Foundations of Physics, Vol. 15, No. 12, 1985

Orthospaces and Quantum Logic

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Received July 6, 1984

In this paper we construct the ortholattices arising in quantum logic starting from the phenomenologically plausible idea of a collection of ensembles subject to passing or failing various "tests." A collection of ensembles forms a certain kind of preordered set with extra structure called an orthospace; *we show that complete ortholattices arise as canonical completions of orthospaces in much the same way as arbitrary complete lattices arise as canonical completions of partially ordered sets. We also show that the canonical completion of an orthospace of ensembles is naturally identifiable as the complete lattice of properties of the ensembles, thereby revealing exactly* why *ortholattices arise in the analysis of "tests" or experimental propositions. Finally, we axiomatize the hitherto implicit concept of "test" and show how they may be correlated with properties of ensembles.*

1. INTRODUCTION

The idea of a "logic of quantum mechanics" or *quantum logic* was, as is well known, originally suggested by Birkhoff and von Neumann in their pioneering paper.[5] Pursuing an analogy with classical logic, they proposed identifying the logic of "experimental propositions" pertaining to a quantum-mechanical system $\mathscr{S}$ with the complete nondistributive lattice (a complete *ortholattice*: for a definition see Section 2 below) of subspaces of the infinite-dimensional Hilbert space associated with $\mathscr{S}$. The intriguing but somewhat *ad hoc* character of this proposal (should something as fundamental and general as a logical system be tied to something as specific as a Hilbert space?) led many later investigators to attempt to derive quantum logic from simpler foundations (for a survey, see Ref. 10). In particular, it has been suggested by Putnam, Finkelstein, and others (see

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0015-9018/85/1200-1179$04.50/0 © 1985 Plenum Publishing Corporation

especially Refs. 7 and 12) that quantum logic should be obtained as a general logic of experimental propositions or "tests." But this suggestion still appears somewhat *ad hoc*, since one must apparently assume that the "tests" or propositions already form an ortholattice (for a critique, see Ref. 3), an assumption not really justified phenomenologically.

In this paper, we attempt to eradicate this deficiency by constructing the ortholattices associated with quantum logic in a *canonical* manner starting from the simple and phenomenologically plausible idea of a collection of *ensembles* (of objects, "particles," or the like) subject to passing or failing various "*tests*." More precisely, we assume given an abstract set P (the set of "ensembles") together with relations $\leqslant$, $\perp$ on P: $p \leqslant q$ is construed to mean that the ensemble p passes every "test" that the ensemble q passes, i.e., that p is *included* in q or is a *subensemble* of q; while $p \perp q$ is contrued to mean that p and q are *mutually exclusive*, i.e., either there is a "test" which p passes but q and all its (nonempty) subensembles fail or *vice versa*. We also assume the presence of an *empty* ensemble 0 which is deemed to pass every "test," and is accordingly a subensemble of *every* ensemble. It is thus natural to require that $\leqslant$ be a *preordering* on P, i.e., a reflexive transitive relation on P, such that $0 \leqslant p$ for all $p \in P$, and that $\perp$ be an *orthogonality* relation on P, i.e., irreflexive and symmetric on $P - \{0\}$ and such that[2] $0 \perp p$ for all $p \in P$. Moreover, it is clear that $\perp$ and $\leqslant$ should be related by the condition

$$p \leqslant q \;\&\; q \perp r \rightarrow p \perp r$$

In our analysis of the concept of a "testable" ensemble, we are thus naturally led to consider structures $\mathbf{P} = (P, \leqslant, \perp, 0)$ satisfying the above conditions: we call such structures *preordered orthogonality spaces* or simply *orthospaces*. In our mathematical investigation of these structures we show that the complete ortholattices of quantum logic may be obtained as canonical completions of orthospaces in much the same way as arbitrary complete lattices are obtainable as canonical completions of partially ordered sets (see, e.g., Ref. 1). We also show that the canonical completion of an orthospace $\mathbf{P}$ is naturally identifiable as the complete lattice of properties of the ensembles in $\mathbf{P}$, thereby revealing exactly *why* ortholattices arise in the analysis of "tests" or experimental propositions. Finally, we axiomatize the hitherto implicit concept of "test" and show how they may be correlated with properties of ensembles.

We begin with an analysis of the concept of *orthospace*.

[2] For this to be the case, we must, strictly speaking, assume that there is a test passed *only* by 0.

2. THE THEORY OF ORTHOSPACES

A *preordering* on a set P is a reflexive transitive relation $\leqslant$ on P; if in addition $\leqslant$ is *antisymmetric*, i.e., if $p \leqslant q$ & $q \leqslant p$ always imply $p = q$ for any $p, q \in P$, then $\leqslant$ is a *partial ordering* on P. A *least element* of a preordered set $(P, \leqslant)$ is an element $p \in P$ such that $p \leqslant q$ for all $q \in P$. A preordered set may in general have more than one least element, but we shall always assume that any preordered set we consider has a *unique* least element, which we shall denote by 0.

By an *orthogonality space* we mean a triple $(P, \perp, e)$ where P is a set, $e \in P$, and $\perp$ is an *orthogonality relation* on P, i.e., a binary relation on P which is symmetric and irreflexive on $P - \{e\}$ and in addition satisfies $e \perp p$ for all $p \in P$. A preordered orthogonality space or simply an *orthospace* is a quadruple $\mathbf{P} = (P, \leqslant, \perp, 0)$ where $(P, \leqslant)$ is a preordered set with least element 0 and $(P, \perp, 0)$ is an orthogonality space such that, for all $p, q, r \in P$

$$p \leqslant q \;\&\; q \perp r \rightarrow p \perp r$$

(It is easily shown that under these conditions 0 is actually the *unique* least element of the preordered set $(P, \leqslant)$.) The set-theoretic complement of the relation $\perp$ in $P \times P$ is then a reflexive symmetric relation on $P - \{0\}$ called the *proximity* relation associated with $\perp$; it will always be denoted by $\approx$: thus we have

$$p \approx q \leftrightarrow p \not\perp q$$

We also define

$$Q_p = \{q \in P: p \approx q\}$$

Examples.

1. If $(P, \perp, e)$ is any orthogonality space, define $\leqslant$ on P by $p \leqslant q \leftrightarrow \forall r[r \perp q \rightarrow r \perp p]$. Then $(P, \leqslant, \perp, e)$ is an orthospace, the orthospace *canonically associated* with $(P, \perp, e)$. In particular if H is a Hilbert space (or any inner product space) then $(H, \leqslant, \perp, 0)$ is an orthospace, where $\perp$ is the relation of perpendicularity of vectors.

2. If $\mathsf{P} = (P, \leqslant)$ is any preordered set with least element 0, call two elements p, q (order) *incompatible* and write pIq if $\forall r[r \leqslant p \;\&\; r \leqslant q \rightarrow r = 0]$ (and (order) *compatible* if $\exists r \neq 0$ $(r \leqslant p \;\&\; r \leqslant q)$). Then $\mathbf{P} = (P, \leqslant, I, 0)$ is an orthospace, the orthospace *canonically associated* with P.

3. An *ortholattice* is a lattice $\mathscr{L} = (L, \leqslant, \wedge, \vee)$ with top and bottom elements 1, 0 equipped also with an operation *: $L \rightarrow L$ satisfying

$$x \vee x^* = 1, \qquad x \leqslant y \rightarrow x^* \geqslant y^*$$

$$x^{**} = x$$

for all $x, y \in L$. It is easily shown that in any ortholattice $x \wedge x^* = 0$, so that x^* is a *complement* for x in the lattice theoretic sense: x^* is called the *orthocomplement* of x. (Equivalently, in the definition of an ortholattice the condition $x \vee x^* = 1$ may be replaced by $x \wedge x^* = 0$.) Note that in any ortholattice de Morgan's laws hold:

$$(x \wedge y)^* = x^* \vee y^*; \qquad (x \vee y)^* = x^* \wedge y^*$$

Each ortholattice $\mathscr{L}$ may be regarded as an orthospace $\mathbf{L} = (L, \leqslant, \perp, 0)$ in which $\leqslant$ is the given lattice ordering and $\perp$ is the relation defined by

$$x \perp y \leftrightarrow x \leqslant y^*$$

$\mathbf{L}$ is called the orthospace *induced* by $\mathscr{L}$: where there is no risk of confusion, we identify $\mathbf{L}$ and $\mathscr{L}$.

An ortholattice $\mathscr{L}$ is *complete* if it is complete as a lattice, i.e., if every subset X has a supremum (join) $\bigvee X$ and an infimum (meet) $\bigwedge X$. (More generally, we use $\bigvee X$ and $\bigwedge X$ to denote the supremum and infimum, if they exist, of a subset X of an arbitrary partially ordered set.) If $\mathscr{L}$ is complete, we observe that the orthocomplement x^* of $x \in L$ is given by

$$x^* = \bigvee \{y: x \perp y\}$$

A *Boolean algebra* is an ortholattice satisfying the distributive laws

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

It is well known (see, e.g., Ref. 4) that an ortholattice $\mathscr{L}$ is a Boolean algebra iff

$$x \leqslant y^* \leftrightarrow x \wedge y = 0$$

for all $x, y \in L$, i.e., iff the orthogonality relation $\perp$ in $\mathbf{L}$ coincides with the incompatibility relation I.

4. Let S be a multiplicative semigroup with an element 0 such that $s0 = 0s = 0$ for all $s \in S$ (e.g., the multiplicative semigroup of a ring). Let E be the set of idempotent elements of S, i.e., the set of elements $e \in S$ such that $e^2 = e$. Define the relations $\leqslant$, $\perp$ on E by $e \leqslant f \leftrightarrow ef = fe = e$ and $e \perp f \leftrightarrow ef = fe = 0$. Then $(E, \leqslant, \perp, 0)$ is an orthospace.

A *morphism* of an orthospace $\mathbf{P}$ into an orthospace $\mathbf{Q}$ is a map $f: P \to Q$ such that $f(0) = 0$ and

$$p \leqslant q \to f(p) \leqslant f(q)$$

$$p \perp q \to f(p) \perp f(q)$$

If a morphism f satisfies $p \perp q \leftrightarrow f(p) \perp f(q)$ (or equivalently, $p \leftrightarrow q \approx f(p) \approx f(q)$, it is called a *weak embedding*; if in addition $p \leqslant q \leftrightarrow f(p) \leqslant f(q)$ it is a *quasiembedding*. An injective quasiembedding is called an *embedding*. A surjective embedding is, of course, called an *isomorphism*. Thus orthospaces and morphisms between them form a category, the *category of orthospaces*.

We shall be chiefly concerned with the problem of embedding orthospaces in complete ortholattices. We first note the following.

2.1. Proposition. Each orthospace is quasiembeddable in a complete Boolean algebra.

Proof. Let **P** be an orthospace and let I be the family of all subsets $J \subseteq P$ such that (i) $p \in J$, $p \leqslant q \to q \in J$ and (ii) $p, q \in J \to p \approx q$. Let $\mathscr{B}$ be the complete Boolean algebra of all subsets of I and define $f: P \to B$ by

$$f(p) = \{J \in I: p \in J\}$$

We claim that f is a quasiembedding. Clearly $f(0) = \varnothing$ and $p \leqslant q \to f(p) \subseteq f(q)$. Writing $J_p = \{q: p \leqslant q\}$ we have $J_p \in I$ and

$$\begin{aligned} p \nleqslant q &\to J_p \in f(p) \;\&\; J_p \notin f(q) \\ &\to f(p) \nsubseteq f(q) \end{aligned}$$

If $p \approx q$, then clearly $J = J_p \cup J_q \in I$ and $J \in f(p) \cap f(q)$, whence $f(p) \approx f(q)$ in **B**. Finally, if $f(p) \approx f(q)$ in **B**, then there is $K \in f(p) \cap f(q)$, whence $\{p, q\} \subseteq K$ and $p \approx q$. This completes the proof. ▮

We now introduce the important notion of *density*. Call a subset X of a preordered set P (join) *dense* if for all $p, q \in P$ we have

$$p \leqslant q \leftrightarrow \forall x \in X[x \leqslant p \to x \leqslant q]$$

If P is partially ordered, this is equivalent to the condition that each element of P bé the supremum of the set of its predecessors in X, i.e.,

$$p = \bigvee \{x \in X: x \leqslant p\}$$

A preorder-preserving map f: P → Q between preordered sets P, Q is called *dense* if the image $f[\mathrm{P}]$ of P is dense in Q.

We shall now turn to the problem of *densely embedding* an orthospace into a complete ortholattice. (Note that, in general, the quasiembedding constructed in 2.1 is not dense.) Given any orthospace **P**, we construct a complete ortholattice $\mathscr{L}(\mathbf{P})$ as follows. For each $p \in P$, recall that $Q_p =$

$\{q \in P: q \approx p\}$; let $L(P)$ be the family of all *unions* of sets of the form Q_p. Then we have

2.2. Proposition. $L(P)$ is a complete ortholattice $\mathscr{L}(\mathbf{P})$ under set inclusion, with set-theoretic union as the supremum operation, and in which the orthocomplement U^* of an element $U \in L(P)$ is given by

$$U^* = \bigcup_{p \notin U} Q_p = \{q: (\exists p \notin U) p \approx q\}$$

Furthermore, the map $i: p \mapsto Q_p$ is a dense weak embedding of $\mathbf{P}$ into $\mathbf{L}(\mathbf{P})$, the orthospace induced by $\mathscr{L}(\mathbf{P})$.

Proof. We first observe that since $L(P)$ is closed under unions, it is automatically a complete lattice under $\subseteq$ with union as supremum and $\varnothing$, P as bottom and top elements. Clearly, also, for any U, $V \in L(P)$ we have $U \cup U^* = P$ and $U \subseteq V \to U^* \supseteq V^*$. And

$$U^{**} = \bigcup_{p \notin U^*} Q_p = \bigcup \{Q_p: (\forall q \approx p) q \in U\}$$

$$= \bigcup \{Q_p: Q_p \subseteq U\} = U$$

Thus $\mathscr{L}(\mathbf{P})$ is an ortholattice.

Finally, we show that i is a dense weak embedding of $\mathbf{P}$ into $\mathbf{L}(\mathbf{P})$. The density of i is obvious; clearly $i(0) = \varnothing$ and $p \leqslant q \to i(p) \subseteq i(q)$. It remains to show that $p \approx q \leftrightarrow i(p) \approx i(q)$ or equivalently that $p \approx q \leftrightarrow Q_p \nsubseteq Q_p^*$. But this follows from the implications

$$p \perp q \to p \notin Q_q \to Q_p \subseteq \bigcup_{r \notin Q_q} Q_r = Q_q^*$$

and

$$p \approx q \to q \in Q_p \to Q_p \nsubseteq Q_q^*$$

since evidently $q \notin Q_q^*$. ∎

The orthospace $\mathbf{L}(\mathbf{P})$ induced by the complete ortholattice $\mathscr{L}(\mathbf{P})$ (more precisely the pair $(\mathbf{L}(\mathbf{P}), i)$) is called the *canonical orthocompletion* of $\mathbf{P}$. We shall see later on that it is uniquely determined up to isomorphism.

When is the map $i: \mathbf{P} \to \mathbf{L}(\mathbf{P})$ an *embedding*? To answer this question we introduce the concept of a *normal* orthospace. An orthospace $\mathbf{P}$ is said to be *normal* if $\leqslant$ is a *partial ordering* of P and for all $p, q \in P$,

$$p \leqslant q \leftrightarrow \forall r[r \perp q \to r \perp p]$$

or equivalently if

$$p \leqslant q \leftrightarrow Q_p \subseteq Q_q$$

(Observe that the implication from left to right holds for any orthospace.) Clearly every ortholattice is a normal orthospace. If P is a partially ordered set, the orthospace $\mathbf{P} = (P, \leqslant, I, 0)$ canonically associated with P is easily seen to be normal iff it satisfies the condition

$$p \nleqslant q \rightarrow (\exists r \leqslant p) r I q$$

In the case of a partially ordered P, this is precisely the condition that P be *refined* (in the terminology of Ref. 2) or *separative* (in the terminology of Ref. 11).

From 2.2 we deduce the following.

2.3. Corollary. The following conditions on an orthospace **P** are equivalent:

(i) **P** is normal;

(ii) the map $i: \mathbf{P} \rightarrow \mathbf{L}(\mathbf{P})$ is an embedding;

(iii) **P** is densely embeddable in a complete ortholattice.

Proof. (i) → (ii) is obvious from the definition of normality, and (ii) → (iii) is an immediate consequence of 2.2.

(iii) → (i). If (iii) holds, then the preordered set underlying **P** may be regarded as a dense subset of a complete ortholattice $\mathscr{L}$: in particular, $\leqslant$ partially orders P. If $p \nleqslant q$ in **P**, then $q^* \nleqslant p^*$ in $\mathscr{L}$, so by the density of **P** there is $r \in P$ such that $r \leqslant q^*$ and $r \nleqslant p^*$, i.e., $r \perp q$ and $r \not\perp p$ in $\mathscr{L}$, hence also in **P**. (i) follows. ∎

If **P** is a normal orthospace, the density of the canonical embedding i: $\mathbf{P} \rightarrow \mathbf{L}(\mathbf{P})$ implies that it automatically preserves any meets that already exist in **P**. That is, $i(\bigwedge X) = \bigwedge i[X]$ for any subset $X \subseteq P$ such that $\bigwedge X$ exists in **P**. Under what conditions does i also preserve joins? This question is readily answered. Call a morphism $f: \mathbf{P} \rightarrow \mathbf{Q}$ (join) *complete* if whenever $X \subseteq P$ and $\bigvee X$ exists in P, then $f(\bigvee X) = \bigvee f[X]$ in **Q**. **P** is said to be $\perp$-*continuous* if whenever $X \subseteq P$ and $\bigvee X$ exists in **P**, then for any $p \in P$

$$(\forall x \in X) p \perp x \rightarrow p \perp \bigvee X$$

Note that (the orthospace induced by) any ortholattice is $\perp$-continuous.

2.4. Proposition. The following conditions on a normal orthospace **P** are equivalent:

(i) **P** is $\perp$-continuous;

(ii) the canonical embedding $i: \mathbf{P} \rightarrow \mathbf{L}(\mathbf{P})$ is complete.

Proof. (i) → (ii). Assume (i), let $X \subseteq P$, and suppose that $\bigvee X$ exists in **P**. Since i preserves order, $\bigvee i[X] \leqslant i(\bigvee X)$. To prove the reverse inequality, put $\bigvee i[X] = a$, $i(\bigvee X) = b$. Then for any $p \in P$,

$$
\begin{aligned}
i(p) \leqslant b &\rightarrow p \leqslant \bigvee X \\
&\rightarrow \forall q \left[q \perp \bigvee X \rightarrow q \perp p \right] \\
&\rightarrow \forall q [(\forall x \in X) q \perp x \rightarrow q \perp p] \qquad \text{(by } \perp\text{-continuity of } P) \\
&\rightarrow \forall q [(\forall x \in X) i(q) \perp i(x) \rightarrow i(q) \perp i(p)] \\
&\rightarrow \forall q [i(q) \perp a \rightarrow i(q) \perp i(p)] \\
&\rightarrow \forall q [i(q) \leqslant a^* \rightarrow i(q) \leqslant i(p)^*] \\
&\rightarrow a^* \leqslant i(p)^* \qquad \text{(by density of } i) \\
&\rightarrow i(p) \leqslant a
\end{aligned}
$$

It now follows from the density of i that $b \leqslant a$ and hence that $a = b$. This gives (ii).

(ii) → (i). Assume (ii), let $X \subseteq P$, and suppose that $\bigvee X$ exists in **P**. Then if $p \in P$,

$$
\begin{aligned}
(\forall x \in X) p \perp x &\rightarrow (\forall x \in X) i(p) \perp i(x) \\
&\rightarrow i(p) \perp \bigvee i[X] = i\left(\bigvee X\right) \\
&\rightarrow p \perp \bigvee X \quad \blacksquare
\end{aligned}
$$

An orthospace **P** is said to be *complete* if the preordered set $(P, \leqslant)$ is a complete lattice (so in particular $\leqslant$ must be a partial ordering). Then a complete orthospace is (the orthospace induced by) a complete ortholattice if (and only if) it is $\perp$-continuous and normal. More precisely, we have the following.

2.5. Lemma. Let **E** be a complete normal $\perp$-continuous orthospace. Define the map $x \mapsto x^*$ on **E** by

$$x^* = \bigvee \{ y \in E : x \perp y \}$$

Then $\mathscr{E} = (E, \leqslant, {}^*)$ is a complete ortholattice and

$$x \perp y \leftrightarrow x \leqslant y^*$$

so that **E** is the orthospace induced by $\mathscr{E}$.

Proof. Let $O_x = \{y \in E: x \perp y\}$; then $x^* = \bigvee O_x$. Clearly $x \leqslant y \rightarrow x^* \geqslant y^*$. Since $x \perp y$ for all $y \in O_x$, the $\perp$-continuity of E implies that $x \perp x^*$. So if $y \leqslant x$ and $y \leqslant x^*$ then $y = 0$, whence $x \wedge x^* = 0$. Now x^* is the largest element of E orthogonal to x, i.e., $y \leqslant x^* \leftrightarrow x \perp y$. Hence

$$
\begin{aligned}
y \leqslant x^{**} &\leftrightarrow y \perp x^* \\
&\leftrightarrow \forall z[z \perp x \rightarrow z \perp y] \qquad \text{(by } \perp\text{-continuity)} \\
&\leftrightarrow y \leqslant x
\end{aligned}
$$

by the normality of E. Thus $*$ is an orthocomplementation on E. ∎

Remark. If P is a (partially ordered set which is a) complete lattice, its canonically associated orthospace **P** is $\perp$-continuous iff P is a complete *pseudocomplemented* lattice, i.e., if for each $p \in P$ there is a *largest* element $q \in P$ such that $p \wedge q = 0$. In particular, this will be the case if P is a *complete Heyting algebra*, i.e., a complete lattice satisfying the distributive law

$$p \wedge \bigvee_{i \in I} q_i = \bigvee_{i \in I} p \wedge q_i$$

(Note, however, that there are complete pseudocomplemented lattices which are not complete Heyting algebras, e.g., the 5-element pentagon lattice.) It follows that P is a complete Boolean algebra iff **P** is a *normal* complete $\perp$-continuous orthospace.

Next we characterize the canonical orthocompletion of a normal orthospace as a *minimal* completion in a way analogous to the characterization of the Dedekind–MacNeille completion of a partially ordered set (cf. Ref. 1).

A *completion* of an orthospace **P** is a pair (**E**, j) in which **E** is a complete $\perp$-continuous orthospace and j is an embedding of P into E. A completion (**E**, j) of **P** is *minimal* if for any completion (**F**, k) of **P** there is an embedding f: **E** → **F** such that $k = f \circ j$. Two completions (**E**, j) and (**F**, k) are *isomorphic over* **P** if there is an isomorphism f: **E** → **F** such that $k = f \circ j$. Finally, a *dense orthocompletion* of **P** is a pair (**E**, j) in which **E** is (the orthospace induced by) a complete ortholattice and j is a dense *weak* embedding of **P** into **E**: it is easily shown that if **P** is normal and (**E**, j) is any dense orthocompletion of **P**, then j is actually an *embedding*.

2.6. Proposition. Let (**E**, j) be a completion of a *normal* orthospace **P**. The following are equivalent:

(i) (**E**, j) is a dense orthocompletion of **P**;

(ii) (**E**, j) is a minimal completion of **P**.

Proof. (i) → (ii). Suppose that (E, j) is a dense orthocompletion of P and that (F, k) is any completion of P. Define $f: E \to F$ by

$$f(x) = \bigvee \{k(p): j(p) \leqslant x\}$$

for $x \in E$. Since j is an embedding, it follows that $f(0) = 0$. Also, for $x, y \in E$, it folows from the density of j that

$$\begin{aligned} x \perp y &\leftrightarrow \bigvee \{j(p): j(p) \leqslant x\} \perp \bigvee \{j(q): j(q) \leqslant y\} \\ &\leftrightarrow \forall p \in P[j(p) \leqslant x \,\&\, j(q) \leqslant y \to j(p) \perp j(q)] \\ &\leftrightarrow \forall p \in P[j(p) \leqslant x \,\&\, j(q) \leqslant y \to p \perp q] \\ &\leftrightarrow \forall p \in P[j(p) \leqslant x \,\&\, j(q) \leqslant y \to k(p) \perp k(q)] \\ &\leftrightarrow \bigvee \{k(p): j(p) \leqslant x\} \perp \bigvee \{k(q): j(q) \leqslant y\} \\ &\leftrightarrow f(x) \perp f(y) \end{aligned}$$

Moreover, $x \leqslant y \to f(x) \leqslant f(y)$ is obvious, and

$$\begin{aligned} x \not\leqslant y &\to \exists z \in E[z \perp y \,\&\, z \not\perp x] \\ &\to \exists z \in E[f(z) \perp f(y) \,\&\, f(z) \not\perp f(x)] \\ &\to f(x) \not\leqslant f(y) \end{aligned}$$

Thus f is an embedding of E into F.

Finally,

$$\begin{aligned} f(j(p)) &= \bigvee \{k(q): j(q) \leqslant j(p)\} \\ &= \bigvee \{k(q): q \leqslant p\} \\ &= k(p) \end{aligned}$$

whence $f \circ j = k$.

(ii) → (i). Let (E, j) be a minimal completion of P and let (F, k) be any dense orthocompletion of P (which exists by 2.3). Then, by the proof of (i) → (ii), (F, k) is minimal, so there is an embedding g: F → E such that $g \circ k = j$. Similarly, there is an embedding f: E → F such that $f \circ j = k$. Now, putting $f \circ g = h$, we have $h \circ k = k$. Then h is the identity on F, because for $x \in F$ and $p \in P$, since h is clearly an embedding, we have

$$k(p) \leqslant x \leftrightarrow k(p) = h(k(p)) \leqslant h(x)$$

and hence, since k is dense, $x = h(x)$.

It follows that $f: E \to F$ is surjective, and hence an order isomorphism. Thus f, and hence also its inverse f', are lattice isomorphisms. Since f' is also a $\perp$-isomorphism it follows that E is normal (since F is) and hence a complete ortholattice (isomorphic to E) by 2.5. Finally, since $j = f' \circ k$, and k is dense, so is j. ▮

In the proof of (ii) → (i) above, we showed that any minimal completion of P is isomorphic over P to any dense orthocompletion of P, in particular, to the canonical orthocompletion of P. Since any dense orthocompletion is minimal, we have

2.7. Corollary. All dense orthocompletions of a normal orthospace P and all minimal completions of P are isomorphic over P to its canonical orthocompletion. ▮

This justifies calling L(P) (or $\mathscr{L}$(P)), for a normal P, the *canonical* orthocompletion of P.

Next, we show that, like complete Boolean algebras (see Ref. 9) or complete lattices (see Ref. 1), complete ortholattices may be characterized in terms of the notions of *injectivity* and *retractiveness.*

Let E be a normal orthospace. E is said to be *injective* if for any morphism f: P → E of a normal orthospace P into E and any embedding g: P → Q of P into a normal orthospace Q, there is a morphism h: Q → E such that $f = h \circ g$. E is an *absolute subretract* if for any embedding f: E → P into a normal P there is a morphism g: P → E such that $g \circ f$ is the identity on E.

2.8. Proposition. The following conditions on a normal orthospace E are equivalent:

(i) E is (the orthospace induced by a) complete ortholattice;

(ii) E is injective;

(iii) E is an absolute subretract.

Proof. (i) → (ii). Suppose that E is a complete ortholattice, P and Q normal orthospaces, f: P → E a morphism, and g: P → Q an embedding. Define h: Q → E by

$$h(x) = \bigvee \{f(p): p \in P \;\&\; g(p) \leqslant x\}$$

Clearly $h(0) = 0$ and h is order-preserving. Suppose, moreover, that $x \perp y$ and $p, q \in P$ satisfy $g(p) \leqslant x$, $g(q) \leqslant y$. Then $g(p) \perp g(q)$, whence $p \perp q$, so that $f(p) \perp f(q)$. It follows that

$$h(x) = \bigvee \{f(p): g(p) \leqslant x\} \perp \bigvee \{f(q): g(q) \leqslant y\} = h(y)$$

Thus h is a morphism. Finally,

$$\begin{aligned} h(g(p)) &= \bigvee \{f(q): g(q) \leqslant g(p)\} \\ &= \bigvee \{f(q): q \leqslant p\} \\ &= f(p) \end{aligned}$$

whence $h \circ g = f$.

(ii) → (iii) is a standard argument; we omit it (see, e.g., Ref. 9, Section 32).

(iii) → (i). Suppose that $\mathbf{E}$ is an absolute subretract; let $(\mathbf{F}, j)$ be the canonical orthocompletion of $\mathbf{E}$. Then there is a morphism $g: \mathbf{F} \to \mathbf{E}$ such that $g \circ j$ is the identity on $\mathbf{E}$. We claim that $\mathbf{E}$ is a complete orthospace and hence (by 2.4) a complete ortholattice.

First, $\mathbf{E}$ is a complete lattice. For let $X \subseteq E$, let $a = \bigvee_{x \in X} j(x)$ in F, and put $g(a) = b$. We claim that b is the supremum of X in E. For certainly b is an upper bound for X, since

$$x \in X \to j(x) \leqslant a \to x = g(j(x)) \leqslant g(a) = b$$

And b is the least upper bound for X since for any $y \in E$ we have

$$\begin{aligned} (\forall x \in X) x \leqslant y &\to (\forall x \in X) j(x) \leqslant j(y) \\ &\to a \leqslant j(y) \\ &\to b = g(a) \leqslant g(j(y)) = y \end{aligned}$$

It remains to show that $\mathbf{E}$ is $\perp$-continuous. So let $e \in E$, $X \subseteq E$ and suppose that $e \perp x$ for all $x \in X$. Then $j(e) \perp j(x)$ whence $j(e) \perp \bigvee_{x \in X} j(x)$, so that

$$e = g(j(e)) \perp g\left(\bigvee_{x \in X} j(x)\right)$$

But we have shown above that $g(\bigvee_{x \in X} j(x))$ is $\bigvee X$ in $\mathbf{E}$. Hence $e \perp \bigvee X$, and the proof is complete. ∎

It follows that the canonical orthocompletion of a normal orthospace $\mathbf{P}$ may be described as the *minimal injective orthospace into which* $\mathbf{P}$ *is embeddable*.

So far we have confined attention chiefly to *normal* orthospaces. We turn now to a discussion of the general case. First, we observe that any (not necessarily normal) orthospace $\mathbf{P}$ can be canonically associated with a normal orthospace $\bar{\mathbf{P}}$ as follows. Define an equivalence relation $\equiv$ on P by

$$p \equiv q \leftrightarrow Q_p = Q_q$$

and for each $p \in P$ let $\bar{p}$ be the equivalence class of p under $\equiv$. Put $\bar{P} = \{\bar{p}: p \in P\}$; define relations $\perp$, $\leqslant$ on $\bar{P}$ by

$$\bar{p} \perp \bar{q} \leftrightarrow p \perp q$$
$$\bar{p} \leqslant \bar{q} \leftrightarrow Q_p \subseteq Q_q$$

It is now readily verified that $\bar{\mathbf{P}} = (\bar{P}, \leqslant, \perp, 0)$ (here 0 is the 0 of $\mathbf{P}$) is a normal orthospace and that the map $j: P \to \bar{P}$ defined by $j(p) = \bar{p}$ is a weak embedding of $\mathbf{P}$ onto $\bar{\mathbf{P}}$. $(\bar{\mathbf{P}}, j)$ (or just $\bar{\mathbf{P}}$) is called the *normalization* of $\mathbf{P}$, and j the *canonical map*.

Remark. When $\mathbf{P}$ is (the orthospace canonically associated with) a complete Heyting algebra (e.g., a topology on a space), the normalization of $\mathbf{P}$ is the complete Boolean algebra of *regular elements* of $\mathbf{P}$, i.e., the $p \in P$ such that $p = p^{**}$.

The normalization of an orthospace may be characterized as follows. Call a morphism $f: \mathbf{P} \to \mathbf{Q}$ between orthospaces *normal* if for all $p, q \in P$,

$$Q_p \subseteq Q_q \to f(p) \leqslant f(q)$$

and

$$Q_p = Q_q \to f(p) = f(q)$$

Observe that any morphism with normal domain is normal, as is the canonical map of an orthospace onto its normalization.

2.9. Lemma. Up to isomorphism over $\mathbf{P}$, the normalization $(\bar{P}, j)$ is the unique normal orthospace such that for any normal morphism $f: \mathbf{P} \to \mathbf{Q}$ there is a unique (normal) morphism $g: \bar{\mathbf{P}} \to \mathbf{Q}$ such that $f = g \circ j$.

Proof. We need only verify that $(\bar{\mathbf{P}}, j)$ satisfies the stated condition; uniqueness up to isomorphism follows in the usual way. Given a normal morphism $f: \mathbf{P} \to \mathbf{Q}$, we define $g: \bar{P} \to Q$ by $g(\bar{p}) = f(p)$ for all $p \in P$; it is now easily verified that g is a well-defined morphism of $\mathbf{P}$ into $\mathbf{Q}$. Clearly $f = g \circ j$ and g is the unique morphism satisfying this condition. ∎

The result of Lemma 2.9 may be expressed in categorical language as follows. Let $\mathcal{OS}$ be the category of orthospaces and *normal* morphisms and $\mathcal{NOS}$ the (full) subcategory of normal orthospaces. Then the assignment $\mathbf{P} \mapsto \bar{\mathbf{P}}$ of the normalization is a functor left adjoint to the inclusion functor $\mathcal{NOS} \hookrightarrow \mathcal{OS}$, so that $\mathcal{NOS}$ is reflective in $\mathcal{OS}$.

We can now show that the canonical orthocompletion of a (not necessarily normal) orthospace is uniquely determined up to isomorphism.

2.10. Proposition. Let $\mathbf{P}$ be an orthospace, let $(\bar{\mathbf{P}}, j)$ be its normalization, let $(\mathbf{E}, k)$ be the canonical orthocompletion of $\bar{\mathbf{P}}$, and put $i = k \circ j$. Then $(\mathbf{E}, i)$ is a dense orthocompletion of $\mathbf{P}$ and any dense orthocompletion of $\mathbf{P}$ is isomorphic over $\mathbf{P}$ to $(\mathbf{E}, i)$. In particular, $\mathbf{E}$ is isomorphic to the canonical orthocompletion $\mathbf{L}(\mathbf{P})$ of $\mathbf{P}$.

Proof. Since k is a dense embedding and j is a surjective weak embedding, $i = k \circ j$ is a dense weak embedding. We claim that i is also normal. For suppose $i(p) \not\leqslant i(q)$. Then (since E is normal) there is $x \in E$ such that $x \approx i(p)$ and $x \perp i(q)$. Since i is dense, $x = \bigvee \{i(r): i(r) \leqslant x\}$ and so there is $r \in P$ such that $i(r) \leqslant x$ and $i(r) \approx i(p)$. It follows that $i(r) \perp i(q)$. Since i is a weak embedding, $r \approx p$ and $r \perp q$, whence $Q_p \nsubseteq Q_q$. It also follows from this that $i(p) = i(q)$ whenever $Q_p = Q_q$.

By 2.9, there is a unique morphism $f: \bar{\mathbf{P}} \to \mathbf{E}$ such that $i = f \circ j$. Since i is dense, f is dense. Observe also that

$$\begin{aligned} f(j(p)) \perp f(j(q)) &\leftrightarrow i(p) \perp i(q) \\ &\leftrightarrow p \perp q \\ &\leftrightarrow j(p) \perp j(q) \end{aligned}$$

so that f is a weak embedding, and hence an embedding since both its domain and range are normal. It follows that $(\mathbf{E}, f)$ is a dense orthocompletion of $\bar{\mathbf{P}}$ and is therefore uniquely determined up to isomorphism over $\bar{\mathbf{P}}$.

To show that $(\mathbf{E}, i)$ is uniquely determined up to isomorphism over $\mathbf{P}$, let $(\mathbf{E}', i')$ be another such pair. Then by the same reasoning as for $(\mathbf{E}, i)$ there is a dense embedding $f': \bar{\mathbf{P}} \to \mathbf{E}'$ such that $i' = f' \circ j$. Thus $(\mathbf{E}', f')$ is a dense orthocompletion of $\bar{\mathbf{P}}$ and so there is an isomorphism $g: \mathbf{E}' \to \mathbf{E}$ such that $f = g \circ f'$. But then $g \circ i' = g \circ f' \circ j = f \circ j = i$. This completes the proof. ∎

When is a dense orthocompletion of an orthospace a Boolean algebra? Our next result provides an answer to this question. Recall that two elements p, q of an orthospace $\mathbf{P}$ are *compatible* if $\exists r \neq 0\ [r \leqslant p \ \&\ r \leqslant q]$.

2.11. Proposition. The following conditions on an orthospace $\mathbf{P}$ are equivalent:

(i) $\mathbf{P}$ satisfies

$$p \approx q \to \exists r \neq 0\ [Q_r \subseteq Q_p \cap Q_q] \qquad (*)$$

(ii) The proximity relation $\approx$ on the normalization $\bar{\mathbf{P}}$ of $\mathbf{P}$ coincides with the compatibility relation on $\bar{\mathbf{P}}$.

(iii) The dense orthocompletion of $\mathbf{P}$ (or, equivalently, of $\bar{\mathbf{P}}$) is a Boolean algebra.

Proof. (i) → (ii). Suppose $\mathbf{P}$ satisfies (*). If $\bar{p} \approx \bar{q}$ in $\bar{\mathbf{P}}$, then $p \approx q$ in $\mathbf{P}$ and so there is $r \neq 0$ in $\mathbf{P}$ such that $Q_r \subseteq Q_p$, $Q_r \subseteq Q_q$. But then $\bar{r} \neq 0$, $\bar{r} \leqslant \bar{p}$, $\bar{r} \leqslant \bar{q}$, giving (ii).

(ii) → (iii). Assume (ii) and let $(\mathbf{E}, i)$ be the dense orthocompletion of $\bar{\mathbf{P}}$. If $x \approx y$ in $\mathbf{E}$, then the density of i implies that there exist $a, b \in \bar{P}$ such that $i(a) \leqslant x$, $i(b) \leqslant y$, and $i(a) \approx i(b)$, whence $a \approx b$. So there is $c \in \bar{P}$ such that $c \neq 0$ & $c \leqslant a$, $c \leqslant b$, whence $i(c) \neq 0$, $i(c) \leqslant i(a) \leqslant x$, $i(c) \leqslant i(b) \leqslant y$. It follows that $\approx$ is the compatibility relation in $\mathbf{E}$ so that $\mathbf{E}$ is a Boolean algebra.

(iii) → (i). Suppose that the dense orthocompletion $(\mathbf{L}(\mathbf{P}), i)$ of $\mathbf{P}$ is a Boolean algebra. If $p \approx q$ in $\mathbf{P}$, then $i(p) \approx i(q)$ in $\mathbf{L}(\mathbf{P})$; since the latter is a Boolean algebra and i is dense, there must be $r \neq 0$ in P such that $i(r) \leqslant i(p)$ and $i(r) \leqslant i(q)$. It follows easily that $Q_r \subseteq Q_p$ and $Q_r \subseteq Q_q$, completing the proof. ∎

Clearly, if $\approx$ is the compatibility relation in $\mathbf{P}$, then $\mathbf{P}$ satisfies (*) of 2.11 so that the canonical orthocompletion of $\mathbf{P}$ is a Boolean algebra. (If $\mathbf{P}$ is *normal*, then the converse holds, i.e., $\approx$ is the compatibility relation in $\mathbf{P}$.) In this case the construction of a dense (Boolean) orthocompletion of $\mathbf{P}$ may also be carried out in the manner familiar to students of set-theoretic forcing (cf. Refs. 2 or 11): we topologize $P - \{0\}$ by taking the subsets $\{q: q \leqslant p\}$ as basic opens; the complete Boolean algebra $\mathrm{RO}(\mathbf{P})$ of regular open subsets of the resulting topological space—the *regular open algebra* of $\mathbf{P}$—is then a dense orthocompletion of P, and so isomorphic to the canonical orthocompletion $\mathbf{L}(\mathbf{P})$ of $\mathbf{P}$.

3. ENSEMBLES, TESTS, AND QUANTUM LOGIC

In the introduction we put forward the idea of regarding an orthospace $\mathbf{P} = (P, \leqslant, \perp, 0)$ as a collection P of ensembles subject to passing or failing various "tests," where $\leqslant$ and $\perp$ are the relations of *inclusion* and *exclusion*. We now attempt to show how the canonical orthocompletion of $\mathbf{P}$ may be regarded as the natural lattice of (formal) properties of $\mathbf{P}$.

Assume for the moment that $\mathbf{P}$ is *normal*. For each $p \in P$, consider the property $\phi(p)$ of being a *subensemble* of p, i.e., for any $q \in P$, q has property $\phi(p)$ if and only if $q \leqslant p$. We have natural relations of entailment ($\leqslant$) and exclusion ($\perp$) among these properties:

$$(3.1).\qquad \begin{cases} \phi(p) \leqslant \phi(q) \leftrightarrow p \leqslant q \leftrightarrow p \text{ has property } \phi(q) \\ \phi(p) \perp \phi(q) \leftrightarrow p \perp q \end{cases}$$

We seek to "close up" the resulting orthospace of properties $\{\phi(p): p \in P\}$ under the logical operations of conjunction and disjunction in such a way that each member of the resulting augmented orthospace E of "properties" may still be construed as a property of the elements of P. In particular, E will be a complete orthospace; the ordering $\leqslant$ on E will again be construed as entailment, the orthogonality relation $\perp$ on E as exclusion, and by analogy with (3.1), for $p \in P$, $a \in E$, the relation $\phi(p) \leqslant a$ will be taken to mean that p (and all its subensembles) have the "property" a.

If each element of E is to be nothing more than a "property" of elements of P, then the relation *a entails b*, i.e., $a \leqslant b$, must mean that for any $p \in P$, whenever p has "property" a, then p has "property" b, i.e.,

$$a \leqslant b \leftrightarrow \forall p \in P[\phi(p) \leqslant a \rightarrow \phi(p) \leqslant b]$$

But this, of course, is precisely the condition that the subset $\{\phi(p): p \in P\}$ be *dense* in E. That is, ϕ must be a *dense embedding* of P into E.

So we see that for the complete orthospace E to qualify as a complete orthospace of "properties" of the elements of P we must require that P be densely embeddable in E. If we further impose the reasonable requirement that E be a *minimal completion* of P (so that E arises by adding the "fewest possible" properties to P), then by 2.6, E is a dense orthocompletion of P which must, by 2.7, be isomorphic to the canonical orthocompletion of P. Thus, starting with a normal orthospace P of ensembles, any dense orthocompletion E of P may be considered a natural candidate for being the complete lattice of "properties" of the elements of P. We shall call the elements of any dense orthocompletion (E, i) of P *formal properties* of the elements of P in order to distinguish them from properties of the elements of P in the usual extensional sense (i.e., those which are represented simply by subsets of P). Any formal property $a \in E$ is correlated with the property $i(p) \leqslant a$ of ensembles $p \in P$. And conversely, any property of ensembles is correlated with the subset $X \subseteq P$ of all ensembles having the property, and X in turn may be correlated with the formal property $a_X = \bigvee i[X] \in E$. This is natural since a_X is the least formal property a such that, for all $p \in X$, p has the property correlated with a.

We observe that since E is *not* in general a Boolean algebra (in view of the fact that the orthogonality relation on P does not in general coincide with the incompatibility relation), it follows that the complete lattices of formal properties of sets of ensembles do not normally embody all the laws of classical logic (e.g., distributivity fails), but rather just the laws of *quantum logic* (see, e.g., Ref. 3). It seems, then, that *quantum logic has a natural origin in the analysis of orthospaces of ensembles.*

The above analysis extends to the case in which P is not normal by replacing P with its normalization $\bar{\mathrm{P}}$: any dense orthocompletion of $\bar{\mathrm{P}}$ may now be considered (albeit in a slightly weaker sense than that described above, since P is now only *weakly* densely embeddable in E) as the complete lattice of formal properties of the elements of P.

We now finally introduce an axiomatization of the concept of *test* which will bring the discussion down to an even more basic phenomenological level. This axiomatization was first proposed in Ref. 3; for the sake of completeness we provide a (somewhat compressed) account of its fundamentals.

We assume that we are given two sets S—called the set of *screens*—and P—called the set of *ensembles* (or beams). Each screen s *acts* on each ensemble x to yield a new ensemble sx (to be interpreted as the ensemble that emerges when x *travels through* s). We assume that each screen is uniquely determined by its action on ensembles: thus for $s, t \in S$,

$$\forall x \in P[sx = tx] \rightarrow s = t \qquad (*)$$

(We note that in making this assumption we incur no essential loss of generality since we can always replace each $s \in S$ by its equivalence class $[s]$ under the equivalence relation $s \equiv t \leftrightarrow (\forall x \in P) sx = tx$ and define $[s]x = sx$; the action of these equivalence classes on P then satisfies (*).) Any pair $s, t \in S$ is assumed to have a *product* $st \in S$ satisfying

$$(\forall x \in P)(st)x = s(tx)$$

Thus st may be regarded as the screen obtained by juxtaposing t and s (in that order). We shall assume that this operation is associative: $(st)u = s(tu)$ for all $s, t, u \in S$.

We also suppose that P contains a unique *empty* ensemble 0 satisfying $s0 = 0$ for all s and that S contains elements 0, 1 such that $0x = 0$ and $1x = x$ for all $x \in P$. Thus 0 is a screen which "absorbs" every ensemble and 1 a screen which has no effect on any ensemble. Clearly $0s = s0 = 0$ and $1s = s1 = s$ for all $s \in S$.

We assume also that each screen s can be "reversed" so as to form a new screen $\tilde{s}$ called its *transpose*. (Thus, for any $x \in P$, $\tilde{s}x$ may be construed as the result of allowing the ensemble x to travel through s "in the opposite direction.") The transpose operation is presumed to satisfy the evident conditions

$$(st)^{\sim} = \tilde{t}\tilde{s}, \qquad (\tilde{s})^{\sim} = s$$

for all $s, t \in S$. Clearly, $\tilde{1} = 1$, $\tilde{0} = 0$.

We may sum up the situation by saying that S *is an involution semigroup, with* 0 *and* 1, *acting on the set* P.

We next determine what properties a screen should possess if it is to correspond to a "test for a property" of ensembles. To begin with, it is natural to suppose that, once an ensemble has been subjected to a "test," an immediate subsequent application of the same "test" should have no further effect on the ensemble. This means that a screen s which corresponds to a "test" must be *idempotent*:

$$s^2 = s$$

We shall also suppose that the effect exerted on any ensemble by a screen corresponding to a "test" is *independent* of the ensemble's "direction of travel" through the screen. That is, a screen s corresponding to a "test" must be *equal to its transpose*:

$$\tilde{s} = s$$

A screen corresponding to a test will be called a *filter*; thus an element $s \in S$ is a filter iff $s^2 = \tilde{s} = s$. This is equivalent to the condition: $s\tilde{s} = s$. We write F for the set of all filters. Notice that $0, 1 \in F$, and that, for $s, t \in F$,

$$st \in F \leftrightarrow st = ts$$

A screen s is said to be *transparent* to an ensemble s, and x is said to *pass* s if $sx = x$, i.e., if passage through s has no effect on x. And s is said to be *opaque* to x or to *block* x if $sx = 0$, i.e., if x is completely absorbed by s. Thus, thinking of a filter s as a *test* for a property P, an ensemble x passes the test corresponding to s iff s is transparent to x, and x (completely) fails the test iff s blocks s.

We turn F into an orthospace in the following natural way. We define, for $s, t \in F$

$$s \leqslant t \leftrightarrow \forall x \in P[sx = x \rightarrow tx = x]$$

and

$$s \perp t \leftrightarrow \forall x \in P[sx = x \rightarrow tx = 0 \;\&\; tx = x \rightarrow sx = 0]$$

These are the relations of *entailment* and *exclusion* between filters (or "tests") respectively. We then have

3.2. Lemma. $\mathbf{F} = (F, \leqslant, \perp, 0)$ is a (partially ordered) orthospace in which

(i) $s \leqslant t \leftrightarrow st = ts = s$

(ii) $s \perp t \leftrightarrow st = ts = 0$

for all $s, t \in F$.

Proof. We verify (i) and (ii).

(i) If $s \leqslant t$, then $sx = x \rightarrow tx = x$ for all $x \in P$. But $ssx = s^2x = sx$, so $tsx = sx$ for all $x \in P$, whence $ts = s$. Therefore $ts \in F$, whence $st = ts = s$. Conversely, suppose $ts = s$; then if $sx = x$, it follows that $x = sx = tsx = tx$, so that $s \leqslant t$.

(ii) If $s \perp t$, then $sx = x \rightarrow tx = 0$ for all $x \in P$. But $ssx = s^2x = sx$, so $tsx = 0$ for all $x \in P$, whence $ts = 0$. Hence $ts \in F$, so that $st = ts = 0$. Conversely, suppose $st = ts = 0$. Then if $sx = x$, we have $tx = tsx = 0$ and if $tx = x$, then $sx = stx = 0$. Hence $s \perp t$.

We leave to the reader the easy verification, using (i) and (ii), that $(F, \leqslant, \perp, 0)$ is an orthospace and that $\leqslant$ is a partial ordering. ∎

It follows easily from 3.2 that, if $s, t \in F$ and $st = ts$, then st is the greatest lower bound of $\{s, t\}$ in F.

Now consider the set P of ensembles. We define the relations $\leqslant$ and $\perp$ on P by

$$x \leqslant y \leftrightarrow \forall s \in F[sy = y \rightarrow sx = x]$$

$$x \perp y \leftrightarrow \exists s \in F[[sx = x \;\&\; (\forall z \leqslant y) sz = 0] \vee [sy = y \;\&\; (\forall z \leqslant x) sz = 0]]$$

Thus $\leqslant$ is the relation of *inclusion*: $x \leqslant y$ means that x passes every "test" that y passes, or simply that x is a *subensemble* of y. And $\perp$ is the relation of *exclusion*: $x \perp y$ means that there is a "test" which x passes but no subensemble of y does, or vice-versa. As usual, we write $x \approx y$ for $x \not\perp y$.

It is now easily verified that $\mathbf{P} = (P, \leqslant, \perp, 0)$ is an orthospace, *the orthospace of ensembles determined by F*. Note that $\leqslant$ is *not* in general a partial ordering: two ensembles may be subensembles of one another yet not identical.

Now let $(\mathbf{E}, i)$ be any dense orthocompletion of $\mathbf{P}$. As we have seen, $\mathbf{E}$ can be considered as being (up to isomorphism) the complete lattice of formal properties of elements of $\mathbf{P}$. Each filter, moreover, is presumed to correspond to a "test" for a property of members of P. It is natural to take this property to be that of *passing* s, so that s is correlated with the subset $P_s = \{x \in P: sx = x\}$ of P. But by the observations above, P_s, and hence also s, is correlated with the formal property (i.e., element of E) $\hat{s} = \bigvee i[P_s] \in E$. Now if $(\mathbf{E}, i)$ is the *canonical orthocompletion* of $\mathbf{P}$, then

$$\hat{s} = \bigvee i[P_s] = \{y \in P: \exists x \in P_s x \approx y\}$$

Thus, although s is supposed to be "testing" ensembles x for the property of passing s, i.e., the property $sx = x$, the (extension of) the corresponding formal property in E is *not* the set P_s of all ensembles that pass s, but rather the set of all ensembles y such that $x \approx y$ for some ensemble x passing s. We shall see later on, however, that when F is *commutative* we may take $\hat{s}$ to be P_s.

We now introduce the concept of *compatibility* of filters, which is intended to give expression to the idea of *simultaneous performability* of the corresponding "tests." (Note that the concept of compatibility of filters should *not* be confused with the notion of order compatibility introduced in Example 2 of Section 2.) For each filter s, let T_s be the corresponding "test." We say that two filters s, t are *compatible*—and the "tests" T_s, T_t *simultaneously performable*—if any ensemble that passes T_s continues to do so *even after it has been subjected to* T_t (cf. Ref. 8, 4.3c). That is, s is compatible with t if and only if

$$\forall x \in P[sx = x \rightarrow stx = tx] \qquad (*)$$

Now it is easily shown that (*) is equivalent to the condition $sts = ts$. (For if (*) holds, then, for any $y \in P$, putting $x = sy$, we have $sx = x$, so $stsy = tsy$; since this holds for any $y \in P$, it follows that $sts = ts$. Conversely, if $sts = ts$, and $sx = x$, then $stx = stsx = tsx = tx$.) Therefore: *Two filters are compatible* (*and the corresponding "tests" simultaneously performable*) *if and only if they commute.*

The notion of compatibility is closely related to the concept of *tightness*. We say that a filter s is *tight* if $sx \leqslant x$ for every ensemble x. Thus a tight filter is one whose effect on any ensemble x is to "filter out" a *subensemble* of x, i.e., an ensemble that will continue to pass any "tests" that x would pass. We note the following.

3.3. Proposition. A filter s is tight iff $st = ts$ for all filters t. Thus F is commutative iff every filter is tight.

Proof. Suppose that s is tight and $t \in F$. Then for any $x \in P$ we have $ty = y$ where $y = tx$; since $sy \leqslant y$ it follows that $tsy = sy$, whence $tstx = stx$, so that $tst = st$ and it follows as above that $st = ts$. Conversely, suppose $st = ts$ for all $t \in F$. Then if $tx = x$, we have $tsx = stx = sx$, so $sx \leqslant x$. ∎

Next, we investigate some of the consequences of the assumption that for each ensemble a there is a filter s_a which "tests" the property of being a *subensemble* of a, i.e., is such that, for any $x \in P$,

$$s_a x = x \leftrightarrow x \leqslant a \qquad (3.4).$$

If, for each $a \in P$, an $s_a \in F$ satisfying (3.4) exists, we say that F acts *adequately* on P.

We note the following.

3.5. Lemma. Suppose that F acts adequately on P. Then

(i) $s_a x \leqslant a$ for any $x \in P$;

(ii) $s_a \leqslant s \leftrightarrow sa = a$ for all $s \in F$, $a \in P$;

(iii) the map $a \mapsto s_a$ is a dense quasiembedding of P into F.

Proof. (i) Since $s_a(s_a x) = s_a x$, this follows from (3.4).

(ii) We have

$$\begin{aligned} s_a \leqslant s &\leftrightarrow \forall x(s_a x = x \rightarrow sx = x) \\ &\leftrightarrow (\forall x \leqslant a) sx = x \\ &\leftrightarrow sa = a \end{aligned}$$

(iii) We have

$$\begin{aligned} s_a \leqslant s_b &\leftrightarrow s_b a = a \qquad \text{(by (ii))} \\ &\leftrightarrow a \leqslant b \qquad \text{(by (3.4))} \end{aligned}$$

and

$$\begin{aligned} s_a \perp s_b &\rightarrow s_a s_b = 0 \\ &\rightarrow (s_a a = a) \ \&\ \forall z \leqslant b(s_a z = s_a s_b z = 0) \\ &\rightarrow a \perp b \end{aligned}$$

If $a \perp b$, then there is $s \in F$ such that $sa = a$ and $(\forall z \leqslant b) sz = 0$. Hence $s_a \leqslant s$ by (ii), so that $s_a = s_a s$. Thus, for any $x \in P$, $s_a s_b x = s_a s s_b x = s_a 0 = 0$. Hence $s_a s_b = 0$ and $s_a \perp s_b$.

Therefore $a \mapsto s_a$ is a quasiembedding. To show that it is dense, suppose that $s, t \in F$ and $s_a \leqslant s \rightarrow s_a \leqslant t$ for all $a \in P$. Then by (ii) $sa = a \rightarrow ta = a$ for all $a \in P$, whence $s \leqslant t$. ∎

If F acts adequately on P, then it follows from this last result that the canonical orthocompletion of **F** is also a dense orthocompletion of **P**, which by 2.10 is isomorphic to the canonical orthocompletion $(\mathbf{L}(\mathbf{P}), i)$ of **P**. In this case it is easily shown that the formal property (i.e., element of $\mathbf{L}(\mathbf{P})$) correlated with a given filter s is

$$\hat{s} = \{x \in P \colon s s_x \neq 0\}$$

If F acts adequately on P and is also *commutative*, then the proximity relation $\approx$ on **P** coincides with the relation of compatibility. (For if $x \approx y$

in P, then $s_x x = x$ implies $\exists z \leqslant y s_x z \neq 0$. But $s_x z \leqslant x$ and $s_x z \leqslant z \leqslant y$.) It follows from 2.11 that in this case L(P) is a complete Boolean algebra which, by the remarks at the end of Section 2, is isomorphic to the regular open algebra RO(P) of P. Thus RO(P) is a complete lattice of formal properties of the elements of P. If P is *normal*, one easily shows that the formal property (i.e., element of RO(P)) correlated with a given $s \in F$ is the set

$$\dot{s} = P_s = \{x \in P: sx = x\}$$

of ensembles that pass s. It is also readily shown that, for $s, t \in F$,

$$\dot{st} = \dot{s} \cap \dot{t}$$

If, further, we correlate each $x \in P$ with its image

$$\dot{x} =_{df} \dot{s}_x = \{y \in P: y \leqslant x\}$$

under the canonical dense embedding of P in RO(P), then the action of F on P transforms into the action by set-theoretic *intersection* of the set $\dot{F} = \{x: \dot{s} \in F\}$ on the set $\dot{P} = \{\dot{x}: x \in P\}$, since it is easily shown that

$$(sx)^{\cdot} = \dot{s} \cap \dot{x}$$

for $s \in F$, $x \in P$.

To sum up, then, if F is commutative, normal, and acts adequately on a normal P, we may regard F and P as families of subsets of the same set (actually P itself) and both the product in F and the action of F on P as set-theoretic intersection. This situation may be obtained by starting with any family I of individuals, taking F (or S) to be any family of subsets of I containing $\varnothing$ and I and closed under finite intersections and P as any family of subsets of I containing $\varnothing$ and closed under intersection with members of F. This is the *classical* situation in which each filter or "test" is correlated with a classical property (considered in extension) of individuals and each ensemble is just a collection of individuals.

When the set of filters is *noncommutative* the complete lattice of formal properties E of P is in general non-Boolean and the "logic" of these properties is accordingly nonclassical. Moreover, in this situation there arise some of the characteristic features of quantum physics. In addition to incompatibility of tests which we have already mentioned, we can formulate a reasonable version of the concept of the quantum mechanical concept of *superposition* (of pure ensembles).

Let us call a nonzero ensemble a *pure* if it is minimal in P, i.e., if it satisfies

$$\forall x \in P[x \leqslant a \leftrightarrow x = 0 \text{ or } x = a]$$

Paraphrasing Dirac's famous 1930 definition of superposition of states (see Ref. 6, Chapter 1, Section 6) we say that a pure ensemble a is a *superposition* of pure ensembles b and c provided that any test which both b and c fail is also failed by a. Thus a is a superposition of b and c if

$$\forall s \in F[sb = 0 \ \& \ sc = 0 \rightarrow sa = 0] \qquad (*)$$

It is now easily shown using 3.3 that if F acts adequately on P and nontrivial superpositions are present in P, i.e., if there are distinct pure ensembles a, b, c satisfying (*) above, then F is noncommutative. *So the presence of nontrivial superpositions of pure ensembles ensures that the lattice of formal properties of ensembles has a nonclassical nature.*

Observe that nowhere have we needed to assume that the set of filters or ensembles forms a lattice, let alone a lattice of projections or subspaces of a Hilbert space, as one customarily does in the foundations of quantum mechanics. It is of interest, however, to mention the "orthodox" quantum mechanical framework in which S is the semigroup of continuous linear operators on a Hilbert space H, F is the set of projections in S, and P is a set of (closed) subspaces of H which includes all the one-dimensional subspaces. In this case the canonical orthocompletion of $\mathbf{P}$ may be identified with the complete ortholattice $\mathbf{E}$ of all closed subspaces of H and for $s \in F$ the correlated formal property $\hat{s} \in E$ is the range of the projection s. Pure ensembles are just one-dimensional subspaces of H, i.e., pure states in the quantum-mechanical sense. Finally, a pure ensemble (state) a is a superposition in the above sense of pure ensembles (states) b, c iff a is in the subspace of H spanned by b and c, i.e., if a is a superposition of b and c in the usual quantum-mechanical sense.

In conclusion, we may assert that, starting with the simple and phenomenologically plausible idea of a collection of filters acting on a collection of ensembles, we obtain in a natural way a complete lattice of (formal) properties of the ensembles. If all filters are compatible, i.e., if every filter is tight, then the resulting lattice is Boolean and the corresponding "logic" of the properties is classical. If, on the other hand, the filters are not all compatible, or if nontrivial superpositions of pure ensembles are present, then the lattice of properties is not in general Boolean, and the corresponding "logic" of the properties is "*non*classical" or "quantum logical." Thus "quantum logic" arises simply from the *incompatibility or noncommutativity* of filters or "tests."

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