Reflections on Bourbaki's Notion of "Structure" and Categories

John L. Bell

1. **Structure in Bourbaki's *Éléments de Mathématique.***

A remarkable passage in Edward Gibbon's *Decline and Fall of the Roman Empire* goes

The mathematics are distinguished by a particular privilege, that is, in the course of ages, they may always advance and can never recede.

Gibbon's assertion could serve as the starting point of an absorbing discussion of whether "the mathematics" has in fact never made a backward step. But I quote it here only to draw attention to the fact that Gibbon uses the plural form "mathematics", even with the (now obsolete use of) the definite article. This may be because classical Greek mathematics - the quadrivium - was a plurality, divided into arithmetic, geometry, astronomy and music. In English the singular form "mathematic" does not exist as a noun, but in French the singular form *la mathématique* and the plural form *les mathématiques* are both acceptable, even if the singular has a whiff of the archaic. Bourbaki adopted the singular form in entitling his masterwork *Éléments de Mathématique*[1]. I have a particular affection for Bourbaki's *Élements* because it - more specifically, the chapters on *Topologie Génerale, Théorie des Ensembles,* and *Algèbre* opened my undergraduate eyes to the world of what I was pleased to identify as "real" mathematics.

It has been suggested, quite plausibly, by Maurice Marshaal[2] that the use of the singular "mathématique" in the title of the *Élements* is tendentious, intended to convey the authors' conviction that mathematics is a unity, contrary to what the use of the plural form of the term "mathematics" might suggest. Marshaal also claims that the use of the

---

[1] Bourbaki (1939 -).
[2] Marshaal (2006).

plural form in Bourbaki's *Éléments d'histoire des mathématiques*[3] is intended to indicate that, before they appeared on the scene, mathematics was a set of scattered practices, and that the modern notion of *structure* enabled these practices to become fused into a single discipline. This claim also has a certain plausibility.

Bourbaki's article *The Architecture of Mathematics* (ghostwritten by J. Dieudonné)[4], begins with a question: *Mathematic or mathematics*? The article continues:

To present a view of the entire field of mathematical science as it exists, - this is an enterprise which presents, at first sight, almost insurmountable difficulties, on account of the extent and the varied character of the subject. As is the case in all other sciences, the number of mathematicians and the number of works devoted to mathematics have greatly increased since the end of the 19th century. The memoirs in pure mathematics published in the world during a normal year cover several thousands of pages. Of course, not all of this material is of equal value; but, after full allowance has been made for the unavoidable tares [weeds}, *it remains true nevertheless that mathematical science is enriched each year by a mass of new results, that it spreads and branches out steadily into theories, which are subjected to modifications based on new foundations, compared and combined with one another. No mathematician, even were he to devote all his time to the task, would be able to follow all the details of this development. Many mathematicians take up quarters in a corner of the domain of mathematics, which they do not intend to leave; not only do they ignore almost completely what does not concern their special field, but they are unable to understand the language and the terminology used by colleagues who are working in a corner remote from their own. Even among those who have the widest training, there are none who do not feel lost in certain regions of the immense world of mathematics; those who, like Poincaré or Hilbert, put the seal of their genius on almost every domain, constitute a very great exception even among the men of greatest accomplishment. It must therefore be out of the question to give to the uninitiated an exact picture of that which the mathematicians themselves cannot conceive in its totality. Nevertheless it is legitimate to ask whether this exuberant proliferation makes for the development of a strongly constructed organism, acquiring ever greater cohesion and unity with its new growths, or whether it is the external manifestation of a tendency towards a progressive splintering, inherent in the very nature of mathematics, whether the domain of mathematics is not becoming a tower of Babel, in which autonomous disciplines are being more and more widely separated from one another, not only in their aims, but also in their methods and even in their language. In other words, do we have today a mathematic or do we have several mathematics?*

[3] Bourbaki (1994)
[4] Bourbaki (1950).

Here Bourbaki/Dieudonné uses the phrase "tower of Babel" with its usual connotation of "place of confusion". But it is worth pointing out that, according to Genesis, it was the fact that human beings possessed only a *single language* that enabled them to embark on the construction of the tower of Babel. God frustrates these aims by the introduction of linguistic *diversity.* Before God's intervention, the Tower of Babel was actually a place of order, not confusion. Thus the tower of Babel might be seen, not as representing the jumble of separate practices that Bourbaki deplores, but rather as the *unity* that they wished to impose on mathematics. In that case, Bourbaki's *Élements* would, ironically perhaps, amount precisely to the attempt to build a mathematical "tower of Babel". The fact that Bourbaki failed - as is well-known - to complete his grandiose project as originally conceived was not, as in Genesis, the result of God sowing linguistic confusion - the Bourbaki members, after all, still spoke a common mathematical language. Rather, Bourbaki's project was simply too ambitious to be brought to completion. Nevertheless, the *Éléments*, unlike the tower of Babel, remains a magnificent plinth.

In *the Architecture of Mathematics* Bourbaki/Dieudonné asserts that the unity of contemporary mathematics rests on the *axiomatic method,* and that the latter, in mathematics at least, rests in turn on the notion of *structure.* Bourbaki identifies three basic types of mathematical structure - *structures mères*" - or "mother structures". These are *algebraic, order,* and *topological* structures, which can be summed up as the "three C's" : *Combination, Comparison* and *Continuity*.

The group concept is presented as a simple, fundamental kind of mathematical structure:

One says that a ***set*** *in which an operation ... has been defined which has the three properties (a), (b), (c) is provided with a* ***group structure****....* (*or, briefly, that it is a* ***group***)*; the properties (a), (b), (c) are called the* ***axioms*** *of the group structures.*

Here we see that a group is a *set*, while group structure is a "something", specified by axioms, imposed on the set. The use of the term "axiom" to specify structure is analogous to the Definitions of geometric objects (as opposed to the axioms and postulates) in Euclid's *Elements.*

The text continues:

It can now be made clear what is to be understood, in general, by a mathematical structure. The common character of the different concepts designated by this generic name, is that they can be applied to sets of elements whose nature has not been specified; to define a structure, one takes as given one or several relations, into which these elements enter, then one postulates that the given relation, or relations, satisfy certain conditions (which are explicitly stated and which are the axioms of the structure under consideration.) To set up the axiomatic theory of a given structure amounts to the deduction of the logical consequences of the axioms of the structure, excluding every other hypothesis on the elements under consideration (in particular, every hypothesis as to their own nature).

Now this passage does not make it entirely clear what is to be understood by a "mathematical structure". It would seem that a structure is to be taken as a *definite set* having some prescribed form. This impression is reinforced by the fact that in the *Théorie des Ensembles* Bourbaki uses the phrase ``*structure of the species T*``. A species is thus a collection of structures sharing a common form.

As to the notion of set itself, we read in a footnote:

We take here a naive point of view and do not deal with the thorny questions, half philosophical, half mathematical, raised by the problem of the "nature" of the mathematical "beings or "objects." Suffice it to say that the axiomatic studies of the nineteenth and twentieth centuries have gradually replaced the initial pluralism of the mental representation of these "beings" thought of at first as ideal "abstractions" of sense experiences and retaining all their heterogeneity-by a unitary concept, gradually reducing all the mathematical notions, first to the concept of the natural number and then, in a second stage, to the notion of set. This latter concept, considered for a long time as "primitive" and "undefinable," has been the object of endless polemics, as a result of its extremely general character and on account of the very vague type of mental representation which it calls forth; ***the difficulties did not disappear until the notion of set itself disappeared*** *(*my boldening) *and with it all the metaphysical pseudo-problems concerning mathematical "beings" in the light of the recent work on logical formalism.*

The observation that ***the difficulties did not disappear until the notion of set itself disappeared*** is striking. What Bourbaki seems to mean is that while the notion of a mathematical structure in the strictest sense is dependent on the concept of set, in using mathematical structures the intrinsic properties of the sets (whatever these are) from which the structures are actually built can be safely ignored. In handling structures all one needs to know is that [the structure in question] "can be applied to sets of elements whose nature has not been specified".

Accordingly, the departure from the scene of the concept of set opened the way for Bourbaki to maintain that the unity of mathematics stems, not from the set concept, but from the concept of structure.

Bourbaki's manifesto can be seen as a declaration, *avant la lettre,* of what has come to be termed *mathematical strcturalism.* Yet, as Leo Corry[5] has pointed out, the concept of mathematical structure *as such* plays a very minor role in Bourbaki's development of mathematics in the *Éléments de Mathematique.* True, Chapter 4 of the *Théorie des Ensembles* is devoted to the presentation of a theory of structures in which, roughly speaking, a structure is defined to be a collection of sets together with functions and relations on them. Similar structures are organized into what are called *species.* (This theory cane to be described by Pierre Cartier, a later Bourbaki member, as "a monstrous endeavor to formulate categories without categories".) But the cumbersome mechanism fashioned there is never again called forth. All of the succeeding volumes of the *Éléments* can be read in complete ignorance of what Bourbaki terms a "structure". In particular, no explanation is provided of the importance of the "mother structures".

The hierarchy of structures as presented in the *Elements* is best understood as an (unconscious) version of *simple type theory*. For, give a collection **C** of base sets (types), Bourbaki's "structures" are essentially just the members of the universe **C*** of sets obtained by closing **C** under the operations of power set and Cartesian product. (Since Bourbaki takes ordered pairs as primitive, rather than defining them as sets, the operation of Cartesian product is required.) The types of simple type theory can, analogously, be obtained by starting with a collection **T** of base types and closing under the operations of power type, product, and subtype. Church's definitive 1940 formulation of simple type theory is actually based on *functions* rather than relations or classes, and incorporates certain features of the λ-calculus which he had already developed. It seems unlikely, given Bourbaki's well-known distaste for logic, that he would have known of Church's contributions, and even if he had, he would likely have regarded it as irrelevant to his concerns.

In any case, as already remarked, Bourbaki's general concept of structure, organized into species, plays only a very minor role in his actual development of mathematics. By and large, only *specific* kinds of structure are discussed: e.g., topological spaces,

[5] E.g. in Corry (1992).

algebraic structures, and combinations of the two such as topological groups. In practice, the role of structure in general is played by the defining axioms of the various species of structures.

2. Category Theory as a Theory of Mathematical Structure and Form

In the middle 1940s, a decade after the launch of the *Éléments,* Eilenberg (later to become a Bourbaki member) and Mac Lane invented *category theory*[6]. Their original intention was to systematize the construction of homology theories, a procedure in algebraic topology which involves the correlation of topological spaces and groups - two of Bourbaki`s mother structures. This correlation between *different* sorts of structure – the key idea underlying category theory - was termed by Eilenberg and Mac Lane a *functor.* The idea of a *category* was introduced to underpin the notion of functor by furnishing it (like a function) with a definite domain and range. They conceived functors as acting not just on the structures themselves but also on the `structure-preserving`` maps, or *morphisms,* between structures. Accordingly categories would have to contain these latter as well. The recognition that categories would have to incorporate as basic constituents not just structures but morphisms marks the fundamental advance of the category concept over Bourbaki's idea of species[7].

Eilenberg and Mac Lane rather played down the notion of category, stating:

It should be observed ... that the whole concept of a category is essentially an auxiliary one; our basic concepts are essentially those of a functor and of natural transformation (…). The idea of a category is required only by the precept that every function should have a definite class as domain and a definite class as range, for the categories are provided as the domains and ranges of functors.

But this view was to change. Starting in the 1950s, the category concept came to be perceived as a nascent embodiment of the idea of mathematical structure in general, in which Bourbaki's conception of mathematical structure as *individual* structures, defined in set-theoretic terms and only then organized into species, is replaced by the category of *all* such structures given in advance. The Bourbaki fraternity became uncomfortably aware that their program of structuralist grounding of mathematics might be better realized through the systematic use of category theory, but by then it was too daunting

---

[6] Eilenberg and Mac Lane (1945).
[7] Roughly speaking, Bourbaki's species of structures correspond to "mapless" categories.

a project to reconstruct the whole of their oeuvre in category- theoretic terms[8]. In any case it is far from clear - even today when category theory has assumed a commanding place in confirming the unity of mathematics- how this could actually have been done[9].

Category theory offers an account of mathematical structure far transcending that pioneered by Bourbaki, opening doors of conception whose very existence was previously undreamt of.

What is a category? Formally, a *category* **C** is determined by first specifying two assemblies $Ob(\mathbf{C})$, $Arr(\mathbf{C})$—the of **C**-*objects* and **C**-*arrows*, **C-** *morphisms, or* **C** -*maps.* These are subject to the following axioms:

- Each **C**-arrow f is assigned a pair of C-objects dom(f), cod(f) called the *domain* and *codomain* of f, respectively. To indicate the fact that **C**-objects X and Y are respectively the domain and codomain of f we write $f\colon X \to Y$ or $X \xrightarrow{f} Y$. The collection of **C-**arrows with domain X and codomain Y is written $\mathbf{C}(X, Y)$.

- Each **C**-object X is assigned a **C**-arrow $1_X\colon X \to X$ called the *identity arrow* on X.

- Each pair f, g of **C**-arrows such that cod(f) = dom(g) is assigned an arrow $g \circ f$: dom(f) $\to$ cod(g) called the *composite* of f and g. Thus if $f\colon X \to Y$ and $g\colon Y \to Z$ then $g \circ f\colon X \to Z$. We also write $X \xrightarrow{f} Y \xrightarrow{g} Z$ for $g \circ f$. Arrows f, g satisfying cod(f) = dom(g) are called *composable.*

- *Associativity law.* For composable arrows (f, g) and (g, h), we have $h \circ (g \circ f) = h \circ (g \circ f)$.

- *Identity law.* For any arrow $f\colon X \to Y$, we have $f \circ 1_X = f = 1_Y \circ f$.

The concept of category may be regarded as vastly generalized and streamlined, yet richer version of Bourbaki's concept of species of structures. While a Bourbakian species is composed solely of structures, the structures (objects) of a category comprise just half of its constituents, the structure-preserving maps (arrows) between them furnishing the other half. In this spirit we can think of a category as an explicit presentation – an

---

[8] It should be noted, however, in Chapter 4 of the *Théorie des Ensembles* Bourbaki does formulate versions of certain concepts - such as universal arrows and the solution set condition for their existence - which were later to become central to category theory. Mac Lane (1971) remarks that Bourbaki's formulation " was cumbersome because [their] notion of 'structure' did not make use of categorical ideas".

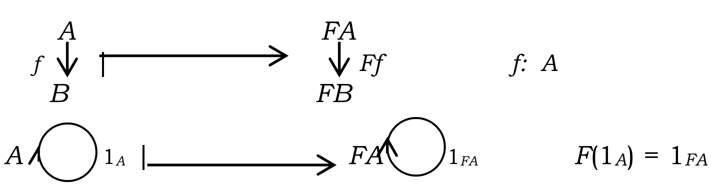
[9] A pioneering first step in this regard at an elementary level was undertaken by Lawvere and Schanuel in their work *Conceptual Mathemtics* (Lawvere and Schanuel 1997).

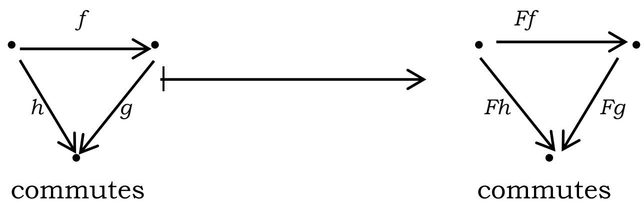
embodiment - of a *mathematical Form* or *Structure*, together with the various ways in which that Form is preserved under transformations. The objects of a category **C** are then naturally identified as *instances* of the associated Form $\mathscr{C}$ and the morphisms or arrows of **C** as transformations of such instances which in some specified sense ``preserves" the Form. If we *identify* categories with Forms, then the specification of a Form requires us to specify, along with its instances, the transformations which "preserve" it. This opens up the possibility that two Forms may have the same instances but *different* Form-preserving transformations. This is illustrated in the first three of the examples below:

Category/Instances of Form	***Form***	***Transformations***
Sets	Pure Discreteness	Functional Correlations
Sets with a Distinguished Point (DP)	Pure Discreteness	DP- Preserving Functional Correlations
Sets with Partial Maps	Pure Discreteness	Functional Correlations on Parts
Groups	Composition/Inversion	Homomorphisms
Topological Spaces	Continuity	Continuous Maps
Differentiable Manifolds	Smoothness	Smooth Maps

In this spirit a functor between two categories may be identified as a pair of correlations (satisfying certain simple conditions): the first between instances of the two Forms embodied by the given categories and the second between Form-preserving transformations of these instances. In short, a functor is a Form-preserving correlation between (the instances of) two Forms.

To be precise, a functor F**:** **C** $\to$ **D** between two categories **C** and **D** is a map that "preserves commutative diagrams", that is, assigns to each **C**-object A a **D**-object FA and to each **C-**arrow $f: A \to B$ a **D**-arrow $Ff: FA \to FB$ in such a way that:





$F(g \circ f) = Fg \circ Ff$

When categories are regarded as Forms, a functor between two Forms is a correlation between instances (transformations) of the first Form with instances (transformations) of the second which preserves composites of morphisms and identity morphisms. Functors considered as acting on Forms will be called *formorphisms.*

If the objects of a category are the instances of a given Form, when should two of these instances be regarded as *identical*? Precisely when they are *isomorphic* (Greek: *equal form*). In a category two objects, are seemed isomorphic, written $\cong$, if there is an invertible morphism, an *isomorphism,* from one to the other. In Bourbaki's formulation isomorphisms and isomorphic structures are defined set-theoretically in terms of bijections.

One of the principal aims of the structuralist approach to mathematics is to take seriously the idea that isomorphic structures should be regarded as in a fundamental sense *identical*. On the set-theoretic account of structures, this is not literally possible. In category theory, however, each category is equivalent (in a sense to be introduced below) to a *skeletal* category, one in which isomorphic objects are always identical. So, if we take the further step of identifying Forms with skeletal categories, isomorphic instances of Forms are literally identical.

If isomorphism is construed as identity of *instances* of a given Form, then how should expression be given to the idea of identity of Forms themselves? In Bourbaki's set-theoretic account of structures this question is never raised, nor would there seem to be any reasonable answer in Bourbaki's framework. But category theory deals with this question most elegantly, through the idea of *equivalence* of categories.

Given two categories **C** and **D**, a functor F: **C**→ **D** is an *equivalence* if it is "an isomorphism up to isomorphism", that is, if it is

- *faithful*: $Ff = Fg \Rightarrow f = g$.
- *full*: for any $h: FA \to FB$ there is $f: A \to B$ such that $h = Ff$.

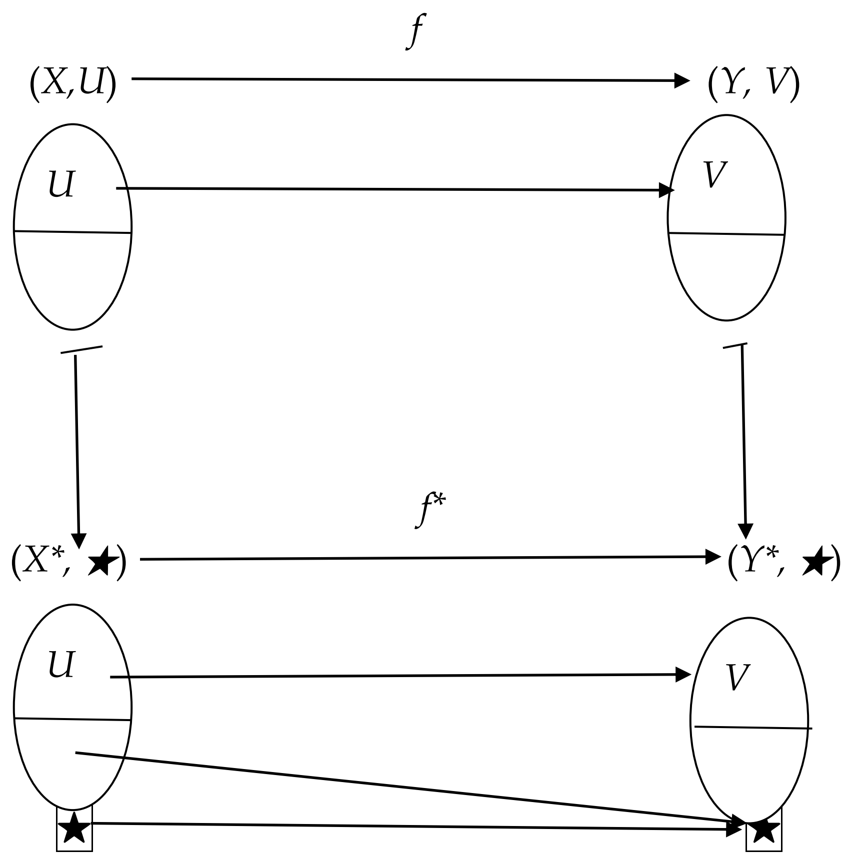
- *dense*: for any **D**-object B there is a **C**-object A such that $B \cong FA$.

If categories are regarded as Forms, then an equivalence between two Forms $\mathscr{C}$ **and** $\mathscr{D}$ is a formorphism from one Form to the other which is bijective on transformations and is such that each instance of $\mathscr{D}$ is isomorphic to the correlate of an instance of $\mathscr{C}$**.**

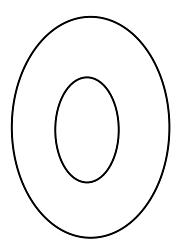
Two categories, or Forms are *equivalent,* written ≈, if there is an equivalence between them. Equivalence of Forms means that, considered purely as Forms, they can be taken as identical.

The idea of equivalence of Forms afforded by category theory is rich and deep. As a simple example, consider the two categories: Sets with Partial Maps (**SPM)** and Sets with a Distinguished Point (**SDP**). Objects of **SPM** are pairs of sets (X, U) with $U \subseteq X$ and an arrow $(X, U) \to (Y, V)$ between two such objects is just a function $f: U \to V$. Objects of **SDP** are *pointed sets,* i.e. pairs (X, a) with X a set and $a \in X$. An arrow $(X, a) \to (Y, b)$ between two such objects is a function $f: X \to Y$ such that $f(a) = b$.

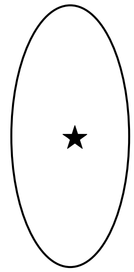
These two categories are equivalent. The equivalence correlates each object (X, U) of **SPM** with the pointed set $(X^*, \bigstar)$ where $X^* = X \cup \{\bigstar\}$ is the set obtained by adding a distinguished "point at infinity" ($\bigstar$) to X. Each arrow $f: (X, U) \to (Y, V)$ of **SPM** is correlated with the arrow $f^*: (X^*, \bigstar) \to (Y^*, \bigstar)$ in **SDP** defined by $f^*(x) = f(x)$ for $x \in U$, $f^*(x) = \bigstar$ for $x \notin U$. This is depicted below.



Adopting the language of Forms, the objects of the category **SPM** can be considered as instances of the Form *Whole and Part*, with transformations *strictly between parts.* (This is to be distinguished from transformations between wholes which preserve parts, which leads to a different category and Form.) The objects of **SDP** can be considered as instances of the Form *Whole and Distinguished Element,* with transformations between Wholes preserving distinguished elements. The equivalence of the two categories, and so of the associated Forms, means that Whole-Part Forms



are Formally the same as Whole-Individual Forms



in which Parts of Wholes are, so to speak, `shrunk`` to points.

Another important concept in category theory which has a satisfying formulation in terms of Forms is that of opposite, or mirror category. Given a category **C,** the *opposite,* or *mirror* category is defined to be the category $\mathbf{C}^{\cup}$ whose objects are those of **C** but whose arrows are those of **C** "reversed", or "viewed in a mirror', if you will. That is, the arrows $X \to Y$ in $\mathbf{C}^{\cup}$ are the arrows $Y \to X$ in **C.**

Simple examples of mirror categories are obtained by considering *preoredered* sets. A *preorder* on a set P is a reflexive transitive relation $\leq$ on P. A *preordered set* is a pair $\mathbf{P} = (P, \leq)$ consisting of a set P and a preorder $\leq$ on P. Preordered sets can be identified with categories in which there is at most one arrow between each pair of objects. Consider the preordered set $\mathbf{N} = (\mathbb{N}, \leq)$ where $\mathbb{N}$ is the set of natural numbers and $\leq$ is the usual equal to or less than relation on it. Regarding $\mathbf{N}$ as a category, its mirror category $\mathbf{N}^{\cup}$ may be identified with the ordered set of *negative* numbers.

Given a category **C** with associated Form $\mathscr{C}$, the *mirror* Form $\mathscr{C}^{\cup}$ is the Form associated with the mirror category $\mathbf{C}^{\cup}$. The Form $\mathscr{N}$ associated with the category $\mathbf{N}$ is *limitless succession.* Its mirror $\mathscr{N}^{\cup}$ may be called *limitless precession.*

For a given category (Form), the associated mirror category (Form) is usually difficult to identify as an autonomous category. But in certain important cases, mirror categories can be shown to be *equivalent* to naturally defined categories. Given two categories **C** and **D,** a *duality* between **C** and **D** is an equivalence between $\mathbf{C}^{\cup}$ and **D** (or, what amounts to the same thing, between **C** and $\mathbf{D}^{\cup}$**).**

3. Duality Theory for Commutative Rings

This sort of duality is the core of the important *representation theory for commutative rings.* Here is a brief history[10].

The concept of *commutative ring* (with identity) provides a basic link between algebra and geometry. Commutative rings arise naturally as *algebras of values of (intensive) quantities* over topological spaces. For example, consider the earth`s atmosphere **A**. There are many intensive quantities defined on **A** - temperature, pressure, density, (wind) velocity, etc. The real number values of these quantities varies continuously from point to point. In general, we can define a (continuously varying value of an) *intensive quantity* on **A** to be a continuous function on **A** to the field ℝ of real numbers. Intensive quantities construed in this way form an *algebra* in which *addition* and *multiplication* can be defined "pointwise": thus, given two intensive quantities f, g, the sum $f + g$ and the product fg are defined by setting, for each point x in **A,**

$$(f+g)(x) = f(x) + g(x) \qquad (fg)(x) = f(x)g(x).$$

In general, given a topological space X, we consider the set C(*X*) of continuous real-valued functions on X, with addition and multiplication defined pointwise as above. This turns C(X) into a *commutative ring,* the *ring of real-valued (continuously varying) intensive quantities over* X**.** We can also consider the subring C*(X) of C(X) consisting of all *bounded* members of C(X), the ring of *bounded intensive quantities over* X**.** When X is *compact*, C*(X) and C(X) coincide.

More generally, given any commutative topological ring T, the ring C(X, T) of continuous T **-** valued functions on X is called the *ring of* T*-valued intensive quantities on* X.

Given a commutative ring, it is natural to raise the question as to whether it can be represented as a ring of intensive quantities (with values in some commutative topological ring) on some topological space.

---

[10]See Johnstone (1986) for a full account of the representation theory for commutative rings.

It was *M. H. Stone* who provided the first answer to this question. In the celebrated *Stone Representation Theorem*, proved in the 1930s, he showed that each member of a certain class of commutative rings, the so-called *Boolean rings,* is representable as a ring of intensive quantities - with values in a fixed simple topological ring (2)- over a certain class of spaces - the *Boolean* or *Stone spaces.* A *Boolean ring* is defined to a ring in which every element is *idempotent*, $x^2 = x$ for every x. A totally disconnected compact Hausdorff space is called a *Boolean* space.

The Stone Representation Theorem establishes the duality between the category of Boolean rings and the category of Boolean spaces.

In 1940 Stone established what amounts to the duality between the category of compact Hausdorff spaces and the category of real C*-algebras - commutative rings equipped rings with an *order* structure and a *norm* naturally possessed by rings of bounded real-valued intensive quantities.

The Russian mathematician I. Gelfand and his collaborators proceeded in another direction, replacing the real field by the complex field ℂ, so introducing rings (or algebras) of *complex-valued* intensive quantities. The abstract versions of these are called *commutative complex C**-algebras. Gelfand and Naimark established a duality between the category of commutative complex C*-algebras and the category of compact Hausdorff spaces.

The representation of rings as rings of intensive quantities to has been extended to *arbitrary* commutative rings, leading to new dualities. In these representations the given commutative ring is represented as the globally defined elements of a collection of *varying* rings of intensive quantities in which the ring of values of the quantities varies with the point in the space - a so-called *ringed space* - at which the quantity is defined. This idea leads to the so-called *Grothendieck duality:*

the category of commutative rings is dual to a certain category of ringed spaces: the category of affine schemes.

The idea of duality, illustrated so beautifully in the case of commutative rings, and naturally and precisely expressed within category theory, transcends Bourbaki's set-based structuralist account of mathematics.

4. The Fate of the "Mother Structures" : Algebraic and Ordered Structues

In the transition from Bourbaki's account of mathematics to its category-theoretic formulation, what is the fate of the "mother structures"? It is remarkable that, given Bourbaki`s distaste for logic, their mother structures came to play a key role in establishing the connection between category theory and logic.

To begin with**, algebraic structures become *algebraic theories* as introduced by Lawvere.** Here the key insight was to view the logical operation of substitution in equational theories as composition of arrows in a certain sort of category. Lawvere showed how models of such theories can be naturally identified as functors of a certain kind, so launching the development of what has come to be known as *functorial semantics*[11].

An *algebraic theory* **T** is a category whose objects are the natural numbers and which for each m is equipped with an m-tuple of arrows, called *projections,* $\pi_i: m \to 1 \quad i = 1, \ldots, m$ making m into the m-fold power of 1: $m = 1^m$. (Here 1 is *not* a terminal object in **T**.)

In an algebraic theory the arrows $m \to 1$ play the role of m-ary operations. Consider, for example, the algebraic theory **Rng** of *rings*. To obtain this, one starts with the usual (language of) the first-order theory Rng of rings and introduces, for each pair of natural numbers (m, n) the set $P(m, n)$ of n-tuples of polynomials in the variables $x_1, \ldots, x_m$. The members of $P(m,n)$ are then taken to be the arrows $m \to n$ in the category **Rng**. Composition of arrows in **Rng** is defined as substitution of polynomials in one aother.

[11] See, e.g. Bell (2018).

The projection arrow π_i: $m \to 1$ is just the monomial x_i considered as a polynomial in the variables $x_1, \ldots, x_m$. Each polynomial in m variables, as an arrow $m \to 1$, may be regarded as an m-ary operation in **Rng**.

In a similar way every equational theory—groups, lattices, Boolean algebras—may be assigned an associated algebraic theory.

Now suppose given a category **C** with finite products. A *model* of an algebraic theory **T** in **C**, or a **T**-*algebra* in **C**, is defined to be a finite product preserving functor A: **T** → **C**. The full subcategory of the functor category $\mathbf{C}^T$ whose objects are all **T**-algebras is called the *category of* **T**-*models* or **T**-*algebras* in C, and is denoted by **Alg(*T*, C).**

For example, if **GRP** is the theory of groups, then a model of GRP in the category of topological spaces is a *topological group*; in the category of manifolds, a *Lie group*; and in a category of sheaves a *sheaf of groups*. In general, modelling a mathematical concept within a category amounts to a kind of refraction or filtering of the concept through the Form associated with the category.

When **T** is the algebraic theory associated with an equational theory *S,* the category of **T**-models in **Set**, the category of sets, is equivalent to the category of algebras axiomatized by *S.*

Lawvere later extended functorial semantics to *first-order* logic. Here the essential insight was that existential and universal quantification can be seen as left and right adjoints, respectively, of substitution.

To see how this comes about, consider two sets A and B and a map $f: A \to B$. The power sets PA and PB of A and B are partially ordered sets under inclusion, and so can be considered as categories. We have the map ("preimage") $f^{-1}: PB \to PA$ given by:

$$f^{-1}(Y) = \{x: f(x) \in Y\},$$

which, being inclusion-preserving, may be regarded as a *functor* between the categories PB and PA. Now define the functors $\exists_f$, $\forall_f$: $PA \to PB$ by

$$\exists_f(X) = \{y: \exists x(x \in X \wedge f(x) = y\} \quad \forall_f(X) = \{y: \forall x(f(x) = y \Rightarrow x \in X\}.$$

These functors $\exists_f$ ("image") and $\forall_f$ ("coimage"), which correspond to the existential and universal quantifiers, are easily checked to be respectively left and right adjoint to f^{-1}; that is, $\exists_f(X) \subseteq Y \Leftrightarrow X \subseteq f^{-1}(Y)$ and $f^{-1}(Y) \subseteq X \Leftrightarrow Y \subseteq \forall_f(X)$. Now think of the members of *PA* and *PB* as corresponding to *attributes* of the members of *A* and *B* (under which the attribute corresponding to a subset is just that of belonging to it), so that inclusion corresponds to entailment. Then, for any attribute *Y* on *B*, the definition of $f^{-1}(Y)$ amounts to saying that, for any $x \in A$, x has the attribute $f^{-1}(Y)$ just when $f(x)$ has the attribute *Y*. That is to say, the attribute $f^{-1}(Y)$ is obtained from *Y* by "substitution" along *f*. This is the sense in which quantification is adjoint to substitution.

Lawvere's concept of *elementary existential doctrine* presents this analysis of the existential quantifier in a categorical setting. Accordingly an elementary existential doctrine is given by the following data: a category **T** with finite products—here the objects of **T** are to be thought of as *types* and the arrows of **T** as *terms*—and for each object *A* of **T** a category **Att**(*A*) called the *category of attributes* of *A*. For each arrow $f: A \to B$ we are also given a functor **Att**(*f*): **Att**(*B*) $\to$ **Att**(*A*), to be thought of as substitution along *f*, which is stipulated to possess a left adjoint $\exists_f$ —existential quantification along *f*.

The category **Set** provides an example of an elementary existential doctrine: here for each set *A*, the category of attributes **Att**(*A*) is just *PA* and for $f: A \to B$, A(*f*) is f^{-1}. This elementary existential doctrine is *Boolean* in the sense that each category of attributes is a Boolean algebra and each substitution along maps a Boolean homomorphism.

Functorial semantics for elementary existential doctrines is most simply illustrated in the Boolean case. Thus a (set-valued) *model* of a Boolean elementary existential doctrine (T, **Att**) is defined to be a product preserving functor *M:* **T** $\to$ **Set** together with, for each object *A* of **T**, a Boolean homomorphism **Att**(*A*) $\to$ *P*(*MA*) satisfying certain natural compatibility conditions.

This concept of model can be related to the usual notion of model for a first-order theory *T* in the following way. First one introduces the so-called "Lindenbaum" doctrine of *T*: this is the elementary existential doctrine (**T**, A) where **T** is the algebraic theory whose arrows are just projections among the various powers of 1 and in which **Att**(*n*) is the Boolean algebra of equivalence classes modulo provable equivalence from *T* of formulas having free variables among $x_1, \ldots, x_n$. For $f : m \to n$, the action of **Att**(*f*) corresponds to syntactic substitution, and in fact $\exists_f$ can be defined in terms of the syntactic $\exists$. Each model of T in the usual sense gives rise to a model of the corresponding elementary existential doctrine (**T**, A).

Ordered structures become identified with categories having at most one arrow between any pair of objects. But they have a further significance, as we shall see.

5. The Fate of the "Mother Structures" : Topological Structures

Topological structures become associated, not with the category of topological spaces, but with the category of *sheaves over a topological space,* the archetypal example of a *topos*[12] in the sense of Grothendieck. Grothendieck saw toposes as "generalized spaces". A sheaf over a topological space may be thought of as a set `varying continuously` over the space. The construction of the topos of sheaves (or presheaves) over a space T depends not on the elements of the underlying set of T, but only on the topology of T, that is, the partially ordered set of opens of T - a so-called *locale* or *pointless space.* In this way ordered structures come to replace topological structures in the construction of sheaf toposes. Observing that ordered structures are themselves categories, Grothendieck generalized these to the concept of a *site,* a (small) category together with a notion of covering, and further extended the concept of a sheaf over a topological space to that of sheaf over a sit. A *Grothendieck topos* is the category of sheaves over a site.

Lawvere and Tierney later generalized Grothendieck toposes to *elementary* toposes. These are the categorical counterparts to *higher-order* logic.

An (elementary) *topos* may be defined as a category possessing a terminal object, products, exponentials, and a truth-value object. Here a *truth-value object* is an object S such that, for each object A, there is a natural correspondence between subobjects of A and arrows $A \to$ S. (Just as, in set theory, for each set X, there is a bijection between subsets of A and arrows $A \to 2$.)

The system of higher-order logic associated with a topos is a generalization of classical set theory within intuitionistic logic: *intuitionistic type theory,* or as it is sometimes called *local set theory.*

[12] See, e.g. Mac Lane and Moerdijk (1992) for an account of topos theory.

The category of sets is a prime example of a topos, and the fact that it is a topos is a consequence of the axioms of classical set theory. Similarly, in a local set theory the construction of a corresponding “category of sets” can also be carried out and shown to be a topos. In fact *any* topos is obtainable (up to equivalence of categories) as the category of sets within some local set theory. Toposes are also, in a natural sense, the *models* or *interpretations* of local set theories. Introducing the concept of *validity* of an assertion of a local set theory under an interpretation, such interpretations are *sound* in the sense that any theorem of a local set theory is valid under every interpretation validating its axioms and *complete* in the sense that, conversely, any assertion of a local set theory valid under every interpretation validating its axioms is itself a theorem. The basic axioms and rules of local set theories are formulated in such a way as to yield as theorems precisely those of higher-order intuitionistic logic. These basic theorems accordingly coincide with those statements that are valid under *every* interpretation.

Once a mathematical comcept is expressed within a local set theory, it can be interpreted in an arbitrary topos. This leads to what I have called i*local mathematics*: here mathematical concepts are held to possess references, not within a fixed absolute universe of sets, but only *relative* to toposes. Absolute truth of mathematical assertions comes then to be replaced by the concept of *invariance*, that is, "local" truth in every topos, which turns out to be equivalent to constructive provability.

In category theory, the concept of *transformation* (morphism or arrow) is an irreducible basic datum. This fact makes it possible to regard arrows in categories as formal embodiments of the idea of *pure variation* or *correlation*, that is, of the idea of *variable quantity* in its original pre-set-theoretic sense. For example, in category theory the variable symbol x with domain of variation X is interpreted as an *identity arrow* (1_X), and this concept is not further analyzable, as, for instance, in set theory, where it is reduced to a set of ordered pairs. Thus the variable x now suggests the idea of pure variation over a domain, just as intended within the usual functional notation $f(x)$. This latter fact is expressed in category theory by the "trivial" axiomatic condition

$$f \circ 1_X = f,$$

in which the symbol x does not appear: this shows that variation is, in a sense, an intrinsic constituent of a category.

In a topos the notion of pure variation is combined with the fundamental principles of construction employed in ordinary mathematics through set theory, viz., forming the *extension of a predicate, Cartesian products*, and *function spaces*. In a topos, as in set theory, every object—and indeed every arrow—can be considered in a certain sense as the extension $\{x: P(x)\}$ of some predicate P. The difference between the two situations is that, while in the set-theoretic case the variable x here can be construed *substitutionally*, i.e. as ranging over (names for) individuals, in a general topos this is no longer the case: the "x" must be considered as a *true variable*. More precisely, while in set theory the rule of inference

$$\frac{P(a) \text{ for every individual } a}{\forall x P(x)}$$

is valid, in general this rule fails in the internal logic of a topos. In fact, assuming classical set theory as metatheory, the correctness of this rule in the internal logic of a topos forces it to be a model of classical set theory: this result can be suitably reformulated in a constructive setting.

In Bourbaki's *Éléments* set theory provides the "raw materials" for the fashioning of mathematical structures, just as stone or clay constitute the materials from which the sculptor's creations are fashioned. In category theory, on the other hand, mathematical structures are not built from sets: they are given *ab initio.* For this reason category theory does much more than merely reorganize the mathematical materials furnished by set theory: its role far transcends the purely cosmetic. This is strikingly illustrated by the various topos models of *synthetic differential geometry* or *smooth infinitesimal analysis*. Here we have an explicit presentation of the Form of the smoothly continuous incorporating actual infinitesimals which is simply *inconsistent* with classical set theory: a form of the continuous which, in a word, *cannot* be reduced to discreteness. In these models, *all* transformations are smoothly continuous, realizing Leibniz's dictum *natura non facit saltus* and Weyl's suggestions in *The Ghost of Modality* and elsewhere. Nevertheless, extensions of predicates, and other mathematical constructs, can still be formed in the usual way (subject to intuitionistic logic). Two startling features of continuity then make their appearance. First, connected continua are *cohesive:* no connected continuum can be split into two disjoimt nonempty parts, echoing Anaxagoras' c. 450 B.C. assertion that the (continuous) world has no parts which can be "cut off by an axe". And, even more importantly, any curve can be regarded as being traced out by the motion, not just of a point, but of an *infinitesimal tangent vector*—an entity embodying the (classically unrealizable) idea of *pure direction*—thus allowing the direct development of the calculus and differential geometry using nilpotent infinitesimal quantities. These near-miraculous, and yet natural ideas, which *cannot* be dealt with coherently by reduction to the discrete or the notion of "set of distinct individuals" (cf. Russell, who in *The Principles of Mathematics* roundly condemned infinitesimals as "unnecessary, erroneous, and self-contradictory"), can be explicitly formulated in category-theoretic terms and developed using a formalism resembling the traditional one.

5. Conclusion

Finally, let me return to Bourbaki's *Élements.* In writing their masterwork the fraternity's members saw themselves qs both sculptors and architects of mathematics. As sculptors, they used set theory to provide the "clay" from which the individual mathematical structures - the "sculptures", so to speak – which were to be exhibited within the grand mathematical edifice (museum, even) they aimed to build. As architects, they constructed this edifice from the same set-theoretic "clay" as the sculptures. Yet at the same time they insisted that, once sculptures and edifice had been formed, the raw materials used in their production could be consigned to oblivion. This is the lordly attitude of the master architect concerned only with the grand design, who in the end ignores the constitution of the bricks from which his edifices are built, as opposed to the sculptor who has a much more intimate relationship – "hands on", so to speak -with the materials from which her creations are shaped.

By contrast, category theorists - those, at least, who are sensitive to such issues (and those constitute the majority) - regard set theory as a kind of ladder leading from pure discreteness to the depiction of the mathematical landscape in terms of pure Form. Categorists are no different from artists in finding the landscape (or its depiction, at least) more interesting than the ladder, which should, following Wittgenstein's advice, be jettisoned after ascent.

References

Bell, John L. (2018) *Categorical logic and model theory.* In *Categories for the Working*

Philosopher, E. Landry, ed. Oxford University Press.

Bourbaki, N. (1939 -) *Éléments de Mathématique.* 10 volumes. Paris: Hermann.

Bourbaki, N. (1950) *The Architecture of Mathematics,* American Mathematical Monthly **67**, 221-232.

Bourbaki, N. (1994) *Éléments d'histoire des mathématiques.* Berlin: Springer.

Corry, L. (1992) *Bourbaki and the concept of mathematical structure.* Synthese **92,** no.3, 311-348.

Eilenberg, S., and Mac Lane, S. (1945) *General theory of natural equivalences*. Trans. Amer. Math. Soc. **58**, 231–294. Reprinted in Eilenberg, S. and Mac Lane, S. (1986) *Eilenberg-Mac Lane: Collected Works.* New York: Academic Press.

Johnstone, P. T. (1896) *Stone Spaces.* Cambridge University Press.

Lawvere, F. W. and Schanuel, S. (1997) *Conceptual Mathematics.* Cambridge University Press.

Mac Lane, S. (1971) *Categories for the Working Mathematician.* Berlin: Springer.

Mac Lane, S. and Moerdijk, I. (1992) *Sheaves in Geometry and Logic; A First Introduction to Topos Theory.* Berlin: Springer.

Marshaal. M. *(2006) Bourbaki: A Secret Society of Mathematicians.* American Mathematical Society.