Russell's Paradox and Diagonalization in a Constructive Context

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Abstract. One of the most familiar uses of the Russell paradox , or, at least, of the idea underlying it, is in proving Cantor's theorem that the cardinality of any set is strictly less than that of its power set. The other method of proving Cantor's theorem—employed by Cantor himself in showing that the set of real numbers is uncountable—is that of diagonalization. Typically, diagonalization arguments are used to show that function spaces are "large" in a suitable sense. Classically, these two methods are equivalent. But constructively they are not: while the argument for Russell's paradox is perfectly constructive, (i.e., employs intuitionistically acceptable principles of logic) the method of diagonalization fails to be so. I describe the ways in which these two methods diverge in a constructive setting.

One of the most familiar uses of the Russell paradox, or, at least, of the idea underlying it, is in proving Cantor's theorem that, for any set E, the cardinality of E is strictly less than that of its power set $\mathcal{P}E$. This, as we all know, boils down to showing that there can be no surjection $E \twoheadrightarrow \mathcal{P}E$. To establish this it is enough to show that, for any map $f\colon E \to \mathcal{P}E$, the assertion

$$\forall X \in \mathcal{P}E\ \exists x \in E.\, X = f(x) \tag{1}$$

leads to a contradiction. Now from a *constructive* standpoint the argument of Russell's paradox establishes *more* than just the negation of (1), since it produces an explicit set R for which it can be proved that

$$\neg\exists x \in E.\, R = f(x)$$

namely the familiar "Russell set"

$$R = \{x \in E : x \notin f(x)\}.$$

This is, of course, because assuming $R = f(e)$ for some $e \in E$ leads instantly to the contradiction $e \in f(e) \Leftrightarrow e \notin f(e)$. For $U \subseteq E$, a similar argument, replacing R above by $R \cap U$, shows that there can be no surjection $U \twoheadrightarrow \mathcal{P}E$. If we agree to say that a set B is *surjective* with a set A provided that there is a surjection $A \twoheadrightarrow B$, then this may be put: for no set E is $\mathcal{P}E$ surjective with a subset of E.

These arguments are constructively valid in that they employ only constructively, or intuitionistically, acceptable principles of logic.

Equally, the (classically equivalent, but not automatically constructively equivalent) form of Cantor's theorem that, for any set E there is no injection $\mathcal{P}E \rightarrowtail E$ can also be given a constructive proof using the idea of Russell's paradox. In fact we can prove more, to wit, that for any set E there can be no injection of $\mathcal{P}E$ into a set *surjective* with E. For suppose given a surjection $f\colon E \twoheadrightarrow A$ and an injection $m\colon \mathcal{P}E \rightarrowtail A$. Define

$$B = \{x \in E : \exists X \in \mathcal{P}E.\, m(X) = f(x) \wedge x \notin X\}.$$

Since f is surjective there is $b \in E$ for which $f(b) = m(B)$. Then we have

$$\begin{aligned} b \in B &\Longleftrightarrow \exists X.\, m(X) = f(b) \wedge b \notin X \\ &\Longleftrightarrow \exists X.\, m(X) = m(B) \wedge b \notin X \\ &\Longleftrightarrow \exists X.\, X = B \wedge b \notin X \\ &\Longleftrightarrow b \notin B, \end{aligned}$$

and we have our contradiction.

As pointed out by George Boolos in [2], one can, classically, produce explicit counterexamples to the injectivity of a given map $m\colon \mathcal{P}E \to E$, that is, subsets X and Y of E for which $X \neq Y$ and $m(X) = m(Y)$. To do this it suffices to define a partial right inverse r of m such that, for $M = \text{dom}(r)$, $m(M) \in M$ and $\forall x \in M.\ x \notin r(x)$. For then, writing $m(M) = a$, and $X = r(a)$, we have $a \notin X$, whence $X \neq M$, and $m(X) = m(r(a)) = a = m(M)$. Using an idea that goes back to Zermelo, Boolos obtains M as the field of the largest partial well-ordering $<$ of E such that $m(\{y : y < x\}) = x$ for all $x \in M$, and defines r by $r(x) = \{y : y < x\}$. The presence of well-orderings in this argument makes it highly nonconstructive; I do not know whether the existence of such an r and M can be established constructively.

Classically, the power set $\mathcal{P}E$ is naturally bijective with 2^E, the set of all maps $E \to 2 = \{0, 1\}$. Constructively, this is no longer the case: here, in general, $\mathcal{P}E \cong \Omega^E$, where Ω is the object of truth values or propositions, which is only $\cong 2$ when the law of excluded middle is assumed. In fact, constructively, 2^E is isomorphic, not to $\mathcal{P}E$, but to its Boolean sublattice $\mathcal{C}E$ consisting of all *complemented* (or *detachable*) subsets of E (a subset U of E is said to be complemented if $\forall x \in E.\ x \in U \vee x \notin U$). What happens when we replace $\mathcal{P}E$ by $\mathcal{C}E$ in the above arguments? Classically, of course, it makes no difference, but do the "Russell paradox" arguments survive the transition to constructivity?

Well, if one takes the first argument, showing that there can be no surjection $f\colon E \twoheadrightarrow \mathcal{P}E$, one finds that, when $\mathcal{P}E$ is replaced by $\mathcal{C}E$, the set $R \notin \text{range}(f)$ is itself complemented and the argument goes through, proving constructively that there can be no surjection $E \twoheadrightarrow \mathcal{C}E$. But the second argument, with $\mathcal{P}E$ replaced by $\mathcal{C}E$ (and E replaced by a subset U of E) only goes through constructively when U is itself complemented. And as for the third argument to go through constructively once $\mathcal{P}E$ is replaced by $\mathcal{C}E$, it is necessary to show that the set B defined there is complemented, and, as we shall see, this cannot in general be done. The failure of these two latter arguments in a constructive context can be easily demonstrated by considering a model $\mathfrak{M}$ of smooth infinitesimal analysis, see, e.g., [1]. In $\mathfrak{M}$ the real line $\mathbb{R}$ has just the two detachable subsets $\emptyset$, $\mathbb{R}$, that is, $\mathcal{C}\mathbb{R}$ has just two elements. *A fortiori* $\mathcal{C}\mathbb{R}$ is injectible into $\mathbb{R}$, showing that the third argument fails constructively. The fact that there is no surjection $\mathbb{R} \to \mathcal{C}\mathbb{R}$ corresponds simply to the fact that, since $\mathbb{R}$ is connected, there are no continuous nonconstant maps $\mathbb{R} \to 2$ (all maps in $\mathfrak{M}$ being smooth, and certainly continuous). But $\mathcal{C}\mathbb{R}$ is trivially surjective with the subset $\{0, 1\}$ of $\mathbb{R}$, refuting the second argument—of course, in $\mathfrak{M}$, $\{0, 1\}$ is not a complemented subset of $\mathbb{R}$! At the end of the paper we supply a quite different constructive example of a set E for which $\mathcal{C}E$ is surjective with a subset of E.

Let us return once again to the argument that there is no surjection $E \twoheadrightarrow \mathcal{P}E$. Classically, we may replace $\mathcal{P}E$ by the isomorphic object 2^E. In that case the use of Russell's paradox is transformed into an application of *diagonalization*, the technique Cantor used to prove that the set of real numbers has strictly larger cardinality than the set of natural numbers. Indeed, if $\phi\colon E \to 2^E$ is the map canonically associated with the given map $f\colon E \to \mathcal{P}E$ via characteristic functions (i.e., defined by $\phi(x)(y) = 1 \Leftrightarrow y \in f(x)$) then the "Russell" set $R \in \mathcal{P}E$ outside the range of f corresponds precisely to the map $r\colon E \to 2$ outside the range of ϕ and defined by "diagonalization":

$$r(x) = \begin{cases} 0 & \text{if } \phi(x)(x) = 1 \\ 1 & \text{if } \phi(x)(x) = 0. \end{cases}$$

This argument is perfectly constructive and parallels that given above for the nonexistence of a surjection $E \twoheadrightarrow \mathcal{C}E$.

Diagonalization appears in one of its most familiar guises in the well-known proof that there can be no surjection of the set $\mathbb{N}$ of natural numbers onto the set $\mathbb{N}^{\mathbb{N}}$ of all maps $\mathbb{N} \to \mathbb{N}$, that, in a word, $\mathbb{N}^{\mathbb{N}}$ is *uncountable*. Here one is given a map ϕ: $\mathbb{N} \to \mathbb{N}^{\mathbb{N}}$; then the map $f\colon \mathbb{N} \to \mathbb{N}$ defined by the prescription

$$f(n) = \begin{cases} 0 & \text{if } \phi(n)(n) \neq 0 \\ 1 & \text{if } \phi(n)(n) = 0. \end{cases}$$

is clearly outside the range of φ. This instance of diagonalization, although not identical with the previous one, also appears very similar to Russell's paradox.

Looked at constructively, this proof depends crucially on the *decidability* of $\mathbb{N}$, i.e., the truth of the assertion

$$\forall m \in \mathbb{N}\ \forall n \in \mathbb{N}.\ m = n \vee m \neq n.$$

Since $\mathbb{N}$ is decidable from a constructive standpoint, the argument is constructively valid, so that $\mathbb{N}^{\mathbb{N}}$ is constructively uncountable. (More generally, the same argument shows that if X is any decidable set with at least two distinct elements, X^X cannot be surjective with X.)

Let us call a set *subcountable* if it is surjective with a subset of $\mathbb{N}$. Classically, it follows trivially from the fact that $\mathbb{N}^{\mathbb{N}}$ is uncountable that it also fails to be *subcount-*

able, that is, $\mathbb{N}^{\mathbb{N}}$ is not surjective with a subset of $\mathbb{N}$. For given $\phi: U \to \mathbb{N}^{\mathbb{N}}$; the map $f: \mathbb{N} \to \mathbb{N}$ defined by the new prescription (still resembling Russell's paradox)

$$f(n) = \begin{cases} 0 & \text{if } n \in U \;\&\; \phi(n)(n) \neq 0 \\ 1 & \text{if } n \in U \;\&\; \phi(n)(n) = 0 \\ 1 & \text{if } n \notin U \end{cases}$$

is again clearly outside the range of ϕ. Now this argument only goes through constructively when U is a detachable subset of $\mathbb{N}$, so that the most "diagonalization" shows, constructively, is that $\mathbb{N}^{\mathbb{N}}$ is not "detachably" *subcountable*. But *prima facie* nothing prevents $\mathbb{N}^{\mathbb{N}}$ from being "nondetachably" subcountable; for if, given $\phi: U \to \mathbb{N}^{\mathbb{N}}$ we repeat the original prescription, the map f so obtained is defined just on U, not on the whole of $\mathbb{N}$, and the argument collapses. So, while it still follows from Russell's paradox that $\mathscr{P}\mathbb{N}$ cannot be constructively subcountable, diagonalization (as well as Russell's paradox) fails to establish the corresponding fact for $\mathbb{N}^{\mathbb{N}}$.

In fact models of constructive mathematics have been produced in which $\mathbb{N}^{\mathbb{N}}$ *is actually subcountable*. Such is the case, notably, in the *effective topos* **Eff**, see, e.g., chap. 23 of [3]. In **Eff**, $\mathscr{P}\mathbb{N}$ and $\mathbb{N}^{\mathbb{N}}$ effectively(!) part company, making Russell's paradox and diagonalization the more easily distinguished. Russell's paradox continues to yield the non-subcountability of $\mathscr{P}\mathbb{N}$, which accordingly remains "large". But diagonalization, while continuing to yield the uncountability of $\mathbb{N}^{\mathbb{N}}$, fails to prevent it from being subcountable in **Eff**, and so from being in some sense "small" there. The reason for this is that, in **Eff**, $\mathbb{N}^{\mathbb{N}}$ consists, not of arbitrary maps $\mathbb{N} \to \mathbb{N}$, but just of the *recursive* ones. The subset $U \subseteq \mathbb{N}$ establishing the subcountability of $\mathbb{N}^{\mathbb{N}}$ is the set of codes of total recursive functions; since, in **Eff**, the complemented subsets of $\mathbb{N}$ are just the recursive subsets, the fact that U cannot be complemented corresponds to the fact that the set of codes of total recursive functions is not itself recursive. The subcountability of $\mathbb{N}^{\mathbb{N}}$ immediately implies that of its subset 2^N, and hence also of the latter's isomorph $\mathcal{C}\mathbb{N}$, showing anew the failure, in a constructive context, of the argument that there can be no set E for which $\mathcal{C}E$ is surjective with a subset of E.

The "divergence" in **Eff** between diagonalization and Russell's paradox can be further pointed up by observing that **Eff** contains *nonsingleton* objects C for which the object C^C of self-maps is actually *isomorphic* to C. Such objects cannot be classical sets, because clearly the only such sets satisfying this condition are singletons. For a classical set C, the condition of being a singleton is equivalent to the condition that there be no injection of 2—the object of truth values in classical set theory—into C. So, in classical set theory, the condition that $C^C \cong C$ implies that there is no injection of Ω into C. In fact Russell's paradox shows that *this* implication continues to hold in the constructive setting. For suppose that i is an isomorphism (even just an injection) of C^C with C and that m is an injection of Ω into C. Then the map

$$X \mapsto i(\{\langle x, m(x \in X)\rangle : x \in C\})$$

is easily seen an injection of $\mathcal{P}C$ into C, which by the Russell's paradox argument above, is impossible.

To summarize: in a constructive context, diagonalization does not fail to prevent the possible presence of an object which is isomorphic to its object of self-maps and yet is not a singleton. But Russell's paradox does preclude the existence of an injection of the object of truth values into any such object C: such a map would yield an injection into C of its power object $\mathcal{P}C$, which by Russell's paradox is too large to be so injectible.

To conclude. We have seen that, in certain constructive contexts, diagonalization may fail to ensure that function spaces are "relatively large". By contrast, Russell's paradox—at least, as properly applied to power sets—retains its potency even in constructive environments, ensuring that power sets, or objects, retain their "size". It seems fitting, therefore, to claim for Russell's paradox a universal applicability which must at the same time be denied diagonalization.

References

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