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ON THE RELATIONSHIP BETWEEN WEAK COMPACTNESS IN  $L_{\omega_1, \omega}$ ,  
 $L_{\omega_1, \omega_1}$ , AND RESTRICTED SECOND-ORDER LANGUAGES\*

By J. L. BELL, London

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1. *Introduction.* Let  $\kappa$  be a cardinal. A set of sentences  $\Sigma$  of a language  $L$  is said to be  $\kappa$ -consistent if each subset of  $\Sigma$  of power  $< \kappa$  has a model. Further,  $L$  is said to be  $\kappa$ -compact if each  $\kappa$ -consistent set of sentences of  $L$  of power  $\kappa$  has a model. In this paper, a sequel to [1], we investigate the property of  $\kappa$ -compactness for the following languages:

- (i) the infinitary languages  $L_{\omega, \omega}$  and  $L_{\omega, \omega_1}$ ;
- (ii) the restricted second-order language  $L(\aleph_1)$  (called  $W_1$  in [1]) which has set variables ranging over countable sets of individuals; and
- (iii) the language  $L(WF)$  for well-founded relations.

Our main result is that, for inaccessible  $\kappa$ , any one of these languages is  $\kappa$ -compact if and only if the others are too. For  $L_{\omega, \omega}$  this result is perhaps somewhat surprising in view of its relatively limited expressive power and the observation that it behaves quite well when only countable sets of sentences are considered.

2. *Terminology and notation.* Throughout this paper the symbol " $L$ " will be used to denote a first-order language with arbitrarily many nonlogical symbols, including in every case an identity symbol " $\approx$ " and a membership symbol " $\in$ ". The infinitary extensions  $L_{\omega, \omega}$  and  $L_{\omega, \omega_1}$  of  $L$  are obtained in the usual way [3]: in  $L_{\omega, \omega}$  one allows countable conjunctions and finite quantifications;  $L_{\omega, \omega_1}$  admits countable quantifications as well. In  $L_{\omega, \omega_1}$  a sentence  $WF$  asserting the well-foundedness of  $\varepsilon$  can be written down, namely

$$\neg \exists (x_n)_{n \in \omega} [ \bigwedge_{n \in \omega} (x_{n+1} \in x_n) ].$$

The language  $L(WF)$  for well-founded relations may now be defined as the smallest class  $F$  of formulas of  $L_{\omega, \omega_1}$  containing  $WF$  and every formula of  $L$ , and such that

$$\varphi, \psi \in F \Rightarrow \neg \varphi, \varphi \wedge \psi, \exists x \varphi \in F.$$

The restricted second-order language  $L(\aleph_\alpha)$  is built up by adding to  $L$  a set  $\{V_n : n \in \omega\}$  of one-place predicate variables which are interpreted as ranging over sets of individuals of power  $< \aleph_\alpha$ . Notice that a sentence  $WF'$  equivalent (under standard interpretations) to  $WF$  can be written down in  $L(\aleph_1)$ , namely,

$$\forall V_0 [ \exists x V_0(x) \rightarrow \exists x [ V_0(x) \wedge \forall y [ V_0(y) \rightarrow \neg (y \in x) ] ] ].$$

Evidently  $L(WF)$  is translatable into  $L(\aleph_1)$  and  $L(\aleph_1)$  into  $L_{\omega, \omega_1}$ .

We use freely the following well-known theorem of Mostowski: if  $A$  is a set,  $R \subseteq A \times A$ , and  $\langle A, R \rangle$  is a model of  $WF$  (or  $WF'$ ) and the axiom of extensionality, then  $\langle A, R \rangle$  is isomorphic to a transitive structure  $\langle B, \in \rangle$  where  $\in$  is the usual membership relation on  $B$ .

Finally, if  $\kappa$  is a cardinal, a family of sets  $F$  is said to have the  $\kappa$ -intersection property if  $\bigcap A \neq \emptyset$  for all  $A \subseteq F$  such that  $\text{Card}(A) < \kappa$ . A field of subsets or a filter in such a field is said to be  $\kappa$ -complete if it is closed under intersections of power  $< \kappa$ .

3. *The main theorem.* We now prove the

*Theorem.* Let  $\kappa$  be a (strongly) inaccessible cardinal  $> \aleph_0$ . The following conditions are equivalent:

- (i)  $L_{\omega_1, \omega_1}$  is  $\kappa$ -compact;
- (ii)  $L(\aleph_1)$  is  $\kappa$ -compact;
- (iii)  $L(WF)$  is  $\kappa$ -compact;
- (iv)  $L_{\omega_1, \omega}$  is  $\kappa$ -compact;
- (v) if  $M$  is a transitive set of power  $\kappa$  and  $X$  is a subset of  $M$  with the  $\kappa$ -intersection property, then there is an elementary embedding  $j$  of  $\langle M, \varepsilon \rangle$  into a transitive structure  $\langle N, \varepsilon \rangle$  such that  $\bigcap j''X \neq \emptyset$ ;
- (vi) if  $A$  is a set of power  $\kappa$ , and  $B$  is a  $\kappa$ -complete field of subsets of  $A$  of power  $\kappa$ , then each  $\kappa$ -complete filter in  $B$  can be extended to an  $\omega_1$ -complete ultrafilter in  $B$ .

*Proof.*

(i)  $\Rightarrow$  (ii) and (ii)  $\Rightarrow$  (iii) are obvious as is (i)  $\Rightarrow$  (iv). We prove (iv)  $\Rightarrow$  (vi), and (iii)  $\Rightarrow$  (v)  $\Rightarrow$  (vi)  $\Rightarrow$  (i).

(iv)  $\Rightarrow$  (vi). Assume (iv); let  $A$  be a set of power  $\kappa$ , let  $B = \{B_\xi : \xi < \kappa\}$  be a field of subsets of  $A$  of power  $\kappa$ , and suppose  $F$  is a  $\kappa$ -complete filter in  $B$ . Let  $\Sigma$  be the union of the  $L_{\omega_1, \omega}$  theory of  $\langle A, (B_\xi)_{\xi < \kappa}, (a)_{a \in A} \rangle$  with the sentences  $\{P_\xi(c) : B_\xi \in F\}$  where  $P_\xi$  is a unary predicate corresponding to  $B_\xi$  and  $c$  is a new constant. Since  $F$  has the  $\kappa$ -intersection property,  $\Sigma$  is  $\kappa$ -consistent. Certainly  $\text{Card}(\Sigma) = \kappa$ , so  $\Sigma$  has a model  $\mathfrak{C} = \langle C, (C_\xi)_{\xi < \kappa} \rangle$ . (We have suppressed reference to the constants of  $L_{\omega_1, \omega}$ ; similar suppressions are made afterwards.) Let  $t$  be the interpretation of  $c$  in  $\mathfrak{C}$ . Since  $\mathfrak{A} <_{\omega_1, \omega} \mathfrak{C}$  up to isomorphism, it is easy to see that the set  $F' = \{B_\xi : t \in C_\xi\}$  is an  $\omega_1$ -complete ultrafilter in  $B$  extending  $F$ .

(iii)  $\Rightarrow$  (v). Assume (iii); let  $A$  be a transitive set of power  $\kappa$ , and  $X$  a subset of  $A$  with the  $\kappa$ -intersection property. For each  $a \in A$  let  $c_a$  be a constant denoting  $a$  in  $A$ , and let  $c$  be a new constant. Let  $\Sigma$  denote the union of the  $L(WF)$  theory of  $\langle A, \varepsilon, (a)_{a \in A} \rangle = \mathfrak{A}$  with the sentences  $\{(c \in c_x) : x \in X\}$ . Then  $\Sigma$  is  $\kappa$ -consistent and of power  $\kappa$ , so has a model  $\mathfrak{B}$ . Since  $\mathfrak{B}$  is a model of  $WF$  and the axiom of extensionality, by Mostowski's theorem we may suppose  $\mathfrak{B} = \langle B, \varepsilon \rangle$  where  $B$  is transitive. The map  $j : \mathfrak{A} \rightarrow \mathfrak{B}$  defined by  $j(a) = c_a^{(\mathfrak{B})}$  is an elementary embedding, and  $c^{(\mathfrak{B})} \in j(X)$  for each  $x \in X$ , i.e.  $\bigcap j''X \neq \emptyset$  as required.

(v)  $\Rightarrow$  (vi). Suppose (v) holds; let  $B$  be a  $\kappa$ -complete field of subsets of power  $\kappa$  of a set  $A$ , also of power  $\kappa$ , and let  $F$  be a  $\kappa$ -complete filter in  $B$ . Without loss of generality we may assume  $A = \kappa$ . Let  $M$  be the transitive closure of  $B \cup {}^\omega B$ ; then  $\text{Card}(M) = \kappa$  and  $\kappa + 1 \subseteq M$ . Let  $j$  be an elementary embedding of  $\langle M, \varepsilon \rangle$  into a transitive  $\langle N, \varepsilon \rangle$  with some  $a \in \bigcap F$ . Defining  $F' = \{x \in B : a \in j(x)\}$  yields an  $\omega_1$ -complete ultrafilter in  $B$  containing  $F$ . For it is a simple matter to verify that  $F'$  is an ultrafilter in  $B$  containing  $F$ . To see that  $F'$  is  $\omega_1$ -complete, suppose  $\{x_n : n \in \omega\} \subseteq F'$ . To show that  $\bigcap_{n \in \omega} x_n \in F'$  it is clearly sufficient to show that  $j(\bigcap_{n \in \omega} x_n) = \bigcap_{n \in \omega} j(x_n)$ , i.e.  $j(\bigcap_{n \in \omega} g(n)) = \bigcap_{n \in \omega} j(g(n))$  where  $g \in {}^\omega B$  is defined by  $g(n) = x_n$  for each  $n \in \omega$ . Since  ${}^\omega B$  is a subset of  $M$ ,  $g \in M$ . Let  $b = \bigcap_{n \in \omega} x_n$ ; then, since  $M$  is transitive, we have

$$\langle M, \varepsilon \rangle \models \forall x [x \in b \leftrightarrow \forall n \in \omega [x \in g(n)]]$$

so that, since  $j$  is elementary,

$$\langle N, \varepsilon \rangle \models \forall x [x \in j(b) \leftrightarrow \forall n \in \omega [x \in j(g(n))]].$$

But since  $N$  is transitive we must have  $j(n) = n$  for each  $n \in \omega$  and  $j(\omega) = \omega$ . (The natural numbers and  $\omega$  are absolute.) Furthermore, by elementarity of  $j$ , we have

$$j(g(n)) = j(g(j(n))) = j(g(n)),$$

for each  $n \in \omega$ . Hence

$$\langle N, \varepsilon \rangle \models \forall x [x \in j(b) \leftrightarrow \forall n \in \omega [x \in j(g(n))]].$$

By transitivity of  $N$ , the statement on the right holds absolutely i.e.

$$j(b) = \bigcap_{n \in \omega} j(g(n)).$$

$\omega_1$ -completeness of  $F'$  follows.

(vi)  $\Rightarrow$  (i). Assume (vi), and let  $\Sigma = \{\sigma_\xi : \xi < \kappa\}$  be a  $\kappa$ -consistent set of sentences of  $L_{\omega_1, \omega_1}$  of power  $\kappa$ . By introducing (infinitary) Skolem functions we may suppose without loss of generality that  $\Sigma$  is a set of universal sentences. For each  $\Delta \in S_\kappa(\kappa)$  (here  $S_\kappa(\kappa)$  is the family of subsets of  $\kappa$  of power  $< \kappa$ ) let  $\mathfrak{A}_\Delta$  be a model of  $\{\sigma_\xi : \xi \in \Delta\}$  in which each of the new function symbols denotes a function. Let  $D$  be a subset of the domain of  $\prod \{\mathfrak{A}_\Delta : \Delta \in S_\kappa(\kappa)\}$  of power  $\kappa$  which is closed under the denotations of the Skolem functions (defined pointwise) in this product. For each open formula  $\varphi(v_\xi)_{\xi < \alpha}$  with  $\alpha < \omega_1$  of  $L_{\omega_1, \omega_1}$  and each sequence  $f = \langle f_\xi : \xi < \alpha \rangle$  from  $D$  let

$$J_{\varphi, f} = \{\Delta \in S_\kappa(\kappa) : \mathfrak{A}_\Delta \models \varphi[\langle f_\xi(\Delta) : \xi < \alpha \rangle]\}.$$

And for each  $\Delta \in S_\kappa(\kappa)$  let  $\hat{\Delta} = \{\Delta' \in S_\kappa(\kappa) : \Delta \subseteq \Delta'\}$ . Let  $B$  be the  $\kappa$ -complete field of subsets of  $S_\kappa(\kappa)$  generated by all  $\hat{\Delta}$  and all  $J_{\varphi, f}$ . The  $\text{Card}(B) = \text{Card}(S_\kappa(\kappa)) = \kappa$ . Now  $\{\hat{\Delta} : \Delta \in S_\kappa(\kappa)\}$  has the  $\kappa$ -intersection property and by (vi) the

ultrafilter  $D$  on a measurable cardinal. Also, it has been pointed out by P. Aczel that, by contrast,  $ZF +$  “there exists a strongly compact cardinal” + “there exist arbitrarily large inaccessibles”  $\vdash \neg P$ .

#### REFERENCES

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