A Representation Theory for Modalized Distributive Lattices

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1. $\Box$-lattices and $\Box$-algebras.

By a *lattice* we shall always mean a *distributive* lattice which is *bounded*, i.e. has both a bottom element 0 and a top element 1. Lattice *homomorphisms* will always be assumed to preserve 0 and 1.

By a *modality* on a (distributive) lattice $L = (L, \wedge, \vee, \leq, 0, 1)$ is meant a map $\Box: L \to L$ satisfying

(1) $\Box 1 = 1$,

(2) $\Box(x \wedge y) = \Box x \wedge \Box y$

for $x, y \in L$. The pair $(L, \Box)$ will be called a *modalized* (distributive) *lattice*, or simply a $\Box$-*lattice.* If $B = (B, \wedge, \vee, \bigstar, \leq, 0, 1)$ is a Boolean algebra, and $\Box$ a modality on B, the pair $(B, \Box)$ will be called a *modalized Boolean algebra*, or simply a $\Box$-*algebra.*

A $\Box$-*morphism* between $\Box$-lattices $(L, \Box)$ and $(M, \triangle)$ is a homomorphism $h: L \to M$ such that $h(\Box x) = \triangle h(x)$ for all $x \in L$. $\Box$-lattices constitute the objects, and $\Box$-morphisms the arrows, of a category $\Box$**Lat**. The full subcategory of $\Box$**Lat** with objects all Boolean algebras is denoted by $\Box$**Bool**.

If L is a Heyting algebra (in particular, a Boolean algebra), we write $\Rightarrow$ for the relative pseudocomplementation operation in L: thus for $x, y \in L$, $x \Rightarrow y$ is the largest element $z \in L$ for which $x \wedge z \leq y$. We also write $x \Leftrightarrow y$ for $(x \Rightarrow y) \wedge (y \Rightarrow x)$. It is easily shown that a self-map h on a Heyting algebra L is a modality if and only if it satisfies $h(1) = 1$ and $h(x \Rightarrow y) \leq h(x) \Rightarrow h(y)$.

By a *filter* in a lattice L we mean a subset $F \subseteq L$ such that 1) $x, y \in F \to x \wedge y \in F$, 2) $x \in F, x \leq y \to y \in F$. *A* filter F in L is *proper* if $F \neq 1$, or equivalently, if $0 \notin F$. $\{1\}$ is a filter called the *trivial filter.* (Dually, an *ideal* in L is a subset $I \subseteq L$ such that $x, y \in I \rightarrow x \vee y \in I$; $x \in I, y \leq x \rightarrow y \in I$; I is *proper* if $1 \notin I$.)

If $X \subseteq L$ the set

$$\{y: \exists x_1 \ldots \exists x_n \in X.\ x_1 \wedge \ldots \wedge x_n \leq y\}$$

(resp.

$$\{y: \exists x_1 \ldots \exists x_n \in X.\ y \leq x_1 \vee \ldots \vee x_n\})$$

is the least filter (resp., ideal) containing X; it is called the filter (ideal) *generated* by X. It is proper iff for each finite subset $\{x_1, \ldots, x_n\}$ of X, $x_1 \wedge \ldots \wedge x_n \neq 0$ (resp. $x_1 \vee \ldots \vee x_n \neq 1$). The filter (ideal) generated by $\{a\}$, for $a \in L$, is written F_a (resp. I_a) thus $F_a = \{x \in L: a \leq x\}$ and $I_a = \{x \in L: x \leq a\}$. Filters (ideals) of the form F_a (I_a) with $a \neq 0$ ($a \neq 1$) are called *principal* filters (ideals). A filter is *prime* if it is proper and if $x \vee y \in F$ implies $x \in F$ or $y \in F$ for all $x, y \in L$. If L is distributive and bounded, then every proper filter is the intersection of the family of proper filters that contain it. As a consequence, two elements are the same iff they are contained in the same prime filters. (For all these facts, see [4].)

It is easy to see that if F is a filter in a $\Box$-lattice $(L, \Box)$, then

$$\Box^{-1}F =_{df} \{x: \Box x \in F\}$$

is also a filter (not necessarily proper) in L. A filter F in $(L, \Box)$ is said to be $\Box$-*prime* if there is a prime filter P in L such that $F = \Box^{-1}P$. Write $\overline{\Box}F$ for the filter generated by $\{\Box x: x \in F\}$. We say that F is $\Box$-*disjunctive* if for any $x_1, \ldots, x_n \in L$,

$$\Box x_1 \vee \ldots \vee \Box x_n \in \overline{\Box}F \rightarrow x_i \in F \text{ for some } i.$$

If the trivial filter is $\Box$-disjunctive, that is, if for any $x_1, \ldots, x_n \in L$,

$$\Box x_1 \vee \ldots \vee \Box x_n = 1 \rightarrow x_i = 1 \text{ for some } i,$$

we shall say that $(L, \Box)$ is *disjunctive.*

1.1. Proposition. For any proper filter F in L, the following are equivalent:

(i) F is $\Box$-prime.

(ii) F is $\Box$-disjunctive.

Proof. **(i)** $\rightarrow$ **(ii)**. Assume $F = \Box^{-1}P$ for some prime P and $\Box x_1 \vee \ldots \vee \Box x_n \in \overline{\Box}F$. Now $\overline{\Box}F = \overline{\Box}(\Box^{-1}P) \subseteq P$, so $\Box x_1 \vee \ldots \vee \Box x_n \in P$, whence $\Box x_i \in P$ for some I, so that $x_i \in F$.

(ii) → (i). Suppose that F is $\square$-disjunctive. Then the set $\{\square x : x \notin F\}$ generates a proper ideal I disjoint from $\overline{\square} F$. By a well-known result (see, e.g., [4], 9.13), there is a prime filter P in L containing $\overline{\square} F$ and disjoint from I. Then $\square^{-1}P = F$, since if $x \in \square^{-1}P$, then $\square x \in P$, so that $\square x \notin I$, whence $x \in F$, while if $x \in F$, then $\square x \in \overline{\square} F \subseteq P$, whence $x \in \square^{-1}P$. ■

In this connection we also note the following

1.2. Proposition. For any $\square$-lattice the following are equivalent:

(i) Every $\square$-prime filter is prime.

(ii) $\square(x \vee y) = \square x \vee \square y$ for all $x, y \in L$.

Proof. (i) → (ii). Assume **(i)** and let P be a prime filter in L. Then $\square^{-1}P$ is prime and we have

$$
\begin{aligned}
\square(x \vee y) \in P &\leftrightarrow x \vee y \in \square^{-1}P \\
&\leftrightarrow x \in \square^{-1}P \text{ or } y \in \square^{-1}P \\
&\leftrightarrow \square x \in P \text{ or } \square y \in P \\
&\leftrightarrow \square x \vee \square y \in P.
\end{aligned}
$$

Therefore $\square(x \vee y)$ and $\square x \vee \square y$ are contained in the same prime filters and are, accordingly, equal.

(ii) → (i). Assuming **(ii)**, if P is prime and $x \vee y \in \square^{-1}P$, then $\square x \vee \square y = \square x \vee \square y \in P$, so that $\square x \in P$ or $\square y \in P$, i.e $x \in \square^{-1}P$ or $y \in \square^{-1}P$. ■

A $\square$-lattice is said to be *well-filtered* (resp. *properly filtered, principally filtered, weakly filtered*) if every filter in it (resp. every proper filter, every principal filter, the trivial filter) is $\square$-prime.

1.3. Proposition. Let $(L, \square)$ be a $\square$-lattice.

(i) $(L, \square)$ is weakly filtered iff it is disjunctive.

(ii) $(L, \square)$ is principally filtered iff for any $a, x_1, \ldots, x_n \in L$,

$$\Box a \leq \Box x_1 \vee \ldots \vee \Box x_n \ \rightarrow a \leq x_i \text{ for some } i.$$

(iii) If $(L, \Box)$ is principally filtered, then $\Box$ is an injective map.

(iv) If $(L, \Box)$ is well filtered, then $\Box 0 \neq 0$ and the map $n \mapsto \Box^n 0$: $\omega \rightarrow L$ (where $\Box^n x = (\Box\circ\ldots\circ\Box)x$ (n times)) is injective, so that L is infinite.

Proof. **(i)** and **(ii)** are immediate consequences of **1.1**.

(iii) If $(L, \Box)$ is principally filtered, it follows from **(ii)** that $\Box x \leq \Box y \rightarrow x \leq y$. The injectivity of $\Box$ is an immediate consequence.

(iv) If $(L, \Box)$ is well filtered, there is a prime filter P for which $L = \Box^{-1}P$. In particular $0 \in \Box^{-1}P$ so that $\Box 0 \in P$ whence $\Box 0 \neq 0$. By **(iii)**, $\Box$ is injective (and order preserving), so $\Box 0 < \Box 0 < \Box^2 0 < \ldots$. ■

2. Ordered topological representation of $\Box$-lattices and $\Box$-algebras.

We now proceed to extend the well-known Priestley representation ([4], [8]) for distributive lattices to $\Box$-lattices. To do this we require some more definitions.

If $(X, \leq)$ is a partially ordered set, an *upper* (*lower*) *set* in X is a subset $U \subseteq X$ such that $x \in U, x \leq y \rightarrow y \in U$ (resp. $x \in U, y \leq x \rightarrow y \in U$). For any $A \subseteq X$, we write $A\uparrow$ for the upper set $\{x \in X: \exists a \in A.\ a \leq x\}$ and $a\uparrow$ for $\{a\}\uparrow$.

If X is a topological space, we write **C**X for the Boolean algebra of clopen (= open-and-closed) subsets of X, and if in addition X carries a partial ordering, we write **A**X for the lattice of clopen upper sets in X.

By a *Boolean space* we mean a compact Hausdorff space X such that **C**X is a base for X. By an *ordered Boolean space* is meant a pair $(X, \leq)$ in which X is a Boolean space and $\leq$ a partial ordering on X such that for any $x, y \in X$ with $x \not\leq y$ there is $U \in$ **A**X with $x \in U$, $y \notin U$. (Under this latter condition we say, following [8] that $(X, \leq)$ is *totally order-separated.*)

We write **OBoolSp** for the category whose objects are all ordered Boolean spaces and whose arrows are all order preserving continuous maps between them. *Priestley's duality theorem* ([4], Ch. 10) asserts that **OBoolSp** is dual to the category **DLat** of distributive lattices and 0,1-preserving lattice homomorphisms. (Two categories **C** and **D** are *dual* if there is an equivalence between **C** and $\mathbf{D}^{op}$.)

We shall require the following fact about ordered Boolean spaces.

2.1. Lemma If A is a closed subset of an ordered Boolean space $(X, \leq)$, then $A\uparrow = \bigcap\{U\in\mathbf{A}X: A \subseteq U\}$, so that $A\uparrow$ is a closed upper set. In particular, if A is a closed upper set, $A = \bigcap\{U\in\mathbf{A}X: A \subseteq U\}$.

Proof. Clearly $A\uparrow \subseteq \bigcap\{U\in\mathbf{A}X: A \subseteq U\}$. For the reverse inclusion, we observe that, given $b \notin A\uparrow$, there is $U_x \in \mathbf{A}X$ such that $x \in U_x$, $b \notin U_x$. Since A is closed it is compact and so is covered by a finite family $\{U_{x_1},...,U_{x_n}\}$. Writing U for the union of this family, we have $A \subseteq U \in \mathbf{A}X$ and $b \notin U$. This proves the reverse inclusion. ■

A relation $\mathsf{R} \subseteq X \times X$ on an ordered Boolean space $(X, \leq)$ will be called *suitable* if

1) $\mathsf{R}[x] =_{df} \{y: (x, y) \in \mathsf{R}\}$ is a closed upper set for each $x \in X$,
2) for any $U \in \mathbf{A}X$, $\Box_{\mathsf{R}}U =_{df} \{x: \mathsf{R}[x] \subseteq U\} \in \mathbf{A}X$.

By an *ordered relspace* is meant a triple $(X, \leq, \mathsf{R})$ in which $(X \leq)$ is an ordered Boolean space and R is a suitable relation on $(X, \leq)$. A *relspace* is a pair (X, R) where X is a Boolean space and R is a suitable relation on the ordered Boolean space $(X, =)$. In other words, a relspace is a pair (X, R) in which X is a Boolean space and $\mathsf{R} \subseteq X \times X$ satisfies: $\mathsf{R}(x)$ is closed for each $x \in X$ and $\Box_{\mathsf{R}}U \in \mathbf{C}X$ for each $U \in \mathbf{C}X$.

By a *morphism of ordered relspaces* $(X, \leq, \mathsf{R}) \to (Y, \leq, \mathsf{S})$ is meant an order preserving continuous map f: $X \to Y$ such that $f[\mathsf{R}[x]]\uparrow = \mathsf{S}[f(x)]$ for all $x \in X$. (Here and in the sequel we write $f[A]$ for the image $\{f(x): x \in A\}$ of A under f.) A *morphism of relspaces* is a morphism of the ordered relspaces $(X, =, \mathsf{R}) \to (Y, =, \mathsf{S})$, i.e. a continuous map f: $X \to Y$ such that $f[\mathsf{R}[x]] = \mathsf{S}[f(x)]$ for $x \in X$.

Ordered relspaces (respectively, relspaces) and morphisms between them form a category **ORelSp** (respectively, **RelSp**).

We now state and prove our central duality theorem for $\Box$-lattices.

2.2. Theorem. $\Box$**Lat** and **ORelSp** are dual categories.

Proof. We define functors D: $\Box$**Lat** $\to$ **ORelSp**^{op} and E: **ORelSp** $\to$ $\Box$**Lat**^{op} which we show to be an equivalence of categories extending the Priestley duality between **DLat** and **OBoolSp**.

For $(L, \Box) = \mathfrak{L}$ in $\Box$**Lat,** $D\mathfrak{L}$ is the triple $(\mathcal{S}L, \subseteq, \mathsf{R})$ in which $\mathcal{S}L$ is the Stone space (= space of prime filters) of L, $\subseteq$ is the inclusion ordering in $\mathcal{S}L$ and $\mathsf{R} \subseteq \mathcal{S}L \times \mathcal{S}L$ is defined by

$$(P, Q) \in \mathsf{R} \leftrightarrow \Box^{-1}P \subseteq Q.$$

R is called the relation on $\mathcal{S}L$ *induced* by $\Box$.

Now $\mathcal{S}L$ has as a base the Boolean subalgebra of the power set $\mathbf{P}\mathcal{S}L$ of $\mathcal{S}L$ generated by the family of sets $\{u(x): x \in L\}$ where u: $L \to \mathbf{P}\mathcal{S}L$ is given by

$$u(x) = \{P \in \mathcal{S}L: x \in P\}.$$

This turns $\mathcal{S}L$ into a Boolean space, and it is easily seen that $(\mathcal{S}L, \subseteq)$ is an ordered Boolean space—the *Priestley space* of L. Moreover u is an isomorphism of L with $\mathbf{A}\mathcal{S}L$. (The proofs of the (duals of) these facts may be found in Ch. 10 of [4].)

For simplicity write X for $\mathcal{S}L$. We need to verify that R as defined above is a suitable relation on $(X, \subseteq)$. Clearly, for any $P \in X$, $\mathsf{R}[P]$ is an upper set (w.r.t. $\subseteq$). To see that $\mathsf{R}[P]$ is closed, we observe that, for $Q \in X$,

$$\begin{aligned} \Box^{-1}P \subseteq Q &\leftrightarrow \forall x \in \Box^{-1}P \,.\, x \in Q \\ &\leftrightarrow \forall x \in \Box^{-1}P \,.\, Q \in u(x). \end{aligned}$$

Therefore

$$\mathsf{R}[P] = \{Q: \Box^{-1}P \subseteq Q\} = \bigcap\{u(x): x \in \Box^{-1}P\}.$$

Since each $u(x)$ is closed, so is $\mathsf{R}[P]$.

To complete the proof of suitability of R, we need to show that $\Box_{\mathsf{R}}U \in \mathbf{A}X$ for each $U \in \mathbf{A}X$, and for this to be the case it suffices to show that, for any $x \in L$,

(2.3) $$\Box_{\mathsf{R}}u(x) = u(\Box x).$$

For any $P \in X$, $\square^{-1}P$, as a filter, is the intersection of the family of prime filters that include it, i.e.,

$$\square^{-1}P = \bigcap\{Q \in X: \square^{-1}P \subseteq Q\}.$$

For any $x \in L$, then,

$$x \in \square^{-1}P \leftrightarrow \forall Q \in X[\square^{-1}P \subseteq Q \to x \in Q\};$$

hence

$$\square x \in P \leftrightarrow \forall Q \in X[\square^{-1}P \subseteq Q \to x \in Q\},$$

so that

$$P \in u(\square x) \leftrightarrow \forall Q \in X[Q \in \mathsf{R}[P] \to Q \in u(x)\}$$

$$\leftrightarrow \mathsf{R}[P] \subseteq u(x)$$

$$\leftrightarrow P \in \square_{\mathsf{R}} u(x),$$

which immediately yields (2.3).

Accordingly (($\mathcal{S}L$, $\subseteq$, R) is an ordered relspace; we shall call it the *dual* of (L, $\square$).

For $h : \mathfrak{L} \to (M, \triangle) = \mathfrak{M}$ an arrow in $\square$**Lat**, we define Dh: $\mathcal{S}M \to \mathcal{S}L$ to be the stone dual of h given by

$$Dh(P) = h^{-1}[P]$$

for $P \in \mathcal{S}M$. Then Dh is a continuous $\subseteq$-preserving map; to show that it is an **ORelSp**-morphism $D\mathfrak{M} \to D\mathfrak{L}$ we argue as follows. Write $Y = \mathcal{S}M$ and let S be the relation on Y induced by $\triangle$. Then for $P \in Y$ the set $Dh[\mathsf{S}[P]]$ is closed in X as the image of the closed subset $\mathsf{S}[P]$ of Y under the continuous map Dh (recall that X and Y are compact Hausdorff). We need to show that $Dh[\mathsf{S}[P]]\uparrow$ coincides with the closed upper set $\mathsf{R}[Dh(P)]$. And by **2.1** to do this it suffices to show that the families of clopen upper sets—that is, sets of the form $u(x)$ for $x \in L$—containing $\mathsf{R}[Dh(P)]$ and $Dh[\mathsf{S}[P]]$ are the same. This follows from the ensuing chain of equivalences, in which we note that $h\square = \triangle h$ since h is a $\square$-morphism:

$$\mathsf{R}[Dh(P)] \subseteq u(x) \leftrightarrow \{Q \in X: \square^{-1} h^{-1}[P] \subseteq Q\} \subseteq u(x)$$

$$\leftrightarrow \{Q \in X: h^{-1}[\triangle^{-1} P] \subseteq Q\} \subseteq u(x)$$

$$\leftrightarrow \forall Q \in X(h^{-1}[\triangle^{-1} P] \subseteq Q\} \to x \in Q)$$

(a) $$\leftrightarrow x \in h^{-1}[\triangle^{-1} P]$$

$$\leftrightarrow \quad h(x) \in \triangle^{-1} P$$

(b) $$\leftrightarrow \; \forall Q \in Y(\triangle^{-1} P \subseteq Q \to h(x) \in Q)$$

$$\leftrightarrow \; \forall Q \in Y(\triangle^{-1} P \subseteq Q \to x \in h^{-1}[Q])$$

$$\leftrightarrow \; \{h^{-1}[Q]): \triangle^{-1} P \subseteq Q\} \subseteq u(x).$$

$$\leftrightarrow \; Dh(\mathsf{S}[P]) \subseteq u(x).$$

(At (a) and (b) above we have once again invoked the fact that any filter in a distributive lattice is the intersection of the family of prime filters that include it.)

We define E: **ORelSp** $\to$ □**Lat** as follows. If $(X, \leq, \mathsf{R}) = \mathscr{X}$ is an object of **ORelSp** let $E\mathscr{X}$ be the pair $(\mathbf{A}X, \Box_{\mathsf{R}})$. It is easily checked that $\Box_{\mathsf{R}}$ is a modality on $\mathbf{A}X$, so $E\mathscr{X}$ is an object of □**Lat** (called the *dual* of $\mathscr{X}$).

If $f\colon \mathscr{X} \to \mathscr{Y} = (Y, \leq, \mathsf{S})$ is an arrow in **ORelSp**, we let $Ef\colon \mathbf{A}Y \to \mathbf{A}X$ be the Stone dual of f, given by $Ef(U) = f^{-1}[U]$ for $U \in \mathbf{A}Y$. Then Ef is a □**Lat** morphism $E\mathscr{Y} \to E\mathscr{X}$, that is, a □-morphism $(\mathbf{A}Y, \Box_{\mathsf{S}}) \to (\mathbf{A}X, \Box_{\mathsf{R}})$, since, for any $x \in X$, $U \in \mathbf{A}Y$,

$$x \in Ef(\Box_{\mathsf{S}}U) \leftrightarrow x \in f^{-1}[\Box_{\mathsf{S}}U]$$

$$\leftrightarrow f(x) \in \Box_{\mathsf{S}}U$$

$$\leftrightarrow \mathsf{S}[f(x)] \subseteq U$$

$$\leftrightarrow f[\mathsf{R}[x]]{\uparrow} = \mathsf{S}[f(x)] \subseteq U$$

$$\leftrightarrow f[\mathsf{R}[x]] \subseteq U$$

$$\leftrightarrow \mathsf{R}[x] \subseteq f^{-1}[U]$$

$$\leftrightarrow x \in \Box_{\mathsf{R}} f^{-1}[U]$$

$$\leftrightarrow x \in \Box_{\mathsf{R}} Ef(U).$$

Hence $Ef(\Box_{\mathsf{S}}U) = \Box_{\mathsf{R}} Ef(U)$ as required.

Finally DE and ED are naturally isomorphic to identity functors. For given $(L, \Box) = \mathfrak{L}$ in □**Lat**, the Stone-Priestley isomorphism $u\colon L \to \mathbf{A}\mathcal{S}L$ is a □**Lat** isomorphism $\mathfrak{L} \to ED(\mathfrak{L})$ by (2.3). And if $\mathscr{X} = (X, \leq, \mathsf{R})$ is an ordered relspace, then the natural order-preserving homeomorphism $v\colon X \to \mathcal{S}\mathbf{A}X$ given by $v(x) = \{U \in \mathbf{A}X\colon x \in U\}$ (cf. [4], 10.19) is an isomorphism of the ordered relspaces $\mathscr{X}$ and $DE(\mathscr{X})$. For, writing S for the relation on $\mathcal{S}\mathbf{A}X$ induced by $\square_{\mathsf{R}}$, we have

$$v(y) \in \mathsf{S}[v(x)] \leftrightarrow \{U \in \mathbf{A}X\colon \{z\colon \mathsf{R}[z] \subseteq U\} \in v(x)\} \subseteq v(y)$$
$$\leftrightarrow \forall U \in \mathbf{A}X[\mathsf{R}[x] \subseteq U \to y \in U]$$
$$\leftrightarrow y \in \mathsf{R}[x],$$

the last sep following from **2.1** and the fact that R[x] is a closed upper set.
The proof is complete. ■

It is easy to see that the duality between □**Lat** and **ORelSp** restricts to a duality between □**Bool** and the full subcategory of **ORelSp** whose objects are of the form (X, =, R). Since the latter is (isomorphic to) **RelSp**, we obtain

2.4. Corollary. □**Bool** and **RelSp** are dual categories. ■

It follows from **2.3** that properties of a modality on a □-lattice or □-algebra) correspond to properties of the induced relation on its Priestley space. More precisely, for each property ℙ of □-lattices (resp. □-algebras) write □$\mathbf{Lat}_{\mathbb{P}}$ (resp. □$\mathbf{Bool}_{\mathbb{P}}$) for the full subcategory of □**Lat** (resp. □**Bool**) whose objects are all □-lattices (resp. □-algebras) possessing ℙ. And for each property ℚ of ordered relspaces (resp. relspaces) let $\mathbf{ORelSp}_{\mathbb{Q}}$ (resp. $\mathbf{RelSp}_{\mathbb{Q}}$) be the full subcategory of **ORelSp** (resp. **RelSp**) whose objects are all ordered relspaces (resp. relspaces) possessing ℚ. Then for each property ℙ of □-lattices (resp. □-algebras) there is a property ℙ* of ordered relspaces (resp. relspaces) such that □$\mathbf{Lat}_{\mathbb{P}}$ and $\mathbf{ORelSp}_{\mathbb{P}*}$ (resp. □$\mathbf{Bool}_{\mathbb{P}}$ and $\mathbf{RelSp}_{\mathbb{P}*}$) are dual categories. The following table gives a few examples of this correspondence: here (X, ≤, R) is the dual of (L, □) and the phrases marked with † apply in the case where (L, □) is a □-algebra.

ℙ	ℙ*
$\square 0 = 0$	Domain(R) = X
$\forall x.\ \square x \le x$	R is reflexive
$\forall x.\ \square x \le \square\square x$	R is transitive

$\forall x\,\forall y.\ \Box(x \vee y) = \Box x \vee \Box y$	each $\mathsf{R}[a]$ is either $\varnothing$ or of the form $x\!\uparrow$ (R is a function[†])
$(L, \Box)$ is weakly filtered	X is of the form $\mathsf{R}[a]$
$(L, \Box)$ is principally filtered	Each nonempty $U \in \mathbf{A}X$ ($\mathbf{C}X$[†]) is of the form $\mathsf{R}[a]$
$(L, \Box)$ is properly filtered	Each nonempty closed upper set (closed set[†]) is of the form $\mathsf{R}[a]$
$(L, \Box)$ is well filtered	Each closed upper set (closed set[†]) is of the form $\mathsf{R}[a]$

We establish just the fourth and sixth of these correspondences. For the fourth, by **1.2**, it suffices to show that, for any prime filter P in a $\Box$-lattice, $\Box^{-1}P$ is prime iff the set $\mathsf{R}[P]$ of prime filters containing $\Box^{-1}P$ has a least member. But since $\Box^{-1}P = \bigcap \mathsf{R}[P]$ this follows immediately.

To establish the sixth of these correspondences, it suffices to note the following chain of equivalences for any ordered relspace $(X, \leq, \mathsf{R})$ and any $U \in \mathbf{A}X$, writing F for the principal filter $\{V\in\mathbf{A}X\colon U \subseteq V\}$ in $\mathbf{A}X$ generated by U:

$$F = \Box_{\mathsf{R}}^{-1}P \quad \text{for some prime filter } P \text{ in } \mathbf{A}X$$

$$\leftrightarrow\ F = \Box_{\mathsf{R}}^{-1}[V\in\mathbf{A}X\colon a \in V\} = \{V\in\mathbf{A}X\colon \mathsf{R}[a] \subseteq V\} \ \text{ for some } a \in X$$

$$\leftrightarrow\ \forall V\in\mathbf{A}X[U\subseteq V \leftrightarrow \mathsf{R}[a] \subseteq U] \text{ for some } a \in X$$

$$\leftrightarrow\ U = \mathsf{R}[a] \ \text{ for some } a \in X.$$

3. Coherent space representation and "pointless" representation of $\Box$-lattices

A topological space is X *coherent* if it satisfies the two following conditions:

(i) the family $\mathbf{K}X$ of compact open subsets of X is closed under finite intersections (so that $\mathbf{K}X$ is a distributive lattice) and forms a base for the topology on X;

(ii) X is *sober*, that is, if each closed set which cannot be written as a union of two proper closed subsets is the closure of a singleton.

A map $f\colon X \to Y$ between topological spaces is *coherent* if it is continuous and $f^{-1}[U] \in \mathbf{K}X$ whenever $U \in \mathbf{K}Y$.

Coherent spaces and coherent maps form a category **CohSp**: it is shown in [8] that **CohSp** is both isomorphic to **OBoolSp** and dual to **DLat.** We show how to equip coherent spaces with additional structure so that the resulting category becomes isomorphic to **ORelSp**, and hence dual to **□Lat.**

If X is a coherent space, a binary relation R on X will be called *appropriate* if

1) for each $x \in X$, $\mathsf{R}[x]$ is an intersection of compact open sets in X;
2) for any $U \in \mathbf{K}X$, $\Box_{\mathsf{R}}U = \{x\colon \mathsf{R}[x] \subseteq U\} \in \mathbf{K}X$.

A *coherent relspace* is a pair (X, R) consisting of a coherent space X and an appropriate relation R on X.

Now for each $a \in X$, write U_a for the intersection of the family of all (open) neighbourhoods of a: note that $x \in U_a$ iff $a \in \overline{\{x\}}$. (In a T_0-space the relation $x \in \overline{\{y\}}$ is a partial ordering called the *specialization ordering*). For each subset $A \subseteq X$, let $A^* = \bigcup_{a \in A} U_a$. By a *coherent morphism of coherent relspaces* $(X, \mathsf{R}) \to (Y, \mathsf{S})$ is meant a coherent map $f\colon X \to Y$ such that $f[\mathsf{R}[x]]^* = \mathsf{S}[f(x)]$ for all $x \in X$.

Coherent relspaces and coherent morphisms between them form a category **CohRelSp**, and we have the

3.1. Theorem. CohRelSp and **ORelSp** are isomorphic categories.

Proof. For the proof we rely heavily on the argument in section 4 of Ch. II of [8] establishing the isomorphism of **CohSp** with **OBoolSp.**

Let $(X, \leq, \mathsf{R})$ be an ordered relspace with topology $\mathbf{T}$. Proposition II. 4.7 of [8] asserts that the family $\mathbf{T}'$ of $\mathbf{T}$-open upper subsets of X is a coherent topology on X (the coherent topology *associated* with $\mathbf{T}$) and that $\leq$ is the specialization ordering for this topology. Let us write X' for the topological space $(X, \mathbf{T}')$. In the proof of that proposition it is also shown that the compact open sets in X' are precisely the clopen upper sets in X. Using this fact it follows easily from the suitability of R for $(X, \leq)$ that R is appropriate for X' and so (X', R) is a coherent relspace, the coherent relspace *associated* with the ordered relspace $(X, \leq, \mathsf{R})$.

Given an **ORelSp**-morphism $f\colon (X, \leq, \mathsf{R}) \to (Y, \leq, \mathsf{S})$, it is easily checked (using the fact that $\leq$ is the specialization ordering for the associated coherent topology) that f is also a coherent morphism between the associated coherent relspaces (X', R) and (Y', S).

Conversely, let (X, R) be a coherent relspace with topology **T**, lattice of compact open sets **A,** and specialization ordering $\leq$. Write **T*** for the *patch topology* on X, that is, the topology having as a base the family of sets

$$\{U \cap (X - V): U, V \in \mathbf{A}\},$$

and write X^* for the topological space (X, **T***). In sections II. 4. 5 and II. 4.6 of [8] it is shown that (X^* , $\leq$) is an ordered Boolean space, and that the **T**-open sets coincide with the **T***-open ($\leq$-) upper sets. From this latter fact it is easily deduced that **A** is the lattice of clopen upper sets in X^*, and this in turn enables one to infer the suitability of R for (X^*, $\leq$) from its appropriateness for X. That is, (X^*, $\leq$, R) is an ordered relspace, the ordered relspace *associated* with coherent relspace (X, R).

It is now easily shown that a coherent morphism between coherent relspaces is automatically an **ORelSp-**morphism between the associated ordered relspaces.

Putting all these facts together, we see that we have an isomorphism between **CohRelSp** and **ORelSp.** ■.

3.2. Corollary. CohRelSp and □**Lat** are dual categories. ■

All the representation theorems proved so far involve the use of prime filters in lattices (the "points" of the representing spaces) and hence the axiom of choice. We next extend to □-lattices the "pointless" representation of distributive lattices presented in [8].

Let us recall that a *frame* is a complete lattice A satisfying the distributivity condition

$$x \wedge \bigvee_{i \in I} y_i = \bigvee_{i \in I} x \wedge y_i .$$

A *frame morphism* is a lattice homomorphism preserving arbitrary joins.

Let A be a frame. An element $a \in A$ is *finite* if it satisfies any of the equivalent conditions ([8], section II.3)

(i) For every subset $S \subseteq A$ with $a \leq \bigvee S$, there is a finite $F \subseteq S$ with $a \leq \bigvee F$.

(ii) For every directed subset $S \subseteq A$ with $a \leq \bigvee S$, there exists $s \in S$ with $a \leq s$.

(iii) For every ideal I of A with $a \le \bigvee I$, we have $a \in I$.

(iv) For every subset $S \subseteq A$ with $a = \bigvee S$, there is a finite $F \subseteq S$ with $a = \bigvee F$.

Note that by Lemma II. 3.2 of [8], the set of finite elements of a frame is closed under $\vee$.

A frame A is *coherent* if

(i) Every element of A is a join of finite elements, and
(ii) The finite elements form a sublattice of A—equivalently, 1 is finite, and the meet of two finite elements is finite.

We write $K(A)$ for the lattice of finite elements of a coherent frame A.

A frame morphism $A \to B$ is *coherent* if it carries $K(A)$ to $K(B)$. Coherent frames and coherent frame morphisms form a category **CohFrm.**

For any distributive lattice L, the set Idl(L) is a coherent frame under the inclusion ordering, and in fact any coherent frame is isomorphic to one of the form Idl(L) (see, e.g. [8], Corollary II.2.11 and Prop. II.3.2.) Given a coherent frame A, it is shown in the proof of Prop. II. 3.2 of [8] that the map φ_A: $A \to \mathrm{Idl}(K(A))$ defined by $\varphi(a) = \{k \in K(A)\colon k \le a\}$ is a (coherent) isomorphism of frames. This is the key step in establishing the equivalence of **CohFrm** with **DLat**, stated (in terms of locales rather than frames) as Corollary II.3.3 of [8].

A modality $\square$ on a coherent frame A is said to be *coherent* if

(i) $k \in K(A) \to \square k \in K(A)$

(ii) $\square a = \bigvee\{\square k\colon k \in K(A),\ k \le a\}$ for all $a \in A$.

It is readily shown that, if A is a coherent frame, any modality $\square$ on $K(A)$ extends uniquely to a coherent modality $\square'$ on A given by $\square' a = \bigvee\{\square k\colon k \in K(A),\ k \le a\}$.

A *coherent modalized frame* is a pair $(A, \square)$ with A a coherent frame and $\square$ a coherent modality on A.

A *coherent* $\square$*-morphism* between coherent modalized frames $(A, \square)$ and $(B, \triangle)$ is a coherent frame morphism $f : A \to B$ satisfying $f(\square a) = \triangle f(a)$ for all $a \in A$.

The category **CohModFrm** has as objects all coherent modalized frames and as arrows all coherent $\Box$-morphisms. We now prove the "pointless" version of **3.2**, namely

3.3. Theorem. $\Box$**Lat** and **CohModFrm** are equivalent categories.

Proof. We define functors G: $\Box$**Lat** $\to$ **CohModFrm** and H: **CohModFrm** $\to$ $\Box$**Lat** which we show constitute an equivalence.

For $\mathfrak{L} = (L, \Box)$ *in* $\Box$**Lat** we take $G\mathfrak{L}$ to be $(\mathrm{Idl}(L), \bar{\Box})$ where $\bar{\Box}$ is defined by:

$$\bar{\Box} I = \text{ideal in } L \text{ generated by } \{\Box x \colon x \in I\}$$

$\bar{\Box}$ is easily shown to be a modality on Idl(L). It is coherent because , first , $\bar{\Box} I_a = I_{\Box a}$ for any $a \in A$, and, as shown in the proof of Proposition II. 3.2 of [8], the finite elements of Idl(L) precisely those of the form I_a; secondly, $\bar{\Box} I = \bigvee_{a \in I} \bar{\Box} I_a$ in Idl(L) for any ideal I. Thus $G\mathfrak{L}$ is an object of **CohModFrm**.

If $f : (L, \Box) \to (M, \triangle)$ is a $\Box$-morphism, we define Gf: $\mathrm{Idl}(L) \to \mathrm{Idl}(M)$ by

$$(Gf)I = \text{ideal generated by } f[I]$$

This is easily verified to be a frame morphism; it is coherent since it sends principal ideals to principal ideals (finite elements in the relevant frames). It is also readily checked that Gf is also a $\Box$-morphism.

H: **CohModFrm** $\to$ $\Box$**Lat** is defined as follows. For a coherent modalized frame $\mathscr{A} = (A, \Box)$ we set $H\mathscr{A} = (K(A), \Box)$. For a coherent $\Box$-morphism $f: \mathscr{A} \to \mathscr{B} = (B, \triangle)$ we define Hf: $H\mathscr{A} \to H\mathscr{B}$ by $Hf = f|K(A)$.

Now, for any coherent modalized frame $(A, \Box)$, the natural isomorphism $\varphi = \varphi_A$: $A \to \mathrm{Idl}(K(A))$ is a $\Box$-morphism. For, noting that $\varphi(\Box a) = \{k \in K(A) \colon k \leq \Box a\}$, and $\bar{\Box}\varphi(a)$ is the ideal in $K(A)$ generated by $\{\Box k \colon k \in K(A) \;\&\; k \leq a\}$, it is clear that $\bar{\Box}\varphi(a) \subseteq \varphi(\Box a)$. For the reverse inclusion, observe that if $k \in \varphi(\Box a)$, then

$$k \leq \Box a = \Box \bigvee \{\ell \in K(A) \colon \ell \leq a\} = \bigvee \{\Box \ell \colon \ell \in K(A) \;\&\; \ell \leq a\}.$$

Since $k \in K(A)$, there are $\ell_1, \ldots, \ell_n \in K(A)$ with each $\ell_i \leq a$ for which

$$k \leq \Box\ell_1 \vee \ldots \vee \Box\ell_n.$$

It follows that $k \in \overline{\Box}\varphi(a)$. Hence $\overline{\Box}\varphi(a) = \varphi(\Box a)$ and φ is a $\Box$-morphism as claimed.

For a $\Box$-lattice $(L, \Box)$, the natural isomorphism $L \to K(\mathrm{Idl}(L))$ is given by $a \mapsto I_a$ and is easily seen to be a $\Box$-morphism.

It follows that GH and HG are naturally isomorphic to identity functors, so that G and H define an equivalence. ■.

4. Interior operators and Heyting algebras.

A modality on a Boolean algebra B satisfying the second two conditions (and hence also the first) in the table in §2, viz. $\forall x.\ \Box x \leq x$ and $\forall x.\ \Box x \leq \Box\Box x$ (whence $\forall x.\ \Box x = \Box\Box x$) is usually called an *interior operator* (and the pair $(B, \Box)$ a *topological Boolean algebra*). The corresponding relspaces—i.e., those carrying a reflexive transitive relation—admit a purely topological description.

Let us define, following [5], an *MT-space* to be a triple $(X, \mathbf{T}_1, \mathbf{T}_2)$ such that

(a) $(X, \mathbf{T}_1)$ is a Boolean space,
(b) $(X, \mathbf{T}_2)$ is a topological space for which $\mathbf{C}_1X \cap \mathbf{T}_2$ is a base, where $\mathbf{C}_1X$ is the family of $\mathbf{T}_1$-closed elements of $\mathbf{T}_1$,
(c) for any $U \in \mathbf{C}_1X$, $\mathrm{Int}_2U \in \mathbf{C}_1X$, where Int_2U is the $\mathbf{T}_2$ – interior of U.

Given two MT-spaces $(X, \mathbf{T}_1, \mathbf{T}_2)$ and $(Y, \mathbf{S}_1, \mathbf{S}_2)$, a map $f\colon X \to Y$ is called an *MT-morphism* if f is $(\mathbf{T}_1, \mathbf{S}_1)$- continuous and

$$f^{-1}[\mathrm{Int}_2 U] = \mathrm{Int}_2\, f^{-1}[U] \quad \text{for all } U \in \mathbf{C}_1 Y.$$

MT-spaces and MT-morphisms form a category **MT-Space.** The full subcategory of **MT-Space** whose objects are those MT-spaces $(X, \mathbf{T}_1, \mathbf{T}_2)$ for which $\mathbf{T}_2$ is a T_0-space will be denoted by **MT₀-Space.**

Let $\mathbb{RT}$ stand for the conjunction of the properties: reflexivity, transitivity, and $\mathbb{PO}$ (partial ordering) for the conjunction of $\mathbb{RT}$ with antisymmetry.

4.1. Theorem. $\mathbf{RelSp}_{\mathbb{RT}} \cong$ **MT-Space** and $\mathbf{RelSp}_{\mathbb{PO}} \cong \mathbf{MT_0}$**-Space**.
Proof. Given an object (X, R) of $\mathbf{RelSp}_{\mathbb{RT}}$**,** let $\mathbf{T}_1$ be the topology on X. It is easy to see that the family $\{U\in\mathbf{C}X: \Box_\mathsf{R}U = U\}$ is a base for a topology $\mathbf{T}_2$ on X. We claim that $(X,\mathbf{T}_1,\mathbf{T}_2)$ is an MT-space. To prove this we must show that, for $U \in \mathbf{C}X$,

(*) $$\mathrm{Int}_2 U = \bigcup\{V\in \mathbf{C}X\colon V\subseteq U \;\&\; \Box_\mathsf{R}V = V\} \in \mathbf{C}X.$$

But since R is reflexive and transitive, for any $V \in \mathbf{C}X$ we have $\Box_\mathsf{R}V \subseteq V$ and $\Box_\mathsf{R}\Box_\mathsf{R}V = V$; it follows easily from this that Int_2U, as defined in (*) is $\Box_\mathsf{R}U$, which is a member of $\mathbf{C}X$.

Given a morphism $f\colon (X, \mathsf{R}) \to (Y, \mathsf{S})$, we know that $f[\mathsf{R}[x]] = \mathsf{S}[f(x)]$ for $x \in X$, and we have the following chain of equivalences:

$$\begin{aligned}
& \forall x\in X.\ f[\mathsf{R}[x]] = \mathsf{S}[f(x)] \\
\leftrightarrow\ & \forall x\in X \forall U\in\mathbf{C}Y[\ f[\mathsf{R}[x]] \subseteq U \leftrightarrow \mathsf{S}[f(x)] \subseteq U] \\
\leftrightarrow\ & \forall x\in X \forall U\in\mathbf{C}Y[\ \mathsf{R}[x] \subseteq f^{-1}[U] \leftrightarrow f(x) \in \Box_\mathsf{S}U] \\
\leftrightarrow\ & \forall x\in X \forall U\in\mathbf{C}Y[\ x \in \Box_\mathsf{R} f^{-1}[U] \leftrightarrow x \in f^{-1}[\Box_\mathsf{S}U] \\
\leftrightarrow\ & \forall U\in\mathbf{C}Y.\ \Box_\mathsf{R} f^{-1}[U] = f^{-1}[\Box_\mathsf{S}U] \\
\leftrightarrow\ & \forall U\in\mathbf{C}Y.\ \mathrm{Int}_2 f^{-1}[U] = f^{-1}[\mathrm{Int}_2U].
\end{aligned}$$

So f is an MT-morphism.

Conversely, given an MT-space $(X,\mathbf{T}_1,\mathbf{T}_2)$, define the induced relation $\mathsf{R} \subseteq X \times X$ by

(**) $$(x, y) \in \mathsf{R} \leftrightarrow \forall U\in\mathbf{C}_1X\ [x \in \mathrm{Int}_2U \to y \in U].$$

Then R is clearly reflexive; its transitivity is established as follows. If $(x, y) \in \mathsf{R}$ and $U \in \mathbf{C}_1X$, then since $\mathrm{Int}_2U \in \mathbf{C}_1X$ for any $U \in \mathbf{C}_1X$, we may substitute "Int_2U" for "U" in the r.h.s. of (**); noting that $\mathrm{Int}_2\,\mathrm{Int}_2U = \mathrm{Int}_2U$, we obtain

$$\forall U \in \mathbf{C}_1X[x \in \mathrm{Int}_2U \to y \in \mathrm{Int}_2U].$$

So if in addition $(y, z) \in \mathsf{R}$, i.e.,

$$\forall U\in\mathbf{C}_1X\ [y \in \mathrm{Int}_2U \to z \in U],$$

(**) now yields $(x, z) \in \mathsf{R}$. Thus R is transitive.

We claim also that $((X,\mathbf{T}_1), \mathsf{R})$ is a relspace, and hence an object of $\mathbf{RelSp}_{\mathbb{RT}}$. First, $\mathsf{R}[x]$ is $\mathbf{T}_1$-closed since, from the definition of R, $\mathsf{R}[x]$ is the intersection of the $U \in \mathbf{C}X$ for which $x \in \mathrm{Int}_2 U$. And $\square_{\mathsf{R}} U = \mathrm{Int}_2 U \in \mathbf{C}_1 X$, since for $x \in X$,

$$x \in \square_R U \leftrightarrow R[x] \subseteq U$$
$$\leftrightarrow \forall y[\forall U \in \mathbf{C}_1 X\, (x \in \mathrm{Int}_2 V \to y \in V) \to y \in U]$$
$$\leftrightarrow x \in \mathrm{Int}_2 U$$

(the last equivalence following by taking "$\mathrm{Int}_2 U$" for "V".

If f: $(X,\mathbf{T}_1,\mathbf{T}_2) \to (Y,\mathbf{S}_1,\mathbf{S}_2)$ is an MT-morphism, then for $U \in \mathbf{C}_1 Y$, and writing S for the induced relation on Y,

$$f^{-1}[\square_S U] = f^{-1}[\mathrm{Int}_2 U] = \mathrm{Int}_2 f^{-1}[U] = \square_R f^{-1}[U].$$

We have already noted, in the proof of **2.2**, that this condition is necessary and sufficient for f to be a morphism $(X, R) \to (Y, S)$.

These correspondences give rise to functors $\mathbf{RelSp}_{\mathbb{RT}} \rightleftarrows$ **MT-Space** which are easily checked to be mutually inverse, so that $\mathbf{RelSp}_{\mathbb{RT}}$ and **MT-Space** are isomorphic categories.

These functors then restrict to an isomorphism $\mathbf{RelSp}_{\mathbb{PO}} \cong \mathbf{MT_0}$-**Space**. To see this, note first that if $(X, \leq)$ is an object in $\mathbf{RelSp}_{\mathbb{PO}}$, then the topology $\mathbf{T}_2$ on the associated MT-space is T_0. For if $x \neq y$ in X, then, say, $x \not\leq y$. Thus $y \notin x\!\uparrow$, so, since $x\!\uparrow$ is $\mathbf{T}_1$-closed, there is $U \in \mathbf{C}_1 X$ such that $x\!\uparrow \subseteq U$, $y \notin U$. It now follows from the definition of R that $(x, y) \notin R$.
■

4.2. Corollary. MT-Space is dual to the category of topological Boolean algebras. ■

We next establish

4.3. Theorem. The following collection of objects coincide:

(a) objects in $\mathbf{RelSp}_{\mathbb{PO}}$,

(b) objects simultaneously in **OBoolSp** and **RelSp**

(c) objects $(X, \leq)$ in **OBoolSp** for which $\mathbf{A}X$ is a Heyting algebra.

Proof. Clearly the objects under (b) are included among those under (a). To establish the reverse inclusion, suppose that $(X, \leq)$ is an object in $\mathbf{Relsp}_{\mathbb{PO}}$. We have to show that $(X, \leq)$ is totally order-separated. If $x \not\leq y$, then $y \notin x\uparrow$, so, since $x\uparrow$ is closed, there is $U \in \mathbf{C}X$ such that $x\uparrow \subseteq U$, $y \notin U$. Then $\Box_{\leq}U \in \mathbf{A}X$, $x \in \Box_{\leq}U$, $y \notin \Box_{\leq}U$, and the claim follows.

We now show that the objects under (b) and (c) coincide. To do this, we prove that, for any ordered Boolean space $(X, \leq)$, $\mathbf{A}X$ is a Heyting algebra iff $(X, \leq)$ is a relspace.

We first note that, in any ordered Boolean space $(X, \leq)$, since $\mathbf{A}X$ separates the points of X (by [3], 4.4.7), the field of subsets of X generated by $\mathbf{A}X$ is $\mathbf{C}X$. Accordingly each clopen set in X is of the form

$$(*) \qquad (U_1 \cup (X - V_1)) \cap \ldots \cap (U_n \cup (X - V_n))$$

for $U_1, \ldots, U_n, V_1, \ldots, V_n \in \mathbf{A}X$.

We next claim that, for $U, V \in \mathbf{A}X$, $V \Rightarrow U$ exists in $\mathbf{A}X$ iff $\Box_{\leq}(U \cup (X - V)) \in \mathbf{C}X$, and in that case $V \Rightarrow U = \Box_{\leq}(U \cup (X - V))$.

Suppose first that $V \Rightarrow U = W$ exists in $\mathbf{A}X$. Then W is the largest member of $\mathbf{A}X$ such that $V \cap W \subseteq U$. If $x \in W$, then $x\uparrow \subseteq W \subseteq U \cup (X - V)$, so that $x \in \Box_{\leq}(U \cup (X - V))$, whence $W \subseteq \Box_{\leq}(U \cup (X - V))$. To establish the reverse inclusion, note that if $x \in \Box_{\leq}(U \cup (X - V))$, then $x\uparrow \subseteq U \cup (X - V)$, so that $x\uparrow \cap V \subseteq U$. Now $x\uparrow$ is a closed upper set, so, by **2.1**,

$$x\uparrow = \bigcap\{Z \in \mathbf{A}X : x\uparrow \subseteq Z\}.$$

Therefore

$$X - U \subseteq (X - x\uparrow) \cup (X - V) = \bigcup\{X - Z : Z \in \mathbf{A}X \;\&\; x\uparrow \subseteq Z\} \cup (X - V).$$

Now $X - U$ is closed, hence compact, and each $X - Z$, as well as $X - V$, is open. Accordingly there is a finite family $\{Z_1, \ldots, Z_n\} \subseteq \mathbf{A}X$ such that $x\uparrow \subseteq Z_i$ for all i and

$$X - U \subseteq (X - Z_1) \cup \ldots \cup (X - Z_n) \cup (X - V).$$

Setting $Z = Z_1 \cap \ldots \cap Z_n \in \mathbf{A}X$, we have $x \in x\uparrow \subseteq Z$ and $Z \cap V \subseteq U$. Since W s supposed to be the largest member of $\mathbf{A}X$ with $V \cap W \subseteq U$, it follows that $Z \subseteq W$, whence $x \in W$. Therefore $\Box_{\leq}(U \cup (X - V)) \subseteq W$ and so $\Box_{\leq}(U \cup (X - V)) = V \Rightarrow U = W \in \mathbf{A}X \subseteq \mathbf{C}X$.

Conversely, suppose that $\Box_{\leq}(U \cup (X - V)) \in \mathbf{C}X$. Then $\Box_{\leq}(U \cup (X - V)) \in \mathbf{A}X$. If $W \in \mathbf{A}X$ satisfies $V \cap W \subseteq U$, the argument above shows that

$W \subseteq \Box_\leq(U \cup (X - V))$, so that the latter is therefore the largest member Z of $\mathbf{A}X$ for which $V \cap Z \subseteq U$, and is accordingly the required $V \Rightarrow U$ in $\mathbf{A}X$.

Now if $\mathbf{A}X$ is a Heyting algebra, then $V \Rightarrow U$ exists in $\mathbf{A}X$ for each $U, V \in W \in \mathbf{C}X$; so that $\Box_\leq(U \cup (X - V)) \in \mathbf{C}X$. But since every member A of $\mathbf{C}X$ is of the form (*) and $\Box_\leq$ distributes over intersections, it follows that $\Box_\leq A \in \mathbf{C}X$ for every $A \in \mathbf{C}X$, i.e. X is a relspace.

Conversely, if $(X, \leq)$ is a relspace, then $\Box_\leq W \in \mathbf{C}X$ for any $W \in \mathbf{C}X$; so, in particular, this is the case when W is of the form $U \cup (X - V)$ with $U, V \in \mathbf{C}X$. But then $V \Rightarrow U$ exists in $\mathbf{A}X$ for arbitrary $U, V \in \mathbf{A}X$, so that $\mathbf{A}X$ is a Heyting algebra. ■

Now write **Heyt** for the category of Heyting algebras and Heyting algebra morphisms (i.e. lattice homomorphisms preserving ⇒). Then we have the

4.4. Corollary. $\mathbf{RelSp}_{\mathbb{PO}}$ and **Heyt** are dual categories.

Proof. Under the Priestley representation the category of Heyting algebras with lattice homomorphisms is dual, by **4.3** (b, c), to the full subcategory of **OBoolSp** whose objects are relspaces. To show that **Heyt** is dual to $\mathbf{RelSp}_{\mathbb{PO}}$ it accordingly suffices, by **4.3** (a, b), to show that arrows in **Heyt** correspond, under the Priestley representation, to arrows in $\mathbf{RelSp}_{\mathbb{PO}}$.

To establish this, it in turn suffices to show that, for any continuous order preserving map $f: (X, \leq) \to (Y, \leq)$ of $\mathbf{RelSp}_{\mathbb{PO}}$ objects, the following are equivalent:

(a) f is a morphism of relspaces,

(b) $f^{-1}: \mathbf{A}Y \to \mathbf{A}X$ is an arrow in **Heyt** (i.e. preserves ⇒)

So assume (a). Then by (the proof of) **2.2**, f^{-1} is a $\Box$-morphism from $(\mathbf{C}Y, \Box_\leq)$ to $(\mathbf{C}X, \Box_\leq)$, so if $U, V \in \mathbf{A}Y$ we have, by (the proof of) **4.3**

$$f^{-1}[V \Rightarrow U] = f^{-1}[\Box_\leq(U \cup (Y - V))] = \Box_\leq (f^{-1}[U] \cup (X - f^{-1}[V]) = f^{-1}[V] \Rightarrow f^{-1}[U].$$

Therefore f^{-1} is a Heyting algebra morphism.

Conversely, assume (b). We claim first that, for any $U \in \mathbf{C}Y$,

$$f^{-1}[\Box_\leq U] = \Box_\leq f^{-1}[U].$$

To prove this, recall that in the proof of **4.3** it was shown that any $U \in$ $\mathbf{C}Y$ can be expressed in the form

$$U = (U_1 \cup (Y - V_1)) \cap \ldots \cap (U_n \cup (Y - V_n))$$

with $U_1, \ldots, U_n, V_1, \ldots, V_n \in \mathbf{A}Y$. In that event,

$$\begin{aligned} \Box_{\leq} U &= \Box_{\leq}(U_1 \cup (Y - V_1)) \cap \ldots \cap \Box_{\leq} (U_n \cup (Y - V_n)) \\ &= (U_1 \Rightarrow V_1) \cap \ldots \cap (U_n \Rightarrow V_n). \end{aligned}$$

Hence

$$\begin{aligned} f^{-1}[\Box_{\leq} U] &= f^{-1} [U_1 \Rightarrow V_1] \cap \ldots \cap f^{-1} [U_n \Rightarrow V_n] \\ &= (f^{-1}[U_1] \Rightarrow f^{-1}[V_1]) \cap \ldots \cap (f^{-1}[U_n] \Rightarrow f^{-1}[V_n]) \\ &= \Box_{\leq}(f^{-1}[U_1] \cup (X - f^{-1}[V_1])) \cap \ldots \cap \Box_{\leq}(f^{-1}[U_n] \cup (X - f^{-1}[V_n])) \\ &= \Box_{\leq} [(f^{-1}[U_1] \cup (X - f^{-1}[V_1])) \cap \ldots \cap (f^{-1}[U_n] \cup (X - f^{-1}[V_n]))] \\ &= \Box_{\leq} f^{-1}[U]. \end{aligned}$$

So now, for $x \in X$, $U \in \mathbf{C}Y$, we have

$$f[x\uparrow] \subseteq U \leftrightarrow f^{-1}[x\uparrow] \subseteq U \leftrightarrow x \in \Box_{\leq} f^{-1}[U] = f^{-1}[\Box_{\leq} U] \leftrightarrow f(x)\uparrow \subseteq U.$$

Thus the closed sets $f[x\uparrow]$ and $f(x)\uparrow$ are included in the same clopen sets, and accordingly coincide. So f is a morphism of relspaces. ■

4.5. Corollary. MT$_0$-Space and **Heyt** are dual categories. ■

Remark. Write **□Heyt** for the subcategory of **□Lat** with objects Heyting algebras and arrows □-morphisms preserving ⇒. It is not hard to extend **4.4** to a duality between **□Heyt** and the full subcategory **ORelSp**$_{\mathbb{PO}}$ of **ORelSp** whose objects are ordered relspaces $(X, \leq, R)$ in which R itself is a partial ordering. These latter in turn may be identified as triples $(X, \leq, \trianglelefteq)$ in which $(X, \leq)$ is an ordered Boolean space and $\trianglelefteq$ is a partial ordering on X such that (i) for any $x \in X$, $\{y\colon x \trianglelefteq y\}$ is a closed subset of X containing $x\uparrow$, and (ii) $\Box_{\trianglelefteq} U \in \mathbf{A}X$ for every $U \in \mathbf{A}X$.

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