BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sciences math., astr.
et phys. — Vol. XIX, No. 3, 1971

MATHEMATICS
(LOGIC AND FOUNDATIONS)

On the Relationship between the Boolean Prime Ideal Theorem and Two Principles in Functional Analysis

by

J. L. BELL and F. JELLETT

Presented by A. MOSTOWSKI on July 29, 1970

Summary. It has been shown in several papers that the Boolean prime ideal theorem effectively implies the Hahn—Banach theorem (i.e. the implication can be proved without using the axiom of choice). Whether the implication can be reversed is still an open question. In this paper it is shown that the Boolean prime ideal theorem can be effectively obtained from the conjunction of the Hahn—Banach theorem and a (slightly modified) version of the Krein—Milman theorem on the existence of extreme points in compact convex sets.

In [3—5] it is shown, by various means, that the Boolean prime ideal theorem effectively*) implies the Hahn—Banach theorem. Whether the implication can be reversed is still an open question. In this paper we show that the Boolean prime ideal theorem can be effectively obtained from the conjunction of the Hahn—Banach theorem and a (slightly modified) version of the Krein—Milman theorem on the existence of extreme points of compact convex sets.

1. Preliminaries

Throughout this paper we shall assume that all linear spaces and algebras have the real number field, $\mathbf{R}$, as their underlying field of scalars. A *linear functional* on a linear space is a linear mapping of the space to $\mathbf{R}$.

DEFINITION 1.1 ([5]). Let L be a (real) linear topological space. A subset A of L is said to be *quasicompact***) if whenever $\mathcal{F}$ is a family of closed convex sets such that $\{F \cap A : F \in \mathcal{F}\}$ has the finite intersection property, then $\cap \{F \cap A : F \in \mathcal{F}\} \neq \emptyset$.

The proof of the following lemma is elementary and accordingly we omit it.

LEMMA 1.2 (i) *A closed convex subset of a quasicompact set is itself quasicompact.* (ii) *A quasicompact subset of a Hausdorff locally convex space is closed.* (iii) *The image of a quasicompact set under a continuous linear mapping is quasicompact.* (iv) *A set of real numbers is quasicompact if and only if it is compact.*

*) i.e. no application of the axiom of choice is made in the proof.
**) In [5] the term *convex compact* is used for this notion.

We next state the version of the Krein—Milman theorem we shall use in the sequel.

THEOREM 1.3. *Any convex quasicompact subset of a locally convex Hausdorff (linear topological) space has extreme points.*

Using the facts enumerated in Lemma 1.2, the proof of Theorem 1.3 is a straightforward modification of the proof of the existence of extreme points in compact convex subsets of locally convex Hausdorff spaces (see, e.g., [2], p. 131).

Theorem 1.3 will be abbreviated in future to VKM.

THEOREM 1.4 ([5]). *The Hahn—Banach theorem is effectively equivalent to the following assertion (a weak form of Alaoglu's theorem): for any normed linear space L, the closed unit sphere of the continuous dual L^* of L is quasicompact in the weak* topology for L^*.*

DEFINITION 1.5 Let X be a completely regular Hausdorff topological space. $C^*(X)$ denotes the normed algebra of *bounded* real valued continuous functions on X, with the supremum norm $\|f\|=\sup\{|f(x)|: x\in X\}$. If A is a subalgebra of $C^*(X)$ which contains the identity element 1 of $C^*(X)$, $K_A(X)$ denotes the set of all linear mappings φ on A to the reals such that (i) φ is positive, i.e. $f\geqslant 0$ in A implies $\varphi(f)\geqslant 0$, (ii) $\varphi(1)=1$. If $A=C^*(X)$, then $K_A(X)$ will be denoted simply by $K(X)$.

THEOREM 1.6 ([7]). *Let A be a subalgebra of $C^*(X)$ containing 1. Then an element φ of $K_A(X)$ is an extreme point of $K_A(X)$ if and only if φ is an algebra homomorphism of A into $\mathbf{R}$.*

We point out that the proof of this theorem is *effective.*

Finally, we shall assume the Hahn—Banach theorem (which we shall refer to as HB) in the following form, which is known to be effectively equivalent to all the other forms in the literature.

THEOREM 1.7 (*Hahn—Banach*). *Let L be a partially ordered linear space and let C be the positive cone $\{x\in L: x\geqslant 0\}$. If L' is a linear subspace of L such that $C\cap L'$ is cofinal in C, then each positive linear functional on L' can be extended to a positive linear functional on L.*

2. The main result

Our intention in this section is to prove the following

THEOREM 2.1. *VKM+HB effectively imply the Boolean prime ideal theorem: every Boolean algebra contains a prime ideal.*

The proof of this result rests on two lemmas.

LEMMA 2.1. *Assuming* VKM *and* HB, *one can effectively prove the following assertion: if X is a completely regular Hausdorff space and A is a subalgebra of $C^*(X)$ containing* 1, *then each positive homomorphism φ_0 of A to $\mathbf{R}$ which sends* 1 *to* 1 *can be extended to a homomorphism on the whole of $C^*(X)$.*

Proof. Assume VKM and HB. Let S be the closed unit sphere of the continuous dual D of $C^*(X)$ and let V be the set of all positive members of D which extend φ_0. Then $V \neq \emptyset$ by HB. Also $V \subseteq K(X) \subseteq S$. The first of these inclusions is trivial. To establish the second, notice that, if $\varphi \in K(X)$ and $\|f\| \leqslant 1$ in $C^*(X)$, then $-\mathbf{1} \leqslant f \leqslant \mathbf{1}$, so that $-1 \leqslant \varphi(f) \leqslant 1$ by positivity of φ and the fact that $\varphi(\mathbf{1}) = 1$. Therefore $|\varphi(f)| \leqslant 1$ whenever $\|f\| \leqslant 1$ so that $\varphi \in S$. It is easy to see that V is a closed convex subset of S (in the weak* topology for D). By Theorem 1.4 HB implies that S is quasicompact (again in the weak* topology for D); therefore, by Lemma 1.2, V is also quasicompact. VKM now implies that V has an extreme point ψ, which is also an extreme point of $K(X)$, because if $\psi = a\psi_1 + (1-a)\psi_2$ with $\psi_1, \psi_2 \in K(X)$ and $0 < a < 1$, then $\varphi_0 = \psi \mid A = a\psi_1 \mid A + (1-a)\psi_2 \mid A$. Clearly $\psi_1 \mid A$ and $\psi_2 \mid A$ are both in $K_A(X)$; by Theorem 1.6, φ_0, being a homomorphism, is an extreme point of $K_A(X)$. Therefore $\varphi_0 = \psi_1 \mid A = \psi_2 \mid A$ and so ψ_1 and ψ_2 are both in V. But ψ was assumed to be an extreme point of V so $\psi = \psi_1 = \psi_2$ and ψ is therefore an extreme point of $K(X)$. It follows, again by Theorem 1.6, that ψ is a homomorphism of $C^*(X)$ which extends φ_0.

LEMMA 2.3. *Lemma 2.2 effectively implies that every proper ideal in $C^*(X)$ is contained in a maximal ideal.*

Proof. Assume Lemma 2.2, and let I be a proper ideal in $C^*(X)$. Then, for each $f \in I$, since f is not invertible in $C^*(X)$ we must have $\inf\{|f(x)| : x \in X\} = 0$. Let A be the subalgebra of $C^*(X)$ generated by $I \cup \{\mathbf{1}\}$; then each element g of A is uniquely expressible as a sum $g = \boldsymbol{\alpha} + f$, where $\boldsymbol{\alpha}$ is a constant function with fixed real value a and $f \in I$. Define the mapping $\varphi_0 : A \to \boldsymbol{R}$ by setting, for each $g \in A$, $\varphi_0(g) = a$ where $g = \boldsymbol{\alpha} + f$ is the decomposition given above. Clearly φ_0 is a homomorphism from A to $\boldsymbol{R}$ which carries $\mathbf{1}$ to 1. Furthermore, φ_0 is positive, for if $g \geqslant 0$ in A and $g = \boldsymbol{\alpha} + f$ with $f \in I$ and $a \in \boldsymbol{R}$ then $a + f(x) \geqslant 0$ for all $x \in X$. If $a < 0$ then, since $\inf\{|f(x)| : x \in X\} = 0$, there is $x \in X$ such that $|f(x)| < |a|$ and so $a + f(x) < 0$, a contradiction. Therefore φ_0 is positive and by Lemma 2.2 can be extended to a homomorphism φ on $C^*(X)$. The kernel of φ is the required maximal ideal containing I.

Proof of Theorem 2.1. By Lemmas 2.2 and 2.3, VKM+HB effectively imply that, for any completely regular Hausdorff space X, every proper ideal in $C^*(X)$ can be extended to a maximal ideal. It follows from this, using a standard (effective) argument (q.v. [1]) that the Stone—Čech compactification βX of X can be constructed as the space of maximal ideals of $C^*(X)$. But it is a well-known fact ([6]) that the existence of βX for all such X effectively implies the Boolean prime ideal theorem. This completes the proof.

3. Concluding remarks

We feel that the result obtained in this paper is not entirely satisfactory for two reasons: first, we do not know whether the strong form of the Krein—Milman theorem we use is effectively equivalent to the usual version; and second, attempts to prove either of the versions of the Krein—Milman theorem from the Boolean prime ideal theorem have so far been unsuccessful. It is fairly easy to show, however, that the Krein—Milman theorem can be effectively obtained from the maximal ideal theorem for distributive lattices: unfortunately it is not known whether this result is really stronger than the Boolean prime ideal theorem. Nonetheless we conjecture that the Boolean prime ideal theorem is effectively equivalent to the conjunction of the Krein—Milman theorem and the Hahn—Banach theorem.

DEPARTMENT OF MATHEMATICS, LONDON SCHOOL OF ECONOMICS AND POLITICAL SCIENCE (GREAT BRITAIN)

DEPARTMENT OF MATHEMATICS, CHELSEA COLLEGE OF SCIENCE AND TECHNOLOGY (GREAT BRITAIN)

REFERENCES

[1] L. Gillman, M. Jerison, *Rings of continuous functions*, New York, 1960.

[2] Kelley, Namioka *et al.*, *Linear topological spaces*, New York, 1963.

[3] J. Łoś, C. Ryll-Nardzewski, *On the applications of Tychonoff's theorem in mathematical proofs*, Fund. Math., **38** (1951), 233—237.

[4] W. A. J. Luxemburg, *Two applications of the method of construction by ultrapowers to analysis*, Bull. Am. Math. Soc., **68** (1962), 416—419.

[5] —, *Reduced powers of the real number system*, in: *Applications of model theory to algebra, analysis and probability*, New York, 1969.

[6] J. Rubin, D. Scott, *Some topological theorems equivalent to the Boolean prime ideal theorem*, Bull. Am. Math. Soc., **60** (1954), 389.

[7] R. R. Phelps, *Extreme positive operators and homomorphisms*, Trans. Am. Math. Soc., **108** (1963), 265—274.

И. Л. Бэлль, Ф. Джэллет, **Заметка о зависимости между булевой теоремой простого идеала и двумя принципами в функциональном анализе.**

Содержание. В нескольких заметках было уже показано, что булевая теорема простого идеала эффективно влечет за собой теорему Хана—Банаха (т. е. можно доказать импликацию не прибегая к аксиоме выбора). Однако проблема: можно-ли обратить это утверждение? — все еще остается открытой. В настоящей заметке доказывается, что булевую теорему простого идеала можно эффективно получить из конъюнкции теоремы Хана—Банаха с некоторым (несколько модифицированным) вариантом теоремы Крайна—Мильмана о существовании экстремальных точек в компактных, выпуклых множествах.