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1. Why were you initially drawn to the philosophy of logic?

I should say at once that I have worked chiefly in mathematical logic and the foundations and philosophy of mathematics, and my interest in the philosophy of logic (but not in philosophy *per se*) is the result of my activity in those areas. My route thereto was somewhat circuitous. In youth I was attracted to physics, especially relativity theory and cosmology—I actually attended one of Fred Hoyle's lecture courses on the subject in Cambridge in the early 1960s. (Parenthetically, I may mention that it was through Hoyle's lectures that I first heard the name Gödel, not of course in connection with his discoveries in logic, of which I was then wholly ignorant, but as the deviser of cosmological models containing closed timelike lines.) While I had become quite adept in mathematical physics, at some point it dawned on me that I had no genuine understanding of what I was actually doing. Accordingly I decided to turn away from physics, my first love, and concentrate on pure mathematics. While mathematics lacked, in my eyes, the romantic appeal of cosmology, it had the compensating merit that its concepts and methods could, in principle at least, be fully presented to the understanding. My flight to mathematics was fuelled by my discovery of John Kelley's classic work *General Topology*. Kelley also furnished my first introduction to set theory. This in turn led me to study Gödel's monograph, *The Consistency of the Axiom of Choice and the Generalized Continuum Hypothesis*. The first two-thirds of this mathematical tour-de-force, in which Gödel presents his axiom system for set theory and develops its essential properties, seemed reasonably clear. But, despite my best efforts, I was unable to fathom the final part of the work, its grand finale, so to speak, in which, accompanied by an inaudible clash of cymbals, the consistency of the generalized continuum hypothesis is established. A good few years were to pass before I felt I truly understood what was going on here.

Another influence was Bourbaki's Éléments *de Mathématique*. On first coming across some volumes of this monumental work in Blackwell's bookshop, I was excited to find that it was intended to be a complete, systematic account of abstract mathematics, precisely the kind of mathematics to which I had already been converted by Kelley's *General Topology*. The *oeuvre Bourbachique* included not only *Topologie Génerale*, but *Algèbre*, *Thèorie des Ensembles*, *Espaces Vectoriels Topologiques*, *Algèbre Commutatif*– magical titles in my eyes. I bought as many volumes as I could afford, often in obsolete—and so cheaper—editions (the whole enterprise seemed to be undergoing constant revision), and commenced to work my way through the collections of challenging exercises at the end of each section. I toiled mightily, in particular, to formulate solutions to the exercises on ordered sets in Chapter 3 of the *Thèorie des Ensembles*. It was from these that I first learned about ordinals, which Bourbaki presents in the original Cantorian manner as order types of well-ordered sets.

Kelley, Bourbaki, Gödel: it was through their influence that I was led to the foundations of mathematics. My interest in philosophy, on the other hand, derived from my being a voracious and eclectic reader. As an undergraduate I recall reading Plato's *The Last Days of Socrates*, William James's "Essays on Pragmatism", G. E. Moore's philosophical essays, Hegel's *Philosophy of History*, Descartes' *Discourse on Method*, Spinoza's *Ethics* (the statements of the theorems at least, since I found the "proofs" unenlightening), Leibniz's delphic *Monadology*, some Locke, Berkeley and Hume, Schopenhauer's *Essays in Pessimism*. And of course Bertrand Russell's breezily brilliant, if irresponsible, *History of Western Philosophy*. My attempts to penetrate the profundities of Kant's *Critique of Pure Reason* were frustrated by the work's apparent indigestibility. (I was only to appreciate its depth and philosophical importance many years later.) I greatly enjoyed Hans Reichenbach's *Philosophy of Space and Time*. On Blackwell's shelves in Oxford I came across Norman Malcolm's *Wittgenstein: A Memoir*. I was deeply moved by Malcolm's portrayal of Wittgenstein, in which he emerges as an intellectual ascetic of compelling moral grandeur. Wittgenstein's tiniest defiances of convention, for example, his refusal to wear a tie at dinner in Trinity College, I found admirable. Reading Malcolm's memoir stimulated me to attempt to read Wittgenstein's philosophical works. I was intrigued by the *Tractatus Logico-Philosophicus*, a masterpiece of sybilline refinement and compression in which Wittgenstein embarks on the heroic effort of reducing philosophy to the expressible, but in the end washes up on the shores of the ineffable. The conventionalism of the later Wittgenstein's *Philosophical Investigations*, I found less appealing.

I was drawn quite early on to the foundations of mathematics, and to general philosophy, but my conscious interest in the philosophy of mathematics per se was comparatively slow to crystallize. As an undergraduate I had developed an interest in set theory and the philosophy of the infinite in general. Later I came to believe that mathematics, while being, like art, a beautiful sublimation of human activity, has, in the last analysis, to be understood as the product of actual human beings living in the world. This led me to see that mathematics actually has a *hidden content*, which can actually be *argued about*. This is the opposite of the unthinking Platonism/realism to which I was, I guess, initially attracted as offering the simplest account of mathematical truth, and which also possessed the additional advantage of avoiding what I then felt to be a certain cynicism inherent in Formalism. (Still, as I have come to learn, Formalism has the great merit of offering the weary ex-Platonist a refuge.) But, like the child's loss of belief in Santa Claus, I came to regard the Platonistic account of mathematical entities as a kind of fairy tale, and in any case as engendering insuperable epistemological difficulties. I may parenthetically remark that I have since come to liken Platonism to a (necessary) disease, which, like measles, must have been contracted in one's youth so as to confer an immunity in later life.

A key stage in the development of my interest in the philosophy of mathematics came through my efforts to understand *topos theory*. I was very struck by Bill Lawvere's insight that a topos is an objective presentation of the idea of *variability*, and that its internal—intuitionistic—logic may be considered as a logic of variation. Later I went so far as to attempt to use the topos concept as the basis for a "local" (as opposed to "absolute") interpretation of mathematical statements. I suggested that the unique absolute universe of sets central to the orthodox set-theoretic account of the foundations of mathematics should be replaced by a plurality of local mathematical frameworks—elementary toposes—defined in category-theoretic terms. Observing that such frameworks possess sufficiently rich internal structure to enable mathematical concepts and assertions to be interpreted within them, I maintained that they can serve as local surrogates for the usual "absolute" universe of sets. On this account mathematical concepts will in general no longer possess absolute meaning, nor mathematical assertions (e.g. the continuum hypothesis) absolute truth values, but will instead possess such meanings or truth values only locally, i.e., relative to local frameworks. The absolute truth of set-theoretical assertions would then, I held, give way to the subtler concept of invariance, that is, validity in all local frameworks. Thus, e.g., while the theorems of constructive arithmetic turn out to possess the property of invariance, the axiom of choice or the continuum hypothesis do not, because they hold true in some local frameworks but not others.

I still find this view attractive, but it is, after all, only one among many possible accounts of mathematics. If I were pressed to characterize my present attitude towards the foundations of mathematics, I would use the word *pluralistic*: no unique foundation, rather an interlocking ensemble of "foundations". My pluralist attitude also extends to logic: instead of a single overarching Logic governing all forms of reasoning, my own experience has led me to conclude that each type of reasoning - classical, intuitionistic, quantum, linear - carries its own logic in the form of rules laid down in accordance with the nature of the objects or concepts being reasoned about.

2. What are your main contributions to the philosophy of logic?

While most of my work has been in technical mathematical logic, I have made some contributions to the philosophy of logic.

The first of these was essentially a contribution to philosophy of science. As an ex-aspiring-physicist I had long been intrigued by quantum theory, with its mysterious superpositions of states and incompatible measurements; and as a logician my curiosity was piqued by the so-called quantum logic, whose characteristic feature is that its algebra of propositions is not a Boolean or Heyting algebra, but a certain kind of nondistributive lattice—an ortholattice. All of these facts can be, and are, formally derived from the standard Hilbert space formalism of quantum theory. I became interested in the problem of formulating some simple principles, free of the technicalities of the theory of Hilbert spaces, from which one could derive the anomalous features of quantum theory, as well as the ortholattices underlying quantum logic. I came up with two approaches. The first, essentially topological, was based on the idea of using what I called a proximity space, a set equipped with a symmetric reflexive relation "close to". The lattice of parts of such a space is an ortholattice. There is a natural way, which I called "manifestation", similar to Paul Cohen's celebrated concept of set-theoretic forcing, of relating propositions (actually attributes) to parts of the space. The propositions manifested over the whole of every proximity space are (essentially) the theses of quantum logic. Given two propositions P, Q, their superposition can be identified with $\neg\neg(P \vee Q)$, and they are incompatible if there is a proximity space with a part manifesting P but not $Q \vee \neg Q$, or vice-versa.

In my other approach to the problem, I showed how to construct the ortholattices arising in quantum logic from what I saw as the phenomenologically plausible idea of a collection of ensembles subject to passing or failing various "tests". A collection of ensembles forms a certain kind of preorderd set with an additional relation I called an orthospace: I showed that the complete ortholattices, in particular those of quantum theory, arise as canonical completions of orthospaces in much the same way as arbitrary complete lattices arise as canonical completions of partially ordered sets. I also showed that the canonical completion of an orthospace of ensembles may be identified with the lattice of properties of the ensembles, thereby showing exactly why ortholattices arise in the analysis of "tests" or experimental propositions. I went on to axiomatize the concept of "test" itself in terms of the more primitive notion of "filters" acting on ensembles. "Passing" an ensemble through a filter s produces the subensemble of entities that have "passed" the test corresponding to s. Two filters s and t can be juxtaposed to produce the compound filter st, but in general st ¹ ts. When the latter is that case, the two tests corresponding to s and t are, like position and momentum measurements in quantum theory, not simultaneously performable, that is, *incompatible*. When (and only when) $st = ts$, the juxtaposition of s and t corresponds to their logical conjunction. In this setting, it is the noncommutativity or incompatibility of filters or "tests" that gives rise to "quantum logic".

A philosophical problem which had long intrigued me was: why is traditional logic *bivalent*, that is, why is it assumed that there just two truth values rather than some other number? What is it about the number 2 that gives it this special position in logic? Wittgenstein seems to take the fact for granted when (in his *Notes on Logic*) he says that propositions have two "poles". It is often claimed that bivalent logic is the "logic of realism", that is, logic in which propositions are construèd as referring to independently existing objects, in contrast with "anti-realist" logics such as intuitionistic logic (I don't agree that intuitionistic logic has to be thought of as anti-realist—but let that pass). However, this begs the question, since the thought immediately arises: what is it about the realm of independently existing objects that confers bivalence on propositions referred to it? Why shouldn't the number of objective truth values be, say, 3, like the number of spatial dimensions? Wittgenstein recognized the possibility of this question arising but simply dismissed it.

One way that occurred to me of explaining the role of the number 2 in logic is by moving from individual propositions to sets of propositions, or *theories*. Frege had suggested that the bivalence of the logic of concepts arises from their having *sharp boundaries*: one can determine with exactitude, for such a concept, when an object falls under it , or when it does not. In other words, a concept's possession of a sharp boundary means that the theory of the concept is complete with in regard to atomic propositions. It is then natural to extend this prescription to arbitrary propositions. So, metaphorically, we may say that (the concept determined by) a theory has sharp boundaries if it is *complete*, that is, if any proposition in the theory's vocabulary is provable or refutable from the theory. But it is well known that, for any complete theory T (in propositional intuitionistic or classical logic), it is possible to assign the *two* truth values 0, 1 to propositions in such a way as to respect the logical operations, and also to assign precisely the propositions in T the value 1. And conversely, if such a bivalent assignment exists, the theory is complete. That is, the number 2 is simply the numerical representative of completeness, or the possession of "sharp boundaries".

The major logical consequence of bivalence (although not equivalent to it) is the *law of excluded middle:* the assertion, for any proposition P, of the disjunction $P \vee \neg P$. This is of course the logical principle which whose affirmation distinguishes classical from intuitionistic logic. Like bivalence the law of excluded middle has been taken to be characteristic of logic in which propositions are construed as referring to independently existing objects. I found that, if one starts with intuitionistic predicate logic, and extends it to include Hilbert's ε-terms (these are essentially objects named by the use of the indefinite article: *a* such-and-such), then the law of excluded middle becomes provable. That is, the law of excluded middle is, after all, derivable from what can reasonably be construed as an ontological principle.

3. What is the proper role of philosophy of logic in relation to other disciplines, and to other branches of philosophy?

I think that the philosophy of logic should, and in fact does, play a dialectical role in relation to its sister disciplines, guiding them and, reciprocally, responding to their internal development. Let me attempt to illustrate what I mean. Cantor's philosophy of the infinite (and his associated, if lesser-known, championship of the reduction of the continuous to the discrete) played a major part in his development of set theory, which, as is well-known, came to permeate mathematics. Partly in reaction to the unrestricted use of Cantorian set theory in mathematics, Brouwer formulated his philosophy of intuitionism with its radical revision of the laws of logic (rejection of the law of excluded middle, etc.). This in turn led to a prolonged discussion of the nature of logic, culminating in Carnap's principle of tolerance, which has become a central doctrine in the philosophy of logic.

4. What have been the most significant advances in the philosophy of logic? Here, in my view, are some of those advances:

- Boole's analysis of logic in mathematical terms.
- Frege's analysis of propositions and the idea that truth values are the references of propositions.

- Russell's type-theoretic account of propositions.
- The emergence of intuitionistic logic and its challenge to the supremacy of classical logic.
- The development of logical pluralism, the idea that there is more than one Logic.
- The recognition, following Gödel's pioneering work, that many logical systems are incomplete.
- Tarski's analysis of the concept of truth - in particular, his undefinability theorem.
- The emergence of many-valued logic, the idea that propositions can possess more than two truth values.
- The idea, arising from topos theory, that logical connectives and quantifiers can be construed as mathematical operations on a domain of truth values.
- The "propositions-as-types" doctrine underlying constructive type theories such as that of Martin-Löf.

5. What are the most important open problems in the philosophy of logic, and what are the prospects for progress?

Here are some questions in the philosophy of logic that strike me as being of some importance (I find them interesting, at least).

> Should the foundations of logic be monistic or pluralistic? In particular should quantum logic and other "deviant" systems be considered logic?
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> What is the meaning of a "true" contradiction? Can there be "objective" contradictions, as Hegel claimed, or "contradictions in nature", as Engels claimed?
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> Is it in the nature of judgments to be bivalent (true or false)? In particular, should bivalence ultimately prevail in metalogic?
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> What is the nature of propositions about the quantum microworld? In what respect do these propositions differ from those concerning "ordinary" objects? In particular, can microobjects such as photons be said to possess properties in the same sense as do "ordinary" objects?

It seems to me that some progress has already been made on all of these questions, and I confidently expect further advances to continue to be made.