{R} \rangle$ to be an E -structure in the sense of the above definition we must *identify* the naturally isomorphic objects $(A \times A)^\sim$ and $\hat{A} \times \hat{A}$. We shall continue to do this in the sequel, usually without comment.

So let $\mathfrak{A}$ and $\mathfrak{B}$ be two partially isomorphic structures and let $P: \mathfrak{A} \cong_p \mathfrak{B}$. We partially order the set P by inclusion and assign it the associated order topology in which the basic open sets are of the form $\{f \in P: p \subseteq f\}$ for $p \in P$. Now let C be the complete Boolean algebra of regular open subsets of P ; for each $a \in A$, $b \in B$ let $V_{ab} = \{f \in P: f(a) = b\}$ and

$$U_{ab} = \{q \in P: \forall p \supseteq q \exists f \in V_{ab} [p \subseteq f]\}.$$

Then U_{ab} is the interior of the closure of V_{ab} in P , and so $U_{ab} \in C$. The fact that P is a partial isomorphism now readily yields the following facts about the U_{ab} : in C ,

- (a) $\bigvee_{b \in B} U_{ab} = 1$ for each $a \in A$;
- (b) $\bigvee_{a \in A} U_{ab} = 1$ for each $b \in B$;
- (c) $U_{ab} \wedge U_{a'b} = 0$ whenever $a \neq a'$;
- (d) $U_{ab} \wedge U_{ab'} = 0$ whenever $b \neq b'$;
- (e) if either $\langle a, a' \rangle \in R$ and $\langle b, b' \rangle \notin S$ or $\langle a, a' \rangle \notin R$ and $\langle b, b' \rangle \in S$, then $U_{ab} \wedge U_{a'b'} = 0$.

Properties (a) and (b) derive from the "back and forth" condition on P , (c) and (d) from the bijectiveness of the members of P , and (e) from the compatibility of each member of P with R and S .

Now consider the topos S_C of C -valued sets. Conditions (a) and (d) mean that the U_{ab} define an arrow $f: \hat{A} \rightarrow \hat{B}$ in S_C , and the remaining conditions ensure that this arrow is an isomorphism in S_C compatible with R and S (this will follow from our general result below but can in fact be easily verified in the special case at hand). Thus, in this case, the existence of a subfamily of $C \cong \text{Sub}(S_C)$ satisfying conditions (a)–(e) above is sufficient for $\mathfrak{A}$ and $\mathfrak{B}$ to be S_C -equivalent. We are actually going to establish the much stronger result that, for any S -topos E , the existence of such a subset of $\text{Sub}(E)$ is not only sufficient but also necessary for $\mathfrak{A}$ and $\mathfrak{B}$ to be E -equivalent. To do this, however, we require a

LEMMA. Let E be an S -topos and let $f: A \rightarrow B$ be an arrow in E . Then

- (i) f is monic iff $\llbracket f\hat{a} = f\hat{a}' \rrbracket = 0$ whenever $a \neq a'$ in A ;
- (ii) f is epic iff $\bigvee_{a \in A} \llbracket f\hat{a} = \hat{b} \rrbracket = 1$ for all $b \in B$.

PROOF. Suppose f is monic. For $a, a' \in A$, consider the pullback

$$\begin{array}{ccc} \llbracket f\hat{a} = f\hat{a}' \rrbracket & \longrightarrow & 1 \\ \downarrow & & \downarrow f\hat{a} \\ 1 & \xrightarrow{f\hat{a}'} & \hat{B} \end{array}$$

Taking coproducts over a' , we get the pullback

$$(1) \quad \begin{array}{ccc} \coprod_{a' \in A} \llbracket fa = fa' \rrbracket & \longrightarrow & 1 \\ \downarrow & & \downarrow f\hat{a} \\ \hat{A} & \xrightarrow{f} & \hat{B} \end{array}$$

Since f is monic, so is the top line of (1). But clearly $\llbracket f\hat{a} = f\hat{a} \rrbracket = 1$, so $\coprod_{a' \neq a} \llbracket fa = fa' \rrbracket = 0$ and hence $\llbracket fa = f\hat{a}' \rrbracket = 0$ when $a \neq a'$.

Conversely, suppose that $\llbracket f\hat{a} = f\hat{a}' \rrbracket = 0$ when $a \neq a'$. Then the pullback (1) becomes

$$\begin{array}{ccc} 1 = \llbracket f\hat{a} = f\hat{a} \rrbracket & \longrightarrow & 1 \\ \downarrow & & \downarrow f\hat{a} \\ \hat{A} & \xrightarrow{f} & \hat{B} \end{array}$$

Taking coproducts over $a \in A$ yields the pullback

$$\begin{array}{ccc} \hat{A} & \xrightarrow{\text{id}} & \hat{A} \\ \text{id} \downarrow & & \downarrow f \\ \hat{A} & \xrightarrow{f} & \hat{B} \end{array}$$

from which it immediately follows that f is monic. This proves (i).

(ii) Suppose f is epic. For fixed $b \in B$ we get by taking coproducts a pullback

$$(2) \quad \begin{array}{ccc} \coprod_{a \in A} \llbracket f\hat{a} = \hat{b} \rrbracket & \longrightarrow & 1 \\ \downarrow & & \downarrow \hat{b} \\ \hat{A} & \xrightarrow{f} & \hat{B} \end{array}$$

Since f is epic, so is the top line of (2), giving $\bigvee_{a \in A} \llbracket f\hat{a} = \hat{b} \rrbracket = 1$.

Conversely, suppose this last condition is satisfied. Then the top line in (2) is epic and so we get, by taking coproducts over $b \in B$, a pullback:

$$\begin{array}{ccc} \coprod_{b \in B} \coprod_{a \in A} \llbracket f\hat{a} = \hat{b} \rrbracket & \longrightarrow & \hat{B} \\ \downarrow & & \downarrow \text{id} \\ \hat{A} & \xrightarrow{f} & \hat{B} \end{array}$$

But the top line of this diagram is clearly epic, and so therefore is f . This proves (ii). ■

We are now in a position to prove the promised

THEOREM. *Let E be an S -topos, and let $\mathfrak{A} = \langle A, R \rangle$, $\mathfrak{B} = \langle B, S \rangle$ be structures. Then the following are equivalent.*

(i) $\mathfrak{A} \equiv_E \mathfrak{B}$.

(ii) *There is a family $\{U_{ab} : \langle a, b \rangle \in A \times B\} \subseteq \text{Sub}(E)$ such that conditions (a)–(e) above hold.*

PROOF. (i) $\Rightarrow$ (ii) Assume (i), and let $f: \hat{\mathfrak{A}} \cong_E \hat{\mathfrak{B}}$. Then $f: \hat{A} \cong \hat{B}$ and

$$(1) \quad \chi_R = \chi_S(f \times f).$$

For each $\langle a, b \rangle \in A \times B$ put $U_{ab} = \llbracket f\hat{a} = \hat{b} \rrbracket$; then $U_{ab} \in \text{Sub}(E)$.

(a) By taking coproducts over $b \in B$, we obtain for each $a \in A$ the pullback

$$\begin{array}{ccc} \coprod_{b \in B} U_{ab} & \longrightarrow & \hat{B} \\ \downarrow & & \downarrow \text{id} \\ 1 & \xrightarrow{f\hat{a}} & \hat{B} \end{array}$$

But $\hat{B} \xrightarrow{\text{id}} \hat{B}$ is epic, and so therefore is $\coprod_{b \in B} U_{ab} \rightarrow 1$, giving (a).

(b) follows immediately from (ii) of the lemma.

(c) If $a \neq a' \in A$, then

$$U_{ab} \wedge U_{a'b} \leq \llbracket f\hat{a} = f\hat{a}' \rrbracket = 0$$

by (i) of the lemma.

(d) If $b \neq b' \in B$ then

$$U_{ab} \wedge U_{ab'} \leq \llbracket \hat{b} = \hat{b}' \rrbracket = 0.$$

(e) Before proving this we again remark that for any set X we have agreed to identify $(X \times X)^\wedge$ and $\hat{X} \times \hat{X}$. Thus, for any $\langle x, x' \rangle \in X \times X$ we shall also agree to identify $1 \xrightarrow{\langle \hat{x}, \hat{x}' \rangle} \hat{X} \times \hat{X}$ with $1 \xrightarrow{\langle x, x' \rangle} (X \times X)$.

Now suppose $\langle a, a' \rangle \in R$ and $\langle b, b' \rangle \notin S$. It is easy to see that, from the definition of U_{ab} , we have the commutative diagram

$$(2) \quad \begin{array}{ccc} U_{ab} \times U_{a'b'} & \longrightarrow & 1 \\ \downarrow & & \downarrow \langle \hat{b}, \hat{b}' \rangle \\ 1 & \xrightarrow{\langle f\hat{a}, f\hat{a}' \rangle} & \hat{B} \times \hat{B} \end{array}$$

Since $\langle b, b' \rangle \notin S$, we have $\chi_S \langle \hat{b}, \hat{b}' \rangle = \text{false}$, and so, putting $V = U_{ab} \times U_{a'b'}$, we get, using (1) and (2)

$$\begin{aligned} V &\longrightarrow 1 \xrightarrow{\text{false}} \Omega \\ &= V \longrightarrow 1 \xrightarrow{\langle \hat{b}, \hat{b}' \rangle} \hat{B} \times \hat{B} \xrightarrow{\chi_S} \Omega \\ &= V \longrightarrow 1 \xrightarrow{\langle f\hat{a}, f\hat{a}' \rangle} \hat{B} \times \hat{B} \xrightarrow{\chi_S} \Omega \\ &= V \longrightarrow 1 \xrightarrow{\langle \hat{a}, \hat{a}' \rangle} \hat{A} \times \hat{A} \xrightarrow{f \times f} \hat{B} \times \hat{B} \xrightarrow{\chi_S} \Omega \\ &= V \longrightarrow 1 \xrightarrow{\langle \hat{a}, \hat{a}' \rangle} \hat{A} \times \hat{A} \xrightarrow{\chi_{\hat{R}}} \Omega \\ &= V \longrightarrow 1 \xrightarrow{\text{true}} \Omega, \end{aligned}$$

since $\langle a, a' \rangle \in R$. But then clearly $V = 0$, whence $U_{ab} \wedge U_{a'b'} = 0$ as required. A similar argument shows that if $\langle a, a' \rangle \notin R$ and $\langle b, b' \rangle \in S$, then $U_{ab} \wedge U_{a'b'} = 0$. This proves (e).

(ii) $\Rightarrow$ (i) Let $\{U_{ab} : \langle a, b \rangle \in A \times B\}$ be a subset of $\text{Sub}(E)$ satisfying (a)–(e). We first claim that, for any $a \in A$,

$$(3) \quad \coprod_{b \in B} U_{ab} \cong 1.$$

Since by (a) we have $\coprod_{b \in B} U_{ab} \rightarrow 1$ epic, it suffices to show that this arrow is monic. We consider the pullback

$$\begin{array}{ccc} U_{ab} \times U_{ab'} & \longrightarrow & U_{ab'} \\ \downarrow & & \downarrow \\ U_{ab} & \longrightarrow & 1 \end{array}$$

Then

$$\begin{array}{ccc} \coprod_{b \in B} U_{ab} \times U_{ab'} & \longrightarrow & U_{ab'} \\ \downarrow & & \downarrow \\ \coprod_{b \in B} U_{ab} & \longrightarrow & 1 \end{array}$$

is a pullback. But by (d), $U_{ab} \times U_{ab'} = 0$ if $b \neq b'$, so the above pullback diagram becomes

$$\begin{array}{ccc} U_{ab'} \cong U_{ab'} \times U_{ab'} & \longrightarrow & U_{ab'} \\ \downarrow & & \downarrow \\ \coprod_{b \in B} U_{ab} & \longrightarrow & 1 \end{array}$$

Hence, taking coproducts over $b' \in B$, we get the pullback

$$\begin{array}{ccc} \coprod_{b' \in B} U_{ab'} & \xrightarrow{\text{id}} & \coprod_{b' \in B} U_{ab'} \\ \downarrow \text{id} & & \downarrow \\ \coprod_{b \in B} U_{ab} & \longrightarrow & 1 \end{array}$$

It follows easily from this that the bottom line of this diagram is monic, and so we get (3).

Given $a \in A$, we let $\coprod_{b \in B} U_{ab} \xrightarrow{j_a} \hat{B}$ be the unique arrow making the diagrams

$$(4) \quad \begin{array}{ccc} U_{ab} & \longrightarrow & 1 \\ \downarrow & & \downarrow \hat{b} \\ \coprod_{b \in B} U_{ab} & \xrightarrow{j_a} & \hat{B} \end{array}$$

commute for each $b \in B$. Using (3), we let i_a be the composition of the arrows $1 \cong \coprod_{b \in B} U_{ab} \xrightarrow{j_a} \hat{B}$. We now define $\hat{A} \xrightarrow{f} \hat{B}$ to be the unique arrow making the diagrams

$$\begin{array}{ccc} 1 & \xrightarrow{\hat{a}} & \hat{A} \\ & \searrow i_a & \downarrow f \\ & & \hat{B} \end{array}$$

commute for all $a \in A$.

We claim that $f: \hat{\mathcal{A}} \cong_E \hat{\mathcal{B}}$. To establish this we first observe that

$$(5) \quad U_{ab} = \llbracket f\hat{a} = \hat{b} \rrbracket$$

for $a \in A, b \in B$. For it follows immediately from (4) and the definition of i_a that the diagrams

$$\begin{array}{ccc} U_{ab} & \longrightarrow & 1 \\ \downarrow & & \downarrow \hat{b} \\ 1 & \xrightarrow{f\hat{a}} & \hat{B} \end{array}$$

commute for all $a \in A, b \in B$. Hence $U_{ab} \leq \llbracket f\hat{a} = \hat{b} \rrbracket = V_{ab}$, say. Clearly we have $V_{ab} \wedge V_{ab'} = 0$ whenever $b \neq b'$, so $V_{ab} \wedge U_{ab'} = 0$ when $b \neq b'$. Now, by (a), we have

$$\begin{aligned} V_{ab} &= V_{ab} \wedge 1 = V_{ab} \wedge \bigvee_{b' \in B} U_{ab'} \\ &= \bigvee_{b' \in B} V_{ab} \wedge U_{ab'} = V_{ab} \wedge U_{ab}. \end{aligned}$$

So $V_{ab} \leq U_{ab}$ and (5) follows.

Now we can show:

f is *monic*. This follows from (5), (c) and (i) of the lemma.

f is *epic*. This follows from (5), (b) and (ii) of the lemma.

(*) $\chi_{\hat{R}} = \chi_S(f \times f)$. Using (a), it is easy to show that

$$\bigvee_{b \in B} \bigvee_{b' \in B} U_{ab} \wedge U_{a'b'} = 1,$$

so that the arrow

$$\coprod_{b \in B} \coprod_{b' \in B} U_{ab} \times U_{a'b'} \longrightarrow 1$$

is epic. In order to prove (*) it suffices to show that, for any $a, a' \in A$,

$$(6) \quad \chi_{\hat{R}}\langle \hat{a}, \hat{a}' \rangle = \chi_S(f \times f)\langle \hat{a}, \hat{a}' \rangle = \chi_S\langle f\hat{a}, f\hat{a}' \rangle.$$

To establish (6) we treat separately the cases $\langle a, a' \rangle \in R$ and $\langle a, a' \rangle \notin R$.

Suppose then that $\langle a, a' \rangle \in R$. Using (5) we get a commutative diagram

$$\begin{array}{ccccc} U_{ab} \times U_{a'b'} & \longrightarrow & 1 & & \\ \downarrow & & \downarrow \langle \hat{b}, \hat{b}' \rangle & & \\ 1 & \xrightarrow{\langle f\hat{a}, f\hat{a}' \rangle} & \hat{B} \times \hat{B} & \xrightarrow{\chi_S} & \Omega \end{array}$$

If $\langle b, b' \rangle \in S$ then $\chi_S\langle \hat{b}, \hat{b}' \rangle = \text{true}$, while if $\langle b, b' \rangle \notin S$ then $U_{ab} \times U_{a'b'} = 0$ by (e). Hence for any $b, b' \in B$ we have

$$\begin{aligned} U_{ab} \times U_{a'b'} &\longrightarrow 1 \xrightarrow{\text{true}} \Omega \\ &= U_{ab} \times U_{a'b'} \longrightarrow 1 \xrightarrow{\langle f\hat{a}, f\hat{a}' \rangle} \hat{B} \times \hat{B} \xrightarrow{\chi_S} \Omega \end{aligned}$$

and hence, putting $X = \coprod_{b \in B} \coprod_{b' \in B} U_{ab} \times U_{a'b'}$,

$$\begin{aligned} X &\longrightarrow 1 \xrightarrow{\text{true}} \Omega \\ &= X \longrightarrow 1 \xrightarrow{\chi_S\langle f\hat{a}, f\hat{a}' \rangle} \Omega. \end{aligned}$$

Thus, since $X \rightarrow 1$ is epic,

$$\chi_S \langle f\hat{a}, f\hat{a}' \rangle = \text{true}.$$

But since $\langle a, a' \rangle \in R$ we have

$$\chi_R \langle \hat{a}, \hat{a}' \rangle = \text{true}$$

and so (6) follows in the case where $\langle a, a' \rangle \in R$.

An argument similar to the one just given establishes (6) in the case where $\langle a, a' \rangle \notin R$. Thus (6) holds for all $a, a' \in A$. This gives (*), and completes the proof. ■

Now we can prove the main result of the paper.

COROLLARY 1. $\mathfrak{A} \cong_p \mathfrak{B}$ if and only if $\mathfrak{A} \equiv_T \mathfrak{B}$, and hence $\mathfrak{A} \equiv_{\omega} \mathfrak{B}$ if and only if $\mathfrak{A} \equiv_T \mathfrak{B}$.

PROOF. If $\mathfrak{A} \cong_p \mathfrak{B}$, let P be a partial isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$ and let C be the complete boolean algebra of regular open subsets of P . At the beginning of this section we remarked that we can find a subfamily $\{U_{ab} : \langle a, b \rangle \in A \times B\}$ of C satisfying conditions (a)–(e) above, so it follows from the theorem that $\mathfrak{A}$ and $\mathfrak{B}$ are S_C -equivalent, whence $\mathfrak{A} \equiv_T \mathfrak{B}$.

Conversely, suppose that $\mathfrak{A} \equiv_T \mathfrak{B}$. Then there is an S -topos E such that $\mathfrak{A} \equiv_E \mathfrak{B}$, as guaranteed by the theorem, let $\{U_{ab} : \langle a, b \rangle \in A \times B\}$ be a subset of $\text{Sub}(E)$ satisfying conditions (a)–(e). For each $0 \neq U \in \text{Sub}(E)$ put

$$\tilde{U} = \{\langle a, b \rangle \in A \times B : U \leq U_{ab}\}.$$

We claim that

$$P = \{\tilde{U} : 0 \neq U \in \text{Sub}(E)\}$$

is a partial isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$.

(1) *Each $\tilde{U} \in P$ is a one-one function.* For if $\langle a, b \rangle \in \tilde{U}$ and $\langle a, b' \rangle \in \tilde{U}$, then $U \leq U_{ab} \wedge U_{ab'} = 0$ if $b \neq b'$ by (d); so since $U \neq 0$ we must have $b = b'$. Thus $\tilde{U}$ is a function. In a similar way, now using (c), we can show that $\tilde{U}$ is one-one.

(2) *Each $\tilde{U} \in P$ is an isomorphism of its domain onto its range.* Suppose $\langle a, b \rangle \in \tilde{U}$, $\langle a', b' \rangle \in \tilde{U}$ and $\langle a, a' \rangle \in R$. Then, if $\langle b, b' \rangle \notin S$ it follows from (e) that $U \leq U_{ab} \wedge U_{a'b'} = 0$, so since $U \neq 0$ we must have $\langle b, b' \rangle \in S$. Similarly, if $\langle b, b' \rangle \in S$, we obtain $\langle a, a' \rangle \in R$.

(3) Suppose $\tilde{U} \in P$ and $a \in A$. By (a), we have $1 = \bigvee_{b \in B} U_{ab}$; hence

$$U = U \wedge \bigvee_{b \in B} U_{ab} = \bigvee_{b \in B} U \wedge U_{ab}.$$

Since $U \neq 0$, for some $b \in B$ we must have $V = U \wedge U_{ab} \neq 0$. Then $\tilde{V} \in P$, $\tilde{U} \subseteq \tilde{V}$ and $\langle a, b \rangle \in \tilde{V}$, whence $a \in \text{dom}(\tilde{V})$.

Similarly, now using (b), for each $b \in B$ we get $\tilde{V} \in P$ such that $\tilde{U} \subseteq \tilde{V}$ and $b \in \text{ran}(\tilde{V})$.

Thus P is a partial isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$, and the proof is complete. ■

REMARK. The definition of $\tilde{U}$ in the preceding proof was suggested by considering the special case in which E is a Boolean extension $V^{(C)}$ containing an element f which is an isomorphism of $\mathfrak{A}$ and $\mathfrak{B}$ with probability 1. In this case we take $U_{ab} = \llbracket \langle \hat{a}, \hat{b} \rangle \in f \rrbracket$, so for each $U \in \text{Sub}(V^{(C)}) \cong C$, $\tilde{U}$ is the set of pairs $\langle a, b \rangle \in A \times B$ (whose canonical images are) “forced” by U to be a member of f .

Finally, we observe that, for certain familiar kinds of topos E , E -equivalence is the *same* as isomorphism. Let us call a topos E *normal* if $\text{Sub}(E)$ is (isomorphic to) a *topology* on a set. This condition is satisfied, for example, when E is a presheaf category $S^{D^{\text{op}}}$ where D is any small category, or when E is the category of sheaves over any topological space. We then have:

COROLLARY 2. *Let E be a normal S -topos. Then $\mathcal{A} \equiv_E \mathcal{B}$ if and only if $\mathcal{A} \cong \mathcal{B}$.*

PROOF. One way round is trivial. Conversely, suppose that $\mathcal{A} \equiv_E \mathcal{B}$ and let $\text{Sub}(E)$ be a topology $\mathcal{T}$ on a set I . Let $\{U_{ab} : \langle a, b \rangle \in A \times B\}$ be a subfamily of $\mathcal{T}$ satisfying conditions (a)–(e) as guaranteed by the theorem. Since E is nondegenerate, I is nonempty and so we can choose a member $i_0 \in I$. Define a function $f: A \rightarrow B$ by putting $f(a) = b$ iff $i_0 \in U_{ab}$. It is now easy to verify, using conditions (a)–(e), that this defines an isomorphism between $\mathcal{A}$ and $\mathcal{B}$. ■

ACKNOWLEDGMENTS. I am grateful to Mike Brockway for providing me with a copy of his (so far) unpublished manuscript [2] which contained results that I employed in my original proof of Corollary 1. (The present proof, however, is considerably more direct and bears little resemblance to the original.) I would also like to thank the referee for his helpful comments and suggestions.

REFERENCES

- [1] J.L. BELL, *Boolean-valued models and independence proofs in set theory*, Clarendon Press, Oxford, 1977.
- [2] M. BROCKWAY, *Soft objects of a topos and sheaves over the global truth values*, unpublished manuscript.
- [3] M. DICKMANN, *Large infinitary languages*, North-Holland, Amsterdam, 1975.
- [4] E. ELLENTUCK, *Categoricity regained*, this JOURNAL, vol. 41 (1976), pp. 639–643.
- [5] D. HIGGS, *A category approach to Boolean-valued set theory*, unpublished typescript, University of Waterloo, 1973.
- [6] P.T. JOHSTONE, *Topos theory*, Academic Press, London 1977.
- [7] S. MACLANE, *Categories for the working mathematician*, Springer, New York, 1971.
- [8] M. NADEL, *An application of model theory to set theory*, *Israel Journal of Mathematics*, vol. 11 (1972), pp. 386–393.

LONDON SCHOOL OF ECONOMICS AND POLITICAL SCIENCE
LONDON WC2A 2AE, ENGLAND