THE MAXIMAL IDEAL THEOREM FOR LATTICES OF SETS

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Klimovsky **[1]** has shown that the maximal ideal theorem for distributive lattices with unit implies the axiom of choice. Our aim in this note is to give a simple direct proof that the maximal ideal theorem for *lattices of sets* implies the axiom of choice.

Definitions. Let $\langle L, \wedge, \vee, \leqslant \rangle$ be a lattice. An *ideal* in L is a subset I of L such that (i) $a, b \in I \Rightarrow a \vee b \in I$. (ii) $a \in I$ and $b \leqslant a \Rightarrow b \in I$. A *filter* in L is a subset F of L such that (i) $a, b \in F \Rightarrow a \wedge b \in F$; (ii) $a \in F$ and $a \leqslant b \Rightarrow b \in F$. A lattice of the form $\langle L, \cap, \cup, \subseteq \rangle$ where L is a family of sets and $\cap, \cup, \subseteq$ are set-theoretic intersection, union, and inclusion, respectively, is called a *lattice of sets*. Clearly every lattice of sets is distributive.

It is easily shown that the axiom of choice implies that any lattice of sets with a greatest element 1 (in fact *any* lattice with a greatest element) has maximal proper ideals, i.e. ideals maximal with respect to the property of not containing 1. We now establish the converse.

THEOREM. *If every lattice of sets with a greatest element has a maximal proper ideal, then the axiom of choice must hold.*

Proof. We observe first that if every lattice of sets with a greatest element has a maximal proper ideal, then by duality every lattice of sets with a least element has a maximal proper filter (and conversely). It is this second form which we shall use below.

Let $\{A_i : i \in I\}$ be any indexed family of non-empty sets. Let us assume that at least one A_i has more than one element; otherwise the problem is trivial. Let X be the set of partial choice functions f such that the domain $D(f)$ of f is a subset of I and $f(i) \in A_i$ for each $i \in D(f)$. We regard a function as a set of ordered pairs, so that $f \subseteq g$ means that $D(f) \subseteq D(g)$ and $f(i) = g(i)$ for all $i \in D(f)$. For each $f \in X$, let

$$S(f) = \{g : g \in X \text{ and } f \subseteq g\},$$

and let L be the sublattice of the power set of X generated by $\{S(f) : f \in X\}$.

For any $f, g \in X$, $S(f) \cap S(g)$ is either $S(f \cup g)$ or $\varnothing$, depending on whether f and g agree on $D(f) \cap D(g)$ or not. Consequently each element of L is either $\varnothing$ or can be expressed in the form $S(f_1) \cup \ldots \cup S(f_n)$ for some $f_1, \ldots, f_n \in X$. Also, $\varnothing$ will certainly belong to L, because we are supposing that there is an A_i with more than one element. Thus L is a lattice of sets with a least element.

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By hypothesis, then, there is a maximal proper filter F in L. Since L is a distributive lattice, F is prime, that is, if $a \cup b \in F$ then at least one of a, b belongs to F.

Let $B = \{f : f \in X \text{ and } S(f) \in F\}$. If $f, g \in B$, then $S(f) \cap S(g) \neq \emptyset$, so $f \cup g$ is a function; consequently, $h = \cup B$ is a function, and $h \in X$. Now $S(h) \subseteq S(f)$ for all $f \in B$. But if a is any member of F, then a can be expressed as $S(f_1) \cup \ldots \cup S(f_n)$ for some $f_1, \ldots, f_n \in X$; since F is prime at least one f_i belongs to B, and $S(h) \subseteq S(f_i) \subseteq a$. Thus $S(h)$, which contains h and is therefore non-empty, is contained in every member of F; it follows at once that F must be the filter in L generated by $S(h)$, so that $S(h)$ is a minimal member of L. Thus h has no proper extensions in X, and the domain of h must be I itself. Therefore h is the required choice function for $\{A_i : i \in I\}$, completing the proof.

Remarks. Recall that an ideal I of a lattice L is called *prime* if $x \wedge y \in I \Rightarrow x \in I$ or $y \in I$. It is a well-known fact that a maximal ideal of a distributive lattice must be prime; the converse is easily shown to be false in general. Now it is easy to manufacture prime ideals in any lattice of sets L without appealing to any form of the axiom of choice; simply pick $a \in \cup L$ and let $I = \{X \in L : a \notin X\}$, I is easily seen to be a prime ideal in L. Thus the existence of *prime* ideals in lattices of sets is a triviality; but, as we have shown, the existence of *maximal* ideals in such lattices is equivalent to the axiom of choice, which is known to be independent of the remaining axioms for set theory. This fact indicates that there is a great difference between maximal and prime ideals, even in such relatively simple algebraic structures as lattices of sets.

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Reference

1. G. Klimovsky, "Zorn's theorem and the existence of maximal filters and ideals in distributive lattices", *Rev. Un.-Mat. Argentina*, 18 (1958), 160–164.

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