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Precovers, Modalities and Universal Closure Operators in a Topos

John L. Bell[a] **and Silvia Gebellato**[b 1)]

[a] Department of Philosophy, The University of Western Ontario,
London, Canada N6A 3K7

[b] Dipartimento di Matematica Pura ed Applicata, Università di Padova,
via G. Belzoni 7, I-35131 Padova, Italia

Abstract. In this paper we develop the notion of formal precover in a topos by defining a relation between elements and sets in a local set theory. We show that such relations are equivalent to modalities and to universal closure operators. Finally we prove that these relations are well characterized by a convenient restriction to a particular set.

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1 Introduction

The concept of *precover* is an essential constituent of that of *pretopology*, which was formulated by GIOVANNI SAMBIN in [4] in order to provide a semantics for linear logic. The concept of pretopology may itself be regarded as a generalization of the notion of *formal topology*, which was introduced by SAMBIN as part of a program for developing an intuitionistic approach to *pointless topology*, that is, topology in which the basic concept is that of neighbourhood and points are defined as particular filters of neighbourhoods (see [3] and [4]).

In this paper we develop the notion of precover in an *elementary topos*. Our arguments will in fact be formulated within the so-called *local set theories* and the *linguistic toposes* associated with them (see [1]). After a brief review of the relevant definitions, we show how each modality within a local set theory induces a relation possessing the reflexivity and transitivity properties of a precover. This relation, defined by JOHN BELL in [2] and called *universal precover*, extends the notion of precover to local set theories, and hence to elementary toposes. We show that the concept of universal precover is in fact *equivalent* to that of modality, and hence to that of *universal closure operation* in a local set theory. We also observe that the notion of universal precover is a natural generalization of the *membership relation* within a local set theory.

1) We would like to express our gratitude to GIOVANNI SAMBIN for his assistance.

To define a universal precover in a local set theory we have to assign a relation to each "set" (or to each object of the associated topos). We show that, when the set is a monoid, the assigned relation actually becomes a precover in the original sense of SAMBIN. Finally, we show that a universal precover may be obtained by assigning such a relation just to the *truth-value set* of the local set theory.

2 Precovers and pretopologies

Definition. Let $(S, \cdot, 1)$ be a commutative monoid. A *precovering relation*, or *precover*, on S is a relation $\triangleleft$ between elements and subsets of S, satisfying the following conditions:

(i) If $a \in U$, then $a \triangleleft U$ (reflexivity);
(ii) if $a \triangleleft U$ and $U \triangleleft V$, then $a \triangleleft V$, where $U \triangleleft V \equiv \forall b \in U (b \triangleleft V)$ (transitivity);
(iii) if $a \triangleleft U$ and $b \triangleleft V$, then $a \cdot b \triangleleft U \cdot V$ (stability).

A *pretopology* is a quadruple $\mathcal{F} \equiv (S, \cdot, 1, \triangleleft_{\mathcal{F}})$, where $(S, \cdot, 1)$ is a commutative monoid and $\triangleleft_{\mathcal{F}}$ is a precovering relation on it.

A precover on S can equivalently be presented as an operator $\mathcal{F} : P(S) \longrightarrow P(S)$ defined by

$$\mathcal{F}(U) = \{a \in S : a \triangleleft_{\mathcal{F}} U\}.$$

It follows from properties (i) and (ii) that $\mathcal{F}$ is a closure operator on S (see [4]). $\mathcal{F}(U)$ is called the *$\mathcal{F}$-saturation* of U and U is called *$\mathcal{F}$-saturated* if $\mathcal{F}(U) = U$.

The collection $\mathrm{Sat}(\mathcal{F})$ of $\mathcal{F}$-saturated subsets of S is then a complete lattice with respect to $\subseteq$ in which arbitrary infima and suprema are defined by

$$\bigwedge_{i \in I} U_i = \bigcap_{i \in I} U_i \quad \text{and} \quad \bigvee_{i \in I} U_i = \mathcal{F}(\bigcup_{i \in I} U_i).$$

Furthermore, if we define the operation $\cdot_{\mathcal{F}}$ by

$$U \cdot_{\mathcal{F}} V = \mathcal{F}(U \cdot V), \text{ for any } U, V \in \mathrm{Sat}(\mathcal{F}),$$

$(\mathrm{Sat}(\mathcal{F}), \cdot_{\mathcal{F}})$ becomes a commutative monoid. The only link between the algebraic and the lattice structure is infinite distributivity, that is to say

$$V \cdot_{\mathcal{F}} \bigvee_{i \in I} U_i = \bigvee_{i \in I} (V \cdot_{\mathcal{F}} U_i).$$

If $\cdot_{\mathcal{F}}$ coincides with $\wedge$ or when we require $(\mathrm{Sat}(\mathcal{F}), \cdot_{\mathcal{F}})$ to be a semilattice with ordering $\subseteq$, $\mathrm{Sat}(\mathcal{F})$ is a complete Heyting algebra [4].

3 Local set theories and linguistic toposes

A *linguistic topos* is a topos built up from a language in a way resembling the construction of Lindenbaum algebras. It may be regarded as a "model" for a *local set theory*, which is a type theoretic system built up from the primitive symbols $\in$, $=$, and $\{ : \}$ in a language whose basic axioms and rules are formulated so as to yield exactly the theorems of intuitionistic logic (see [1]).

To sketch the construction of a linguistic topos, we begin by describing a local language. A *local language* is a typed language with no logical operations among its primitive symbols, in which are specified two particular type symbols: $\mathbf{1}$ and $\mathbf{\Omega}$, called the *unity type* and the *truth value type*, respectively. Its terms must include the following:

1) $*$ of type $\mathbf{1}$;
2) $\{x_A : \alpha\}$ of type $\boldsymbol{P}(\boldsymbol{A})$, where x_A is a variable of any type $\boldsymbol{A}$ and α a term of type $\mathbf{\Omega}$;
3) $\sigma = \tau$ of type $\mathbf{\Omega}$, where σ and τ are terms of the same type;
4) $\sigma \in \tau$ of type $\mathbf{\Omega}$, where σ and τ are terms of type $\boldsymbol{A}$ and $\boldsymbol{P}(\boldsymbol{A})$, respectively;
5) $\langle \tau_1, \dots, \tau_n \rangle$ of type $\boldsymbol{A}_1 \times \cdots \times \boldsymbol{A}_n$, where τ_i is of type $\boldsymbol{A}_i$ $(i = 1, \dots, n)$;
6) $(\tau)_i$ of type $\boldsymbol{A}_i$, where τ is of type $\boldsymbol{A}_1 \times \cdots \times \boldsymbol{A}_n$.

An occurrence of a variable x in a term τ is *bound* if it appears within a context of the form $\{x : \alpha\}$, otherwise it is *free*. A term of type $\mathbf{\Omega}$ is called a *formula*.

The logical operations among the terms in the language can be defined as follows, where α, β are terms of type $\mathbf{\Omega}$.

$\alpha \leftrightarrow \beta$	for	$\alpha = \beta$,
true	for	$* = *$,
$\alpha \wedge \beta$	for	$\langle \alpha, \beta \rangle = \langle \mathit{true}, \mathit{true} \rangle$,
$\alpha \to \beta$	for	$(\alpha \wedge \beta) = \alpha$,
$\forall x \alpha$	for	$\{x : \alpha\} = \{x : \mathit{true}\}$,
false	for	$\forall \omega . \omega$, ω a variable of type $\mathbf{\Omega}$,
$\neg\alpha$	for	$\alpha \to \mathit{false}$,
$\alpha \vee \beta$	for	$\forall \omega [((\alpha \to \omega) \wedge (\beta \to \omega)) \to \omega]$, ω a variable of type $\mathbf{\Omega}$ not occurring in α or β,
$\exists x \alpha$	for	$\forall \omega [\forall x (\alpha \to \omega) \to \omega]$, ω a variable of type $\mathbf{\Omega}$ not occurring in α.

To describe the axioms and the inference rules of a local set theory we use the sequent notation $\Gamma : \alpha$ in which α is a formula and Γ is a (possibly empty) finite set of formulas.

The *basic axioms* of a local set theory are the following:

tautology	$\emptyset : \alpha = \alpha$,
unity	$\emptyset : x_1 = *$,
equality	$x = y, \alpha(z/x) : \alpha(z/y)$, with x and y free for z in α,
products	$\emptyset : (\langle x_1, \dots, x_n \rangle)_i = x_i$ and $\emptyset : x = \langle (x)_1, \dots, (x)_n \rangle$,
comprehension	$\emptyset : x \in \{x : \alpha\} \leftrightarrow \alpha$.

The *rules of inference* of a local set theory are the following: thinning, cut (with the limitation that any free variable of the cut-formula is free in the antecedents or in the succedents of the sequents, different from α), and

substitution $\dfrac{\Gamma : \alpha}{\Gamma(x/\tau) : \alpha(x/\tau)}$, τ free for x in Γ and α,

extensionality $\dfrac{\Gamma : x \in \sigma \leftrightarrow x \in \tau}{\Gamma : \sigma = \tau}$, x not free in Γ, σ, τ,

equivalence $\dfrac{\alpha,\ \Gamma : \beta \qquad \beta,\ \Gamma : \alpha}{\Gamma : \alpha \leftrightarrow \beta}$

A *proof* from S is defined in the same way as in the sequent calculus and a sequent $\Gamma : \alpha$ is said to be *derivable* from S if there is a proof from S of which $\Gamma : \alpha$ is the conclusion. We will write $\Gamma \vdash_S \alpha$ to indicate that $\Gamma : \alpha$ is derivable from S, and $\vdash_S \alpha$ for $\emptyset \vdash_S \alpha$. A *local set theory* is a theory in a local language, that is to say a collection of sequents, S, which is closed under derivability.

Linguistic toposes are constructed as follows. In a local language the terms of power type (if $\boldsymbol{A}$ is a type symbol $\boldsymbol{P}(\boldsymbol{A})$ is called the *power type* of $\boldsymbol{A}$) are called *set-like terms* and closed set-like terms are called $\mathcal{L}$-sets or simply *sets*. We consider the equivalence classes, called *S-sets*, of $\mathcal{L}$-sets under the relation $\sim_S$ defined in the following way:

$$X \sim_S Y \quad \text{iff} \quad \vdash_S X = Y.$$

An *S-map* $f : X \longrightarrow Y$ is a triple (f, X, Y) of S-sets such that $\vdash_S f \in Y^X$, where Y^X is defined as

$$\{u : u \subseteq X \times Y \wedge \forall x(x \in X \rightarrow \exists! y(y \in Y \wedge \langle x, y\rangle \in u))\}.$$

The collection of S-sets and maps then forms a topos which is called a *linguistic topos*. It can be shown (see [1]) that any elementary topos is equivalent to a linguistic topos.

4 Modalities

In this paragraph we will show how, given a local set theory S and a modality on S (defined below), we can obtain a new relation between elements and S-sets which generalizes the membership relation. We will also see that saturating the S-sets with respect this new relation yields a universal closure operation.

Definition 4.1. Let S be a local set theory and Ω the S-set $\{\omega : \mathit{true}\}$. A *modality* is an S-map $\mu : \Omega \longrightarrow \Omega$ such that

(i) $\alpha \vdash_S \mu(\alpha)$;

(ii) if $\alpha \vdash_S \beta$, then $\mu(\alpha) \vdash_S \mu(\beta)$;

(iii) $\mu(\mu(\alpha)) \vdash_S \mu(\alpha)$.

We shall usually write $\alpha^\star$ for $\mu(\alpha)$.

Note that this definition does not specify any relation between a modality and the logical operators, but it is consistent with them as we state in the next proposition, which is Corollary 5.3 of [1].

P r o p o s i t i o n 4.2. *If μ is a modality in a local set theory S, we have:*

1. $\vdash_S (\alpha \wedge \beta)^\star = \alpha^\star \wedge \beta^\star$;
2. $(\alpha \to \beta)^\star \vdash_S \alpha^\star \to \beta^\star$;
3. $\alpha \to \beta \vdash_S \alpha^\star \to \beta^\star$;
4. $\exists x \alpha^\star \vdash_S (\exists x \alpha)^\star$, *whence* $(\exists x \alpha^\star)^\star \vdash_S (\exists x \alpha)^\star$;
5. $(\forall x \alpha)^\star \vdash_S \forall x \alpha^\star$. □

Modalities enable us to generalize the membership relation. Thinking of $\mu(\alpha)$ as meaning "*α is possibly true*", $\mu(x \in U)$ means "*it is possible that x is in U*". Thus, given a modality μ on a local set theory S, for each S-set X we define a relation $\lhd_X \subseteq X \times P(X)$ by

$$x \lhd_X U \equiv \mu(x \in U),$$

for any S-set U such that $\vdash_S U \subseteq X$.

T h e o r e m 4.3. *The relation $\lhd_X$ has the following properties:*

1. *If $\vdash_S U \subseteq X$, then $x \in U \vdash_S x \lhd_X U$.*
2. *If $\vdash_S U \subseteq X$ and $\vdash_S V \subseteq X$, then $y \lhd_X U, \forall x(x \in U \to x \lhd_X V) \vdash_S y \lhd_X V$.*
3. *If $f : X \longrightarrow Y$ is an S-function and $\vdash_S U \subseteq Y$, then*
$$x \in X \vdash_S f(x) \lhd_Y U \leftrightarrow x \lhd_X f^{-1}[U],$$

where $f^{-1}[U]$ is the term $\{x : x \in X \wedge f(x) \in U\}$.

P r o o f.

1. This follows immediately from the first property of modalities.
2. According to the definition of $\lhd_X$, we need to show:

$$(*) \qquad \forall x[x \in U \to (x \in V)^\star] \vdash_S (y \in U)^\star \to (y \in V)^\star.$$

Now $y \in U \to (y \in V)^\star \vdash_S (y \in U)^\star \to (y \in V)^{\star\star}$ by 4.2.3, so, since $(y \in V)^{\star\star} \vdash_S (y \in V)^\star$, it follows that $y \in U \to (y \in V)^\star \vdash_S (y \in U)^\star \to (y \in V)^\star$, from which $(*)$ follows immediately.

3. As $x \in X \vdash_S (x \in f^{-1}[U])^\star \leftrightarrow (f(x) \in U)^\star$ follows from

$$x \in X, (x \in f^{-1}[U])^\star : (f(x) \in U)^\star \quad \text{and} \quad x \in X, (f(x) \in U)^\star : (x \in f^{-1}[U])^\star$$

by the equivalence rule, we need to infer the last two sequents. Regarding the first one, we have, by substituting into the equality axiom,

$$x \in f^{-1}[U] = (f(x) \in U \wedge x \in X), (x \in f^{-1}[U])^\star \vdash_S (f(x) \in U \wedge x \in X)^\star$$

so that, using 4.2.1,

$$x \in f^{-1}[U] = (f(x) \in U \wedge x \in X), (x \in f^{-1}[U])^\star \vdash_S (f(x) \in U)^\star \wedge (x \in X)^\star.$$

Hence, by the comprehension axiom we get $(x \in f^{-1}[U])^\star \vdash_S (f(x) \in U)^\star \wedge (x \in X)^\star$ so that $(x \in f^{-1}[U])^\star \vdash_S (f(x) \in U)^\star$. Hence, $x \in X, (x \in f^{-1}[U])^\star \vdash_S (f(x) \in U)^\star$.

Similarly, we show

From the first property of modalities we have $x \in X \vdash_S (x \in X)^\star$ so concluding that $(x \in X),\ (f(x) \in U)^\star \vdash_S (x \in f^{-1}[U])^\star$. □

The three properties in Theorem 4.3 embody the minimal conditions which a putative membership-relation has to satisfy. We also note that when the S-set is a monoid, the reflexive and transitive properties correspond to those of a precover.

The relation $\lhd_X$ enables us to generalize the subset-relation by writing $U \lhd_X V$ for $\forall x(x \in U \rightarrow x \lhd_X V)$. With this new definition we can rewrite Theorem 4.3.2 as follows:

2′. *If* $\vdash_S U \subseteq X$ *and* $\vdash_S V \subseteq X$, *then* $x \lhd_X U,\ U \lhd_X V \vdash_S x \lhd_X V$.

In particular, we can "saturate" an S-set $U, \vdash_S U \subseteq X$, with respect to the relation $\lhd_X$ writing $U^\triangle$ for $\{x : x \lhd_X U\}$ and calling it the *saturation of* U. It follows easily from Theorem 4.3 that the saturation operation $^\triangle$ is a *universal closure operation* in the sense of [1].

5 Universal precovers

We are now going to define a relation between elements and S-sets in a local set theory which will be characterized by the properties established in Theorem 4.3.

Definition 5.1. A *universal precover* $\lhd$ on a local set theory S is an assignment of an S-set $\lhd_X$ to each S-set X such that $\vdash_S \lhd_X \subseteq X \times P(X)$ and

(i) if $\vdash_S U \subseteq X$, then $x \in U \vdash_S x \lhd_X U$,

(ii) if $\vdash_S U \subseteq X$ and $\vdash_S V \subseteq X$, then $x \lhd_X U,\ U \lhd_X V \vdash_S x \lhd_X V$,

(iii) if $f : X \longrightarrow Y$ is an S-function and $\vdash_S U \subseteq Y$, then
$$x \in X \vdash_S f(x) \lhd_Y U \leftrightarrow x \lhd_X f^{-1}[U].$$

We will show that, given a type $\boldsymbol{A}$, a universal precover is "uniform" on every S-set of type $\boldsymbol{P(A)}$, and so also on every subset of a given S-set X. For this reason we can write $\lhd_{\boldsymbol{A}}$ in place of $\lhd_X$ if $\boldsymbol{A}$ is the type of X, and we can omit the subscript whenever it is clear from the context.

Proposition 5.2. *If* X *is an* S*-set of type* $\boldsymbol{P(A)}$ *and* $\vdash_S U \subseteq X$, *then*
$$x \in X \vdash_S x \lhd_X U \leftrightarrow x \lhd_A U,$$
where we write A *for the* S*-set* $\{x_{\boldsymbol{A}} : true\}$.

Proof. Consider the insertion $j : X \longrightarrow A$ defined as $j(x) \equiv x$. Since $\vdash_S U \subseteq A$ it follows from the third property of universal precovers that
$$x \in X \vdash_S x \lhd_X j^{-1}[U] \leftrightarrow j(x) \lhd_A U.$$

It remains to prove $\vdash_S j^{-1}[U] = U$, yielding $x \in X \vdash_S x \lhd_X U \leftrightarrow x \lhd_A U$. Noting that $j^{-1}[U] = \{x : x \in X \wedge x \in U\}$ we infer $x \in X,\ x \in U \vdash_S x \in j^{-1}[U]$, whence $x \in U \vdash_S x \in j^{-1}[U]$, since $\vdash_S U \subseteq X$. From $x \in j^{-1}[U] \vdash_S x \in X \wedge x \in U$ we obtain $x \in j^{-1}[U] \vdash_S x \in U$, and $\vdash_S j^{-1}[U] = U$ follows. □

From this we easily obtain

Corollary 5.3. *If* X *is an* S*-set of type* $\boldsymbol{P(A)}$ *and* U *is an* S*-set such that* $\vdash_S U \subseteq X$, *then* $x \in U \vdash_S x \lhd_U U \leftrightarrow x \lhd_X U$. □

We want next to show that universal precovers respect the operations of function image and cartesian product. To do this we require the following

Lemma 5.4.

(a) $x \lhd X, y \in Y \vdash_S \langle x, y\rangle \lhd X \times Y, \quad x \in X, y \lhd Y \vdash_S \langle x, y\rangle \lhd X \times Y.$

(b) $x \lhd X, y \lhd Y \vdash_S \langle x, y\rangle \lhd X^{\triangle} \times Y, \quad x \lhd X, y \lhd Y \vdash_S \langle x, y\rangle \lhd X \times Y^{\triangle}.$

(c) $\vdash_S X^{\triangle} \times Y \lhd X \times Y, \quad \vdash_S X \times Y^{\triangle} \lhd X \times Y.$

Proof.

(a) Let X be of type $\boldsymbol{A}$ and consider $\sigma : A \times Y \longrightarrow A$ defined by $\sigma(\langle x, y\rangle) \equiv x$. The third property of universal precovers yields

$$X \subseteq A, \langle x, y\rangle \in A \times Y, x \lhd X \vdash_S \langle x, y\rangle \lhd X \times Y$$

and so, since the cartesian product is defined by $A \times Y \equiv \{\langle x, y\rangle : x \in A \wedge y \in Y\}$, we obtain $X \subseteq A, x \in A, y \in Y, x \lhd X \vdash_S \langle x, y\rangle \lhd X \times Y$. From the hypothesis $\vdash_S X \subseteq A$, we obtain $x \in A, y \in Y, x \lhd X \vdash_S \langle x, y\rangle \lhd X \times Y$, i.e. $y \in Y,\ x \lhd X \vdash_S \langle x, y\rangle \lhd X \times Y$, recalling that $A = \{x_{\boldsymbol{A}} : \mathit{true}\}$ so that $\vdash_S x \in A$. The second sequent in (a) is an easy consequence.

(b) This follows from (a) by substituting $X^{\triangle}$ for X and observing that

$$\vdash_S x \in X^{\triangle} \leftrightarrow x \lhd X$$

follows from the comprehension axiom, so that $x \lhd X \vdash_S x \in X^{\triangle}$. In the same way we deduce the second sequence in (b).

(c) The comprehension axiom implies $\langle x, y\rangle \in X^{\triangle} \times Y \vdash_S x \in X^{\triangle} \wedge y \in Y$, and by (a), $x \lhd X, y \in Y \vdash_S \langle x, y\rangle \lhd X \times Y$, whence $\langle x, y\rangle \in X^{\triangle} \times Y \vdash_S \langle x, y\rangle \lhd X \times Y$. Hence $\vdash_S \forall x, y(\langle x, y\rangle \in X^{\triangle} \times Y \rightarrow \langle x, y\rangle \lhd X \times Y)$ which may be written in the form $\vdash_S X^{\triangle} \times Y \lhd X \times Y$. In the same way we deduce $\vdash_S X \times Y^{\triangle} \lhd X \times Y$. □

Proposition 5.5. *If $\lhd$ is a universal precover on a local set theory S, then*

(i) *if $f : X \longrightarrow Y$ is an S-function and $\vdash_S U \subseteq X$, then*

$$x \in X,\ x \lhd_X U \vdash_S f(x) \lhd_Y f[U];$$

(ii) $\vdash_S \langle x, y\rangle \lhd X \times Y \leftrightarrow (x \lhd X \wedge y \lhd Y)$.

Proof.

(i) Let $f : X \longrightarrow Y$ be an S-function. Since $U \subseteq X \vdash_S U \subseteq f^{-1}[f[U]]$ it follows that $U \subseteq X \vdash_S U \lhd f^{-1}[f[U]]$. By the second property of universal precovers, we have $x \lhd_X U, U \lhd f^{-1}[f[U]] \vdash_S x \lhd_X f^{-1}[f[U]]$ which yields

(1) $\quad U \subseteq X,\ x \lhd_X U \vdash_S x \lhd_X f^{-1}[f[U]].$

The third property of universal precovers gives

$$x \in X,\ f[U] \subseteq Y,\ x \lhd_X f^{-1}[f[U]] \vdash_S f(x) \lhd_Y f[U]$$

so that, since $U \subseteq X \vdash_S f[U] \subseteq Y$, we infer

(2) $\quad U \subseteq X,\ x \in X,\ x \lhd_X f^{-1}[f[U]] \vdash_S f(x) \lhd_Y f[U].$

From (1) and (2) follows

$$U \subseteq X,\ x \in X,\ x \lhd_X U \vdash_S f(x) \lhd_Y f[U],$$

whence $x \in X,\ x \lhd_X U \vdash_S f(x) \lhd_Y f[U]$.

(ii) Let X, Y be of types $\boldsymbol{P(A)}$, $\boldsymbol{P(B)}$, respectively, write A, B for $\{x_{\boldsymbol{A}} : true\}$, $\{x_{\boldsymbol{B}} : true\}$, respectively, and $\pi_1 : A \times B \longrightarrow A$, $\pi_2 : A \times B \longrightarrow B$ for the S-functions $\langle x, y\rangle \mapsto x$, $\langle x, y\rangle \mapsto y$, respectively. We first show $\langle x, y\rangle \lhd X \times Y \vdash_S (x \lhd X \wedge y \lhd Y)$. As we saw in (i) we can get

$$X \times Y \subseteq A \times B,\ \langle x, y\rangle \in A \times B,\ \langle x, y\rangle \lhd X \times Y \vdash_S \pi_1(\langle x, y\rangle) \lhd \pi_1(X \times Y)$$

so that

$$(3) \qquad X \times Y \subseteq A \times B,\ \langle x, y\rangle \lhd X \times Y \vdash_S \pi_1(\langle x, y\rangle) \lhd \pi_1(X \times Y),$$

recalling the definitions of A and B and observing that $\vdash_S \langle x, y\rangle \in A \times B$. From the two hypotheses $\vdash_S X \subseteq A$ and $\vdash_S Y \subseteq B$ we infer $\vdash_S X \times Y \subseteq A \times B$, and $\langle x, y\rangle \lhd X \times Y \vdash_S x \lhd X$ follows from (3). In the same way, using π_2 in place of π_1, we have $\langle x, y\rangle \lhd X \times Y \vdash_S y \lhd Y$, whence $\langle x, y\rangle \lhd X \times Y \vdash_S x \lhd X \wedge y \lhd Y$. Lastly, we have $(x \lhd X \wedge y \lhd Y) \vdash_S \langle x, y\rangle \lhd X \times Y$. For from Lemma 5.4.(c), we obtain $\vdash_S X^{\triangle} \times Y^{\triangle} \lhd X \times Y$ from which follows

$$\langle x, y\rangle \in X^{\triangle} \times Y^{\triangle} \vdash_S \langle x, y\rangle \lhd X \times Y$$

so that $x \lhd X,\ y \lhd Y \vdash_S \langle x, y\rangle \lhd X \times Y$. □

We finally show that a universal precover is uniquely determined by the relation between formulas and the S-set $Tr \equiv \{\omega : \omega = true\}$.

Proposition 5.6. *If U is an S-set of type $\mathbf{P(A)}$ and $\vdash_S U \subseteq X$, then*

$$\vdash_S x \lhd_A U \leftrightarrow (x \in U) \lhd_\Omega Tr.$$

Proof. Let $\mathcal{X} : A \longrightarrow \Omega$ be the S-function defined by $\mathcal{X}(x) = x \in U$. Then $\vdash_S \mathcal{X}^{-1}[Tr] = U$ so that $\vdash_S x \lhd_A U \leftrightarrow x \lhd_A \mathcal{X}^{-1}[Tr]$ and by property (iii) of universal precovers, $\vdash_S x \lhd_A \mathcal{X}^{-1}[Tr] \leftrightarrow \mathcal{X}(x) \lhd_\Omega Tr$. Putting these equivalences together gives the required result. □

6 Equivalence between modalities and universal precovers

Our next task is to show that a modality is equivalent to a universal precover. In fact we will show that a universal precover induces a modality which, in its turn, gives rise to a universal precover which turns out to be the same as the original one. We will prove the same thing starting with a modality.

First, we show how to obtain a modality from a universal precover.

Proposition 6.1. *Let $\lhd$ be a universal precover on a local set theory S and $\mu : \Omega \longrightarrow \Omega$ the S-function defined by $\mu(\omega) \equiv \omega \lhd_\Omega Tr$. Then μ is a modality.*

Proof. We have to prove that μ satisfies the three properties of a modality.

(i) We need to show that $\alpha \vdash_S \alpha \lhd_\Omega Tr$. Since $\vdash_S \omega = (\omega = true)$, it follows that $\vdash_S \omega \leftrightarrow \omega \in Tr$ so that $\omega \vdash_S \omega \in Tr$. From the first property of a universal precover we have $\omega \in Tr \vdash_S \omega \lhd_\Omega Tr$, whence $\omega \vdash_S \omega \lhd_\Omega Tr$, as required.

(ii) We have to show that from $\alpha \vdash_S \beta$ we can infer $\alpha \lhd_\Omega Tr \vdash_S \beta \lhd_\Omega Tr$. From the hypothesis $\alpha \vdash_S \beta$ we infer $\vdash_S \alpha \rightarrow \beta$ and hence, recalling the definition of $\alpha \rightarrow \beta$, the equality axiom yields $\alpha \lhd_\Omega Tr \vdash_S \alpha \wedge \beta \lhd_\Omega Tr$. So our task reduces to showing that $\alpha \wedge \beta \lhd_\Omega Tr \vdash_S \beta \lhd_\Omega Tr$, and, to this end, we consider the S-function

$\& : \Omega \times \Omega \longrightarrow \Omega$ defined by $\&(\langle \omega, \omega' \rangle) \equiv \omega \wedge \omega'$. By the third property of universal precovers, we have $\vdash_S (\omega \wedge \omega') \lhd_\Omega Tr \leftrightarrow \langle \omega, \omega' \rangle \lhd \&^{-1}[Tr]$. Now, if we can show that

(4) $\vdash_S \&^{-1}[Tr] = Tr \times Tr$,

we may infer $\vdash_S (\omega \wedge \omega') \lhd_\Omega Tr \leftrightarrow \langle \omega, \omega' \rangle \lhd Tr \times Tr$, so that

$$(\omega \wedge \omega') \lhd_\Omega Tr \vdash_S \langle \omega, \omega' \rangle \lhd Tr \times Tr.$$

Proposition 5.5.(ii) then implies $(\omega \wedge \omega') \lhd_\Omega Tr \vdash_S \omega \lhd_\Omega Tr \wedge \omega' \lhd_\Omega Tr$ from which follows $(\omega \wedge \omega') \lhd_\Omega Tr \vdash_S \omega' \lhd_\Omega Tr$, so that, substituting α for ω and β for ω', we get what we want.

So we turn our attention to proving (4). To do this, we observe that, by the comprehension axiom, $\vdash_S \langle \omega, \omega' \rangle \in \&^{-1}[Tr] \leftrightarrow \langle \omega, \omega' \rangle \in \Omega \times \Omega \wedge (\omega \wedge \omega') \in Tr$, so that $\vdash_S \langle \omega, \omega' \rangle \in \&^{-1}[Tr] \leftrightarrow \langle \omega, \omega' \rangle \in \Omega \times \Omega \wedge (\omega \wedge \omega') = true$. Therefore, we have $\vdash_S \langle \omega, \omega' \rangle \in \&^{-1}[Tr] \leftrightarrow \langle \omega, \omega' \rangle \in \Omega \times \Omega \wedge (\omega \wedge \omega')$ from $\vdash_S \omega \wedge \omega' = ((\omega \wedge \omega') = true)$. Now, $\omega \wedge \omega'$ is $\langle \omega, \omega' \rangle = \langle true, true \rangle$, whence $\vdash_S \omega \wedge \omega' \leftrightarrow \omega \in Tr \wedge \omega' \in Tr$. So we have $\vdash_S \langle \omega, \omega' \rangle \in \&^{-1}[Tr] \leftrightarrow \langle \omega, \omega' \rangle \in \Omega \times \Omega \wedge (\omega \in Tr \wedge \omega' \in Tr)$, whence

$$\vdash_S \langle \omega, \omega' \rangle \in \&^{-1}[Tr] \leftrightarrow \langle \omega, \omega' \rangle \in Tr \times Tr,$$

from which we infer $\vdash_S \&^{-1}[Tr] = Tr \times Tr$ by extensionality.

(iii) Finally, we must prove $\mu(\mu(\alpha)) \vdash_S \mu(\alpha)$, i.e. $(\alpha \lhd_\Omega Tr) \lhd_\Omega Tr \vdash_S \alpha \lhd_\Omega Tr$. Since $\vdash_S \alpha \lhd_\Omega Tr \leftrightarrow \alpha \in Tr^\triangle$ it follows that

$$(\alpha \lhd_\Omega Tr) \lhd_\Omega Tr \vdash_S (\alpha \in Tr^\triangle) \lhd_\Omega Tr.$$

By proposition 5.6, we have $\vdash_S (\alpha \in Tr^\triangle) \lhd_\Omega Tr \leftrightarrow \alpha \lhd_\Omega Tr^\triangle$. Hence

(5) $\vdash_S (\alpha \lhd_\Omega Tr) \lhd_\Omega Tr \leftrightarrow \alpha \lhd_\Omega Tr^\triangle$.

From the second property of universal precovers, we have

$$\alpha \lhd_\Omega Tr^\triangle,\ Tr^\triangle \lhd Tr \vdash_S \alpha \lhd_\Omega Tr.$$

Since clearly $\vdash_S Tr^\triangle \lhd Tr$, we obtain $\alpha \lhd_\Omega Tr^\triangle \vdash_S \alpha \lhd_\Omega Tr$, which, together with (5) gives the required conclusion. □

Now we establish the equivalence between modalities and universal precovers.

Theorem 6.2. *Every modality in a local set theory S is equivalent to a universal precover in S, and further, every universal precover is equivalent to a modality.*

Proof. Let μ be a modality in S. As we saw in Section 4, we can define a universal precover, $x \lhd U \equiv \mu(x \in U)$, which, in its turn, induces a modality $\mu_\lhd$ as follows:

$$\mu_\lhd(\alpha) \equiv \alpha \lhd Tr \equiv \mu(\alpha \in Tr).$$

We show that $\vdash_S \mu(\alpha) \leftrightarrow \mu_\lhd(\alpha)$. Since $\vdash_S \alpha \leftrightarrow \alpha \in Tr$, we can apply the second property of modalities and obtain

$$\vdash_S \mu(\alpha) \leftrightarrow \mu(\alpha \in Tr),$$

as claimed.

Conversely, given a universal precover $\triangleleft$, we can define a modality μ as in the previous proposition and then a universal precover $\triangleleft_\mu$ by

$$x \triangleleft_\mu U \equiv \mu(x \in U) \equiv (x \in U) \triangleleft_\Omega Tr.$$

By Proposition 5.6 we have $\vdash_S x \triangleleft_X U \leftrightarrow (x \in U) \triangleleft_\Omega Tr$ which we can rewrite as $\vdash_S x \triangleleft_X U \leftrightarrow x \triangleleft_\mu U$ so that $\triangleleft_\mu$ is the same as $\triangleleft$. □

7 Stability

We have defined a universal precover on a local set theory by assigning to each S-set X a relation $\triangleleft_X$ satisfying certain properties. When the S-set X is a monoid, we have noted that the first two properties correspond to the reflexive and transitive properties of a precover, respectively. Here we will show that, when X is a monoid, $\triangleleft_X$ also satisfies stability so that $\triangleleft_X$ is a *precover* on X.

Let $\mathcal{L}$ be a local language in which we can write the monoid axioms and let $\boldsymbol{M}$ be a type of a monoid. We write M for the S-set $\{x_{\boldsymbol{M}} : true\}$ of type $\boldsymbol{P}(\boldsymbol{M})$, and if U and V are S-sets of type $\boldsymbol{P}(\boldsymbol{M})$, we write $U \cdot V$ for the term $\{x \cdot y : x \in U \wedge y \in V\}$.

We will say that a universal precover satisfies *stability* if

$$a \triangleleft_M U,\ b \triangleleft_M V \vdash_S a \cdot b \triangleleft_M U \cdot V.$$

Proposition 7.1. *If $\triangleleft$ is a universal precover on a local set theory S, then $\triangleleft_M$ satisfies stability.*

Proof. Let $f : M \times M \longrightarrow M$ be the S-function defined by $f(\langle x, y\rangle) \equiv x \cdot y$. From 5.5.(i) we infer $\langle x, y\rangle \in M \times M,\ \langle x, y\rangle \triangleleft U \times V \vdash_S f(\langle x, y\rangle) \triangleleft_M f(U \times V)$ and so, since $\vdash_S \langle x, y\rangle \triangleleft U \times V \leftrightarrow x \triangleleft_M U \wedge y \triangleleft_M V$ by Proposition 5.5.(ii), we obtain

$$\langle x, y\rangle \in M \times M,\ x \triangleleft_M U \wedge y \triangleleft_M V \vdash_S x \cdot y \triangleleft_M U \cdot V.$$

Since clearly $\vdash_S \langle x, y\rangle \in M \times M$, the conclusion follows. □

8 Precovers on Ω

We have seen in Theorem 6.2 that specifying a modality on a local set theory S is equivalent to specifying a universal precover, but to specify a universal precover we have to define a relation for *every* S-set X whereas, to specify a modality we have to define an S-function *only* on the S-set Ω. This fact leads one to suspect that a universal precover may be obtained just by defining a relation on the S-set Ω, which is what we finally show.

Definition 8.1. A *precover-on-Ω* in a local set theory S is an S-set $\triangleleft_\Omega$ such that $\triangleleft_\Omega \subseteq \Omega \times P(\Omega)$, and

(i) if $\vdash_S U \subseteq \Omega$, then $x \in U \vdash_S x \triangleleft_\Omega U$;
(ii) if $\vdash_S U \subseteq \Omega$ and $\vdash_S V \subseteq \Omega$, then $x \triangleleft_\Omega U,\ U \triangleleft_\Omega V \vdash_S x \triangleleft_\Omega V$;
(iii) $\vdash_S (\omega \wedge \omega') \triangleleft_\Omega Tr \rightarrow \omega \triangleleft_\Omega Tr \wedge \omega' \triangleleft_\Omega Tr$;
(iv) if $f : \Omega \longrightarrow \Omega$ is an S-function and $\vdash_S U \subseteq \Omega$, then
$\vdash_S \omega \triangleleft_\Omega f^{-1}[U] \leftrightarrow f(\omega) \triangleleft_\Omega U$.

We now show that this relation determines a universal precover.

Theorem 8.2. *Let S be a local set theory and $\lhd_\Omega$ a precover-on-Ω in S. If for each S-set X we define the relation $\lhd_X$, $\lhd_X \subseteq X \times P(X)$, by*

$$x \lhd_X U \equiv (x \in U) \lhd_\Omega Tr,$$

then we obtain a universal precover on S.

Proof.

(i) Since $\omega \vdash_S \omega \in Tr$, we obtain $x \in U \vdash_S (x \in U) \in Tr$ from which follows $x \in U \vdash_S (x \in U) \lhd_\Omega Tr$ by the first property of a precover-on-Ω, and if we rewrite the conclusion using the definition of $\lhd_X$, we get $x \in U \vdash_S x \lhd_X U$.

(ii) We must show

$$(y \in U) \lhd_\Omega Tr, \forall x\,((x \in U) \rightarrow (x \in V) \lhd_\Omega Tr) \vdash_S (y \in V) \lhd_\Omega Tr.$$

Now, from the definition of $\alpha \rightarrow \beta$ we infer

$$(x \in U) \rightarrow (x \in V) \lhd_\Omega Tr, (x \in U) \lhd_\Omega Tr \vdash_S ((x \in U) \wedge (x \in V) \lhd_\Omega Tr) \lhd_\Omega Tr$$

and the third property of $\lhd_\Omega$ implies

$$(x \in U) \rightarrow (x \in V) \lhd_\Omega Tr, (x \in U) \lhd_\Omega Tr$$
$$\vdash_S (x \in U) \lhd_\Omega Tr \wedge ((x \in V) \lhd_\Omega Tr) \lhd_\Omega Tr.$$

Hence $(x \in U) \rightarrow (x \in V) \lhd_\Omega Tr, (x \in U) \lhd_\Omega Tr \vdash_S ((x \in V) \lhd_\Omega Tr) \lhd_\Omega Tr$ so that

$$\forall x((x \in U) \rightarrow (x \in V) \lhd_\Omega Tr), (y \in U) \lhd_\Omega Tr \vdash_S ((y \in V) \lhd_\Omega Tr) \lhd_\Omega Tr.$$

Now, from Proposition 5.6 we infer $((y \in V) \lhd_\Omega Tr) \lhd_\Omega Tr \vdash_S (y \in V) \lhd_\Omega Tr^\triangle$ and so, since $\vdash_S Tr^\triangle \lhd Tr$, we deduce

$$\forall x((x \in U) \rightarrow (x \in V) \lhd_\Omega Tr), (y \in U) \lhd_\Omega Tr \vdash_S (y \in V) \lhd_\Omega Tr,$$

as required.

(iii) Let $f : X \longrightarrow Y$ be an S-function and $\vdash_S U \subseteq Y$. The equality axiom gives us $(f(x) \in U) = (x \in f^{-1}[U])$, $(f(x) \in U) \lhd_\Omega Tr \vdash_S (x \in f^{-1}[U]) \lhd_\Omega Tr$, and then, observing that $x \in X \vdash_S f(x) \in U \leftrightarrow x \in f^{-1}[U]$ follows from the definition of $f^{-1}[U]$, we obtain $x \in X, (f(x) \in U) \lhd_\Omega Tr \vdash_S (x \in f^{-1}[U]) \lhd_\Omega Tr$. Similarly we infer $x \in X, (x \in f^{-1}[U]) \lhd_\Omega Tr \vdash_S (f(x) \in U) \lhd_\Omega Tr$ from which follows

$$x \in X \vdash_S (x \in f^{-1}[U]) \lhd_\Omega Tr \leftrightarrow (f(x) \in U) \lhd_\Omega Tr.$$

If we rewrite the conclusion using $\lhd_X$, we get $x \in X \vdash_S x \lhd_X f^{-1}[U] \leftrightarrow f(x) \lhd_Y U$. □

References

[1] BELL, J. L., Toposes and Local Set Theories: An introduction. Clarendon Press, Oxford 1988.

[2] BELL, J. L., Notes from a course on Local Set Theory. Padova, June 1991.

[3] SAMBIN, G., Intuitionistic formal spaces - a first communication. In: Mathematical Logic and its Applications (G. SKORDEV, ed.), Plenum Press, New York 1987, pp. 187 – 204.

[4] SAMBIN, G., Intuitionistic formal spaces and their neighbourhood. In: Logic Colloquium '88 (R. FERRO et al., eds.), North Holland Publ. Comp., Amsterdam 1989, pp. 261 – 285.

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