NOTES ON TOPOSES AND LOCAL SET THEORIES

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This book is written for those who are in sympathy with its spirit. This spirit is different from the one which informs the vast stream of European and American civilization in which all of us stand. That spirit expresses itself in an onwards movement, in building ever larger and more complicated structures; the other in striving in clarity and perspicuity in no matter what structure. The first tries to grasp the world by way of its periphery – in its variety; the second at its centre – in its essence. And so the first adds one construction to another, moving on and up, as it were, from one thing to the next, while the other remains where it is and what it tries to grasp is always the same.

Wittgenstein

CONTENTS

PREFACE: FROM SET THEORY TO TOPOS THEORY

The reigning concepts of set theory are the *membership relation* $\in$ and the *extension* $\{x: A(x)\}$ of an arbitrary predicate A. These are related by the *comprehension principle*

Comp $\qquad A(y) \leftrightarrow y \in \{x: A(x)\}.$

In set theory $\{x: A(x)\}$ is taken to be an actual collection of individuals, namely, the class of individuals having the property associated with the predicate A, and the relation $\in$ to be the concrete membership relation obtaining between a class and the individuals comprising it.

Now from a purely formal standpoint it is not necessary to construe the comprehension principle in the concrete manner prescribed by set theory. One has the option of regarding that principle as asserting a purely formal connection between the symbols involved. This opens up the possibility of conferring new meanings on those symbols, while at the same time continuing to affirm the principle. Topos theory—or local set theory—offers just such a possibility.

In the universe of sets every entity is a set (or a class) and so also an extension of a predicate since the comprehension principle trivially implies that, for each set X, $X = \{x: x \in X\}$. This remains the case in topos theory. A topos is a category every entity of which—that is, each object and arrow—can formally be construed as an extension of a "predicate" (suitably defined) in such a way as to preserve the comprehension principle. The difference between set-theoretic extensions and their "formal" counterparts can then be seen to rest on just how symbols for variables ($x, y, \ldots$) are to be understood. In the set-theoretic case these symbols are construed *substitutionally*—i.e. as placeholders for names of fixed individuals. Thus, for example, $\forall x A(x)$ is understood to mean the conjunction $A(a)$ & $A(b)$ & $A(c)$ & ... where $a, b, c, \ldots$ is a list of names of the distinct individuals constituting the universe of discourse. In the formal case as realized by topos theory, on the other hand, symbols for variables ultimately denote *correspondences* and so have to be regarded as truly variable entities. Thus while in set theory the rule of inference

$$\frac{A(a) \quad \text{for every individual } a}{\forall x A(x)}$$

is affirmed, this rule *fails* in the "formal" case. Indeed, the correctness of the rule *singles out* the set-theoretic case.

The fact that the whole apparatus of extensions is applicable within a topos is what makes topos theory a "generalization" of set theory.

I.

FREE INTUITIONISTIC LOGIC AND COMPLETE HEYTING ALGEBRAS

The system of *free intuitionistic logic* has the following axioms and rules of inference.

Axioms

$\alpha \to (\beta \to \alpha)$

$[\alpha \to (\beta \to \gamma) \to [(\alpha \to \beta) \to (\alpha \to \gamma)]$

$\alpha \to (\beta \to \alpha \wedge \beta)$

$\alpha \wedge \beta \to \alpha \qquad \alpha \wedge \beta \to \beta$

$\alpha \to \alpha \vee \beta \qquad \beta \to \alpha \vee \beta$

$(\alpha \to \gamma) \to [(\beta \to \gamma) \to (\alpha \vee \beta \to \gamma)]$

$(\alpha \to \beta) \to [(\alpha \to \neg\beta) \to \neg\alpha]$

$\neg\alpha \to (\alpha \to \beta)$

$\alpha(t) \to \exists x\alpha(x) \qquad \forall x\alpha(x) \to \alpha(y)$ (x free in α and t free for x in α)

$x = x \qquad \alpha(x) \wedge x = y \to \alpha(y)$

Rules of Inference

Restricted modus ponens $\dfrac{\alpha,\ \alpha \to \beta}{\beta}$ (all free variables of α free in β)

$\dfrac{\beta \to \alpha(x)}{\beta \to \forall x\alpha(x)} \qquad \dfrac{\alpha(x) \to \beta}{\exists x\alpha(x) \to \beta}$ (x not free in β)

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A *lattice* is a partially ordered set with partial ordering $\leq$ in which each two-element subset $\{x, y\}$ has a supremum or *join*—denoted by $x \vee y$—

and an infimum or *meet*—denoted by $x \wedge y$. A lattice is *complete* if every subset X (including $\varnothing$) has a supremum or *join*—denoted by $\bigvee X$—and an infimum or *meet*—denoted by $\bigwedge X$. Note that $\bigvee\varnothing = 0$, the least or *bottom* element of the lattice, and $\bigwedge\varnothing = 1$, the largest or *top* element of the lattice.

A *Heyting algebra* is a lattice H with top and bottom elements such that, for any elements $x, y \in H$, there is an element—denoted by $x \Rightarrow y$—of H such that, for any $z \in H$,

$$z \leq (x \Rightarrow y) \equiv^{1} z \wedge x \leq y.$$

Thus $x \Rightarrow y$ is the *largest* element z such that $z \wedge x \leqslant y$. So in particular, if we write x^* for $x \Rightarrow 0$, then x^* is the largest element z such that $x \wedge z = 0$: it is called the *pseudocomplement* of x. We also write $x \Leftrightarrow y$ for $(x \Rightarrow y) \wedge (y \Rightarrow x)$.

A *Boolean algebra* is a Heyting algebra in which $x^{**} = x$ for all x, or equivalently, in which $x \vee x^* = 1$ for all x.

Heyting algebras are related to intuitionistic propositional logic in precisely the same way as Boolean algebras are related to classical propositional logic. That is, suppose given a propositional language L; let P be its set of propositional variables. Given a map f: $\mathsf{P} \to H$ to a Heyting algebra H, we extend f to a map $\alpha \mapsto [\![\alpha]\!]$ of the set of formulas of L to H à la Tarski

$$[\![\alpha \wedge \beta]\!] = [\![\alpha]\!] \wedge [\![\beta]\!] \quad [\![\alpha \vee \beta]\!] = [\![\alpha]\!] \vee [\![\beta]\!], \quad [\![\neg\alpha]\!] = [\![\alpha]\!]^*$$

$$[\![\alpha \to \beta]\!] = [\![\alpha]\!] \Rightarrow [\![\beta]\!].$$

[1] Throughout, we use "≡" for "if and only if" and "⇛" for "implies".

A formula α is said to be (Heyting) *valid*—written $\vDash\alpha$—if $[\![\alpha]\!] = 1$ for any such map *f*. It can then be shown that α is valid iff $\vdash\alpha$ in the intuitionistic propositional calculus, i.e., α is deducible from the propositional axioms listed above.

A basic fact about *complete* Heyting algebras is that the following identity holds in them:

$$(*) \qquad x \wedge \bigvee_{i\in I} y_i = \bigvee_{i\in I} x \wedge y_i.$$

And conversely, in any complete lattice satisfying (*), defining the operation $\Rightarrow$ by $x \Rightarrow y = \bigvee\{z\colon z \wedge x \leq y\}$ turns it into a Heyting algebra.

To prove this, we observe that in any complete Heyting algebra,

$$\begin{aligned} x \wedge \bigvee_{i\in I} y_i \leq z \quad &\equiv \quad \bigvee_{i\in I} y_i \leq (x \Rightarrow z) \\ &\equiv \quad y_i \leq (x \Rightarrow z), \\ &\equiv \quad y_i \wedge x \leq z, \text{ all } i \\ &\equiv \quad \bigvee_{i\in I} x \wedge y_i \leq z. \end{aligned}$$

Conversely, if (*) is satisfied and $x \Rightarrow y$ is defined as above, then

$$\begin{aligned} (x \Rightarrow y) \wedge x &\leq \bigvee\{z\colon z \wedge x \leq y\} \wedge x \\ &= \bigvee\{z \wedge x\colon z \wedge x \leqslant y\} \\ &\leq y\ . \end{aligned}$$

So $z \leqslant (x \Rightarrow y) \;\Rightarrow\; z \wedge x \leqslant (x \Rightarrow y) \wedge x \leq y$. The reverse inequality is an immediate consequence of the definition.

In view of this result a complete Heyting algebra is frequently defined to be a complete lattice satisfying (*).

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Complete Heyting algebras are related to (free) intuitionistic logic in the same way as complete Boolean algebras are to classical logic. To be precise, let L be a first-order language whose sole extralogical symbol is a binary predicate symbol P. An L-*structure* is a quadruple $\mathbf{M} = (M, eq, Q, L)$, where M is a (not necessarily nonempty!) set, H is a complete Heyting algebra and eq and Q are maps $M^2 \to M$ satisfying, for all $m, n, m', n' \in M$,

$$eq(m, m) = 1, \;\; eq(m, n) = eq(n, m), \;\; eq(m, n) \wedge eq(n, n') \leqslant eq(m, n'),$$

$$Q(m, n) \wedge eq(m, m') \leqslant Q(m', n), \quad Q(m, n) \wedge eq(n, n') \leqslant Q(m, n').$$

For any formula α of L and any finite sequence $\boldsymbol{x} = <x_1, \ldots, x_n>$ of variables of L containing all the free variables of α, we define for any L-structure $\mathbf{M}$ a map

$$[\![\alpha]\!]^{\mathbf{M}}{}_{\boldsymbol{x}}\colon M^n \to H$$

recursively as follows.:

$[\![x_p = x_q]\!]^{\mathbf{M}}{}_{\boldsymbol{x}} = <m_1 \ldots, m_n> \mapsto eq(m_p, m_q)$,

$[\![Px_p\, x_q]\!]^{\mathbf{M}}{}_{\boldsymbol{x}} = <m_1 \ldots, m_n> \mapsto Q(m_p, m_q)$,

$[\![\alpha \wedge \beta]\!]^{\mathbf{M}}{}_{\boldsymbol{x}} = [\![\alpha]\!]^{\mathbf{M}}{}_{\boldsymbol{x}} \wedge [\![\beta]\!]^{\mathbf{M}}{}_{\boldsymbol{x}}$, and similar clauses for the other connectives,

$$[\![\exists y\alpha]\!]^{\mathbf{M}}{}_{\boldsymbol{x}} = <m_1 \ldots, m_n> \mapsto \bigvee_{m \in M} [\![\alpha\,(y/u)]\!]^{\mathbf{M}}{}_{u\boldsymbol{x}}(m, m_1 \ldots, m_n)$$

$$[\![\forall y\alpha]\!]^{\mathbf{M}}{}_{\boldsymbol{x}} = <m_1 \ldots, m_n> \mapsto \bigwedge_{m \in M} [\![\alpha\,(y/u)]\!]^{\mathbf{M}}{}_{u\boldsymbol{x}}(m, m_1 \ldots, m_n)$$

Call α $\mathbf{M}$-*valid* if $[\![\alpha]\!]^{\mathbf{M}}{}_{\boldsymbol{x}}$ is identically 1, where $\boldsymbol{x}$ is the sequence of all free variables of α. Note that, if M is empty, then *any* formula containing a free variable is $\mathbf{M}$-valid, but $[\![\exists x.\ x = x]\!]^{\mathbf{M}} = 0$.

Then it can be shown that α *is* **M-***valid for all* **M** *iff* α *is provable in free intuitionistic logic.* This is the *completeness theorem* for this system of logic.

II

LOCAL SET THEORIES /INTUITIONISTIC TYPE THEORIES

LOGIC IN A LOCAL LANGUAGE

A *local set theory* is a type-theoretic system built on the same primitive symbols =, ∈, {:} as classical set theory, in which the set-theoretic operations of forming products and powers of types can be performed, and which in addition contains a "truth value" type acting as the range of values of "propositional functions" on types. A local set theory is determined by specifying a collection of *axioms* formulated within a *local language* defined as follows.

A *local language* $\mathscr{L}$ has the following *basic symbols:*

- **1** (*unit type*) **Ω** (*truth value type*)
- **S, T, U,…** (*ground types*: possibly none of these)
- **f, g, h,…** (*function symbols*: possibly none of these)
- $x_{\mathbf{A}}$, $y_{\mathbf{A}}$, $z_{\mathbf{A}}$, … (*variables of each type* **A**, where a *type* is as defined below)
- ⋆ (*unique entity of type* **1**)

The *types* of $\mathscr{L}$ are defined recursively as follows:

1, Ω are types

any ground type is a type

$\mathbf{A}_1 \times \ldots \times \mathbf{A}_n$ is a type whenever $\mathbf{A}_1, \ldots, \mathbf{A}_n$ are, where, if $n = 1$,

$\mathbf{A}_1 \times .. \times \mathbf{A}_n$ is $\mathbf{A}_1$, while if $n = 0$, $\mathbf{A}_1 \times .. \times \mathbf{A}_n$ is **1** (*product types*)

- **PA** is a type whenever **A** is (*power types*)

Each function symbol **f** is assigned a *signature* of the form $\mathbf{A} \to \mathbf{B}$, where **A, B** are types; this is indicated by writing $\mathbf{f}: \mathbf{A} \to \mathbf{B}$.

Terms of $\mathscr{L}$ and their associated *types* are defined recursively as follows. We write τ: **A** to indicate that the term τ has type **A.**

Term: type	**Proviso**
$\star$: **1**	
$x_{\mathbf{A}}$: **A**	
$\mathbf{f}(\tau)$: **B**	**f: A → B** τ: **A**
$<\tau_1, \ldots, \tau_n>: \mathbf{A}_1 \times \ldots \times \mathbf{A}_n$, where $<\tau_1, \ldots, \tau_n>$ is τ_1 if $n = 1$, and $\star$ if $n = 0$.	$\tau_1: \mathbf{A}_1, \ldots, \tau_n: \mathbf{A}_n$
$(\tau)_i: \mathbf{A}_i$ where $(\tau)_i$ is τ if $n = 1$	$\tau: \mathbf{A}_1 \times \ldots \times \mathbf{A}_n$, $1 \leq i \leq n$
$\{x_{\mathbf{A}}: \alpha\}$: **PA**	$\alpha : \mathbf{\Omega}$
$\sigma = \tau: \mathbf{\Omega}$	σ, τ of same type
$\sigma \in \tau: \Omega$	σ: **A,** τ: **PA** for some type **A**

Terms of type $\mathbf{\Omega}$ are called *formulas, propositions,* or *truth values.* Notational conventions we shall adopt include:

$\omega, \omega', \omega''$	variables of type Ω
α, β, γ	formulas
$x, y,, z \ldots$	$x_{\mathbf{A}}, y_{\mathbf{A}}, z_{\mathbf{A}} \ldots$
$\tau(x/\sigma)$ or $\tau(\sigma)$	result of substituting σ at each free occurrence of x in τ: an occurrence of x is *free* if it does not appear within $\{x: \alpha\}$
$\alpha \leftrightarrow \beta$	$\alpha = \beta$
$\Gamma : \alpha$	sequent notation: Γ a finite set of formulas
$: \alpha$	$\varnothing: \alpha$

A term is *closed* if it contains no free variables; a closed term of type $\mathbf{\Omega}$ is called a *sentence.*

The *basic axioms* for $\mathscr{L}$ are as follows:

Unity $: x_1 = \star$

Equality $x = y, \alpha(z/x) : \alpha(z/y)$ (x, y free for z in α)

Products $: (<x_1, \ldots, x_n>)_i = x_i$

$: x = <(x)_1, \ldots, (x)_n>$

Comprehension $: x \in \{x: \alpha\} \leftrightarrow \alpha$

The *rules of inference* for $\mathscr{L}$ are:

Thinning

$$\frac{\Gamma : \alpha}{\beta, \Gamma: \alpha}$$

Restricted Cut

$$\frac{\Gamma : \alpha \quad \alpha, \Gamma : \beta}{\Gamma : \beta} \qquad \text{(any free variable of } \alpha \text{ free in } \Gamma\text{) or } \beta\text{)}$$

Substitution

$$\frac{\Gamma : \alpha}{\Gamma(x/\tau) : \alpha(x/\tau)} \qquad (\tau \text{ free for } x \text{ in } \Gamma \text{ and } \alpha)$$

Extensionality

$$\frac{\Gamma : x \in \sigma \leftrightarrow x \in \tau}{\Gamma : \sigma = \tau} \qquad (x \text{ not free in } \Gamma, \sigma, \tau)$$

Equivalence

$$\frac{\alpha, \Gamma : \beta \quad \beta, \Gamma : \alpha}{\Gamma : \alpha \leftrightarrow \beta}$$

These axioms and rules of inference yield a system of *natural deduction* in $\mathscr{L}$. If S *is* any collection of sequents in $\mathscr{L}$, we say that the sequent $\Gamma : \alpha$ is *deducible from S*, and write $\Gamma \vdash_S \alpha$ provided there is a derivation of $\Gamma : \alpha$ using the basic axioms, the sequents in *S*, and the rules of inference. We shall also write $\Gamma \vdash \alpha$ for $\Gamma \vdash_\varnothing \alpha$ and $\vdash_S \alpha$ for $\varnothing \vdash_S \alpha$.

A *local set theory* in $\mathscr{L}$ is a collection *S* of sequents closed under deducibility from *S*. Any collection of sequents *S generates* the local set theory S^* comprising all the sequents deducible from *S*. The local set theory in $\mathscr{L}$ generated by $\varnothing$ is called *pure* local set theory in $\mathscr{L}$.

The *logical operations* in $\mathscr{L}$ are defined as follows:

Logical Operation	**Definition**[2]
$\mathbf{T}$ (true)	$\bigstar = \bigstar$
$\alpha \wedge \beta$	$\langle\alpha, \beta\rangle = \langle\mathbf{T}, \mathbf{T}\rangle$
$\alpha \rightarrow \beta$	$(\alpha \wedge \beta) \leftrightarrow \alpha$
$\forall x\, \alpha$	$\{x : \alpha\} = \{x : \mathbf{T}\}$
$\perp$ (false)	$\forall\omega.\, \omega$

[2] For the intuitive justification of the last pair of definitions, observe that: $\alpha \vee \beta$ holds iff, for any proposition ω, if ω follows both from α and from β, then ω holds. Similarly, $\exists x\alpha$ holds iff, for any ω, if ω follows from $\alpha(x)$ for any value of x, then ω holds.

$\neg\alpha$	$\alpha \to \bot$
$\alpha \vee \beta$	$\forall\omega[(\alpha \to \omega \wedge \beta \to \omega) \to \omega]$[3]
$\exists x\, \alpha$	$\forall\omega[\forall x(\alpha \to \omega) \to \omega]$[4]

We also write $x \neq y$ for $\neg(x = y)$, and $x \notin y$ for $\neg(x \in y)$.

It can now be shown that the logical operations on formulas just defined satisfy the axioms and rules of free intuitionistic logic. We present some of the relevant derivations. In general, we write

$$\dfrac{\Gamma_1 : \alpha_1, \dots , \Gamma_n : \alpha_n}{\Delta : \beta}$$

for derivability of $\Delta : \beta$ from $\Gamma_1 : \alpha_1, \dots , \Gamma_n : \alpha_n$.

1. $\vdash x = x.$

Derivation: $: (x)_1 = x.$

2. $\alpha \vdash \alpha.$

Derivation:

$$\dfrac{\dfrac{\omega, \omega = \omega : \omega \qquad \dfrac{\vdots}{: \omega = \omega}}{\omega : \omega}}{\alpha : \alpha}$$

3. $x = y \vdash y = x, \quad x = y \vdash \tau(z/x) = \tau(z/y).$

4. $\vdash \top, \ \alpha \vdash \alpha = \top, \ \alpha = \top \vdash \alpha.$

Derivations:

$$\dfrac{\top, \alpha : \alpha \qquad \dfrac{\vdots}{\alpha, \alpha : \top}}{\alpha : \alpha = \top}$$

$$\dfrac{\dfrac{\omega = \omega', \omega' : \omega}{\alpha = \top, \top : \alpha} \qquad \dfrac{\vdots}{: \top}}{\alpha = \top : \alpha}$$

5. $\dfrac{\Gamma : \alpha \quad \Gamma : \beta}{\Gamma : \alpha \wedge \beta}$

Derivation:

$$\dfrac{\Gamma : \alpha \qquad \dfrac{\dfrac{\vdots}{\beta : \beta = \top} \qquad \dfrac{\dfrac{\vdots}{\alpha : \alpha = \top} \qquad \dfrac{\vdots}{\alpha = \top, \beta = \top : \alpha \wedge \beta}}{\alpha, \beta = \top : \alpha \wedge \beta}}{\alpha, \beta : \alpha \wedge \beta}}{}$$

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[3] Here ω must nt occur in α or β.
[4] Here ω must nt occur in α.

$$\frac{\Gamma : \beta \qquad \Gamma, \beta : \alpha \wedge \beta}{\Gamma : \alpha \wedge \beta}$$

6. $\dfrac{\alpha, \Gamma : \gamma}{\alpha \wedge \beta, \Gamma : \gamma} \qquad \dfrac{\beta, \Gamma : \gamma}{\alpha \wedge \beta, \Gamma : \gamma}$

7. $\dfrac{\alpha, \Gamma : \beta}{\Gamma : \alpha \to \beta} \qquad \dfrac{\Gamma : \alpha \to \beta}{\alpha, \Gamma : \beta}$

8. $\dfrac{\alpha, \beta : \gamma}{\alpha \wedge \beta : \gamma} \qquad \dfrac{\alpha \wedge \beta : \gamma}{\alpha, \beta : \gamma}$

9. $\dfrac{\Gamma : \alpha \leftrightarrow \beta}{\Gamma : \{x : \alpha\} = \{x : \beta\}}$ (x not free in Γ)

Derivation:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\Gamma : \alpha \leftrightarrow \beta}{}}{\alpha, \Gamma : \beta} \qquad \cfrac{\cfrac{\Gamma : \alpha \leftrightarrow \beta}{}}{\beta, \Gamma : \alpha}}{x \in \{x : \alpha\},\ \Gamma : x \in \{x : \beta\} \qquad x \in \{x : \beta\},\ \Gamma : x \in \{x : \alpha\}}}{\Gamma : x \in \{x : \alpha\} \leftrightarrow x \in \{x : \beta\}}}{\Gamma : \{x : \alpha\} = \{x : \beta\}}$$

10. $\dfrac{\Gamma : \alpha}{\Gamma : \forall x\, \alpha}$ provided x is not free in (a) Γ or (b) α

Derivation: (a)

$$\cfrac{\cfrac{\Gamma : \alpha \qquad \cfrac{\vdots}{\alpha : \alpha \leftrightarrow \top}}{\Gamma : \alpha \leftrightarrow \top}}{\cfrac{\vdots}{\Gamma : \forall x\, \alpha}}$$

(b)

$$\cfrac{\cfrac{\cfrac{\Gamma : \alpha}{\Gamma(x/v) : \alpha}}{\Gamma(x/v) : \forall x\, \alpha}}{\Gamma : \forall x\, \alpha} \qquad (v \text{ new})$$

11. $\dfrac{\Gamma : \{x : \alpha\} = \{x : \beta\}}{\Gamma : \alpha \leftrightarrow \beta}$ provided x is free in α or β

12. $\forall x\, \alpha \vdash \alpha$ provided x is free in α

Derivation:

$$\dfrac{\dfrac{\dfrac{:}{\forall x\,\alpha : \forall x\,\alpha}}{\forall x\,\alpha : \alpha = \top} \qquad \alpha = \top : \boldsymbol{\alpha}}{\forall x\,\alpha : \alpha}$$

13. $\vdash\ \forall u\, \alpha(x/u) \leftrightarrow \forall x\, \alpha$ provided u is free for x, and not free in, α

(This shows that the definitions of $\perp$, $\vee$, and $\exists$ do not depend on the choice of bound variable ω.)

14. $\dfrac{\Gamma : \alpha(x/u)}{\Gamma : \forall x\, \alpha}$ provided that either (a) u is free for x in α, and not free in Γ or $\forall x\, \alpha$ or (b) x is not free in α.

15. $\dfrac{\alpha(x/\tau), \Gamma : \beta}{\forall x\alpha, \Gamma : \boldsymbol{\beta}}$ provided that τ is free for x in α, x is free in α, and any free variable of τ is free in $\forall x\alpha$, Γ, or $\boldsymbol{\beta}$**.**

16. $\perp \vdash \alpha \qquad \dfrac{\alpha, \Gamma : \perp}{\Gamma : \neg\alpha} \qquad \dfrac{\Gamma : \alpha}{\neg\alpha, \Gamma : \perp}$

17. $\dfrac{\alpha, \Gamma : \gamma \qquad \beta, \Gamma : \boldsymbol{\gamma}}{\alpha \vee \beta, \Gamma : \gamma}$

Derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\alpha, \Gamma : \gamma}{\boldsymbol{\Gamma : \alpha \to \gamma}} \qquad \dfrac{\beta, \Gamma : \gamma}{\boldsymbol{\Gamma : \beta \to \gamma}}}{\boldsymbol{\Gamma : \alpha \to \gamma \wedge \beta \to \gamma}} \qquad \dfrac{:}{\gamma : \boldsymbol{\gamma}}}{(\boldsymbol{\alpha} \to \gamma \wedge \boldsymbol{\beta} \to \gamma) \to \gamma, \Gamma : \gamma}}{\forall\omega[(\alpha \to \omega \wedge \beta \to \omega) \to \omega], \Gamma : \boldsymbol{\gamma}} \text{ (from 15)}}{\alpha \vee \beta, \Gamma : \gamma}$$

18. $\dfrac{\Gamma : \alpha}{\Gamma : \alpha \vee \beta} \qquad \dfrac{\Gamma : \beta}{\Gamma : \alpha \vee \beta}$ (Derivation uses 14.)

19. $\alpha \vdash \exists x\alpha$ provided x is free in α

Derivation (with ω not occurring in α):

$$\dfrac{\alpha : \boldsymbol{\alpha} \qquad \boldsymbol{\omega : \omega}}{}$$

$$\cfrac{\cfrac{\cfrac{\boldsymbol{\alpha \to \omega, \alpha : \omega}}{\forall x\,(\alpha \to \omega), \alpha : \boldsymbol{\omega}}\ \text{(from 15)}}{\alpha : \forall x\,(\alpha \to \omega) \to \boldsymbol{\omega}}}{\alpha : \exists x\alpha}$$

Notice that *it does not follow from* **12.***and* **19.** *that* $\forall x\ \alpha \vdash \exists x\ \alpha$, because the free variable x in α is free in neither premise nor conclusion.

20. $$\frac{\alpha, \Gamma : \beta}{\exists x\alpha, \Gamma : \beta}$$ provided x is (a) not free in Γ or β or (b) not free in α.

21. $$\frac{\Gamma : \alpha(x/\tau)}{\Gamma : \exists x\alpha}$$ provided τ is free for x in α, x is free in α and any free variable of τ is free in Γ or $\exists x\alpha$.

22. $\vdash\ \exists u\alpha(x/u) \leftrightarrow \exists x\alpha$ provided u is free for x, but not free in, α.

23. *Modified Cut Rule*

(i) $$\frac{\Gamma : \alpha \qquad \alpha, \Gamma : \beta}{\exists x_1(x_1 = x_1), \ldots, \exists x_n(x_n = x_n), \Gamma : \boldsymbol{\beta}}$$

where $x_1, \ldots, x_n$ are the free variables of α not occurring freely in Γ or β.

(ii) $$\frac{\Gamma : \alpha \qquad \alpha, \Gamma : \beta}{\Gamma : \beta}$$ provided that, whenever **A** is the type of a free variable of α with no free occurrences in Γ or β, there is a closed[5] term of type **A.**

*

We define the *unique existential quantifier* $\exists!$ in the familiar way, namely,

$$\exists! x\alpha \ \equiv\ \exists x[\alpha \wedge \forall y(\alpha(x/y) \to x = y).$$

The *Eliminability of Descriptions for Propositions*[6] can now be established:

[5] A term is *closed* if it contains no free variables.

$$\exists!\omega\alpha \vdash \alpha(\omega/\alpha\ (\omega/\top)).$$

Consequently, if $\vdash_S \exists!\omega\alpha$ then there is an explicit sentence σ for which $\vdash_S \alpha(\sigma)$. Here is the proof.

For simplicity write $\alpha(\tau)$ for $\alpha(\omega/\tau)$. Then we have

$$\exists!\omega\alpha,\ \alpha(\top),\ \alpha\ \vdash\ \omega = \top\ \vdash\ \omega; \quad \omega,\ \alpha\ \vdash\ \omega = \top \wedge \alpha\ \vdash\ \alpha(\top).$$

Hence

$$\exists!\omega\alpha,\ \alpha \vdash \omega = \alpha(\top);\ \ \exists!\omega\alpha,\ \alpha \vdash \omega = \alpha(\top) \wedge \alpha \vdash \alpha(\alpha(\top)),$$

so that $\exists!\omega\alpha, \exists\omega\alpha \vdash \alpha(\alpha(\top))$. Since $\exists!\omega\alpha \vdash \exists\omega\alpha$, the result follows.

SET THEORY IN A LOCAL LANGUAGE

We can now introduce the concept of *set* in a local language. A *set-like* term is a term of power type; a *closed* set-like term is called an ($\mathscr{L}$-) *set.* We shall use upper case italic letters *X, Y, Z, ...* for sets, as well as standard abbreviations such as $\forall x \in X.\alpha$ for $\forall x(x \in X \to \alpha)$. The set theoretic *operations* and *relations* are defined as follows. Note that in the definitions of $\subseteq$, $\cap$, and $\cup$, *X* and *Y must be of the same type*:

Operation	**Definition**
$\{x \in X\colon \alpha\}$	$\{x\colon\ x \in X \wedge \alpha\}$
$X \subseteq Y$	$\forall x \in X.\ x \in Y$
$X \cap Y$	$\{x\colon x \in X \wedge x \in Y\}$
$X \cup Y$	$\{x\colon x \in X \vee x \in Y\}$
$x \notin X$	$\neg(x \in X)$
$U_{\mathbf{A}}$ or A	$\{x_{\mathbf{A}}\colon \top\}$
$\varnothing_{\mathbf{A}}$ or $\varnothing$	$\{x_{\mathbf{A}}\colon \bot\}$
$E - X$	$\{x\colon x \in E \wedge x \notin X\}$
PX	$\{u\colon u \subseteq X\}$

[6] By this is meant that any proposition which can be identified by means of a description of the sort *the unique proposition for which such-and such* can already be explicitly named.

$\bigcap U$ $(U : \mathbf{PPA})$	$\{x\colon \forall u \in U.\ x \in u\}$
$\bigcup U$ $(U : \mathbf{PPA})$	$\{x\colon \exists u \in U.\ x \in u\}$
$\bigcap_{i \in I} X_i$	$\{x\colon \forall i \in I.\ x \in X_i\}$
$\bigcup_{i \in I} X_i$	$\{x\colon \exists i \in I.\ x \in X_i\}$
$\{\tau_1, \ldots, \tau_n\}$	$\{x : x = \tau_1 \vee \ldots \vee x = \tau_n\}$
$\{\tau : \alpha\}$	$\{z : \exists x_1 \ldots \exists x_n (z = \tau \wedge \alpha)\}$
$X \times Y$	$\{<x,y>\colon x \in X \wedge y \in Y\}$
$X + Y$	$\{<\{x\},\varnothing>\colon x \in X\} \cup \{<\varnothing,\{y\}>.\colon y \in Y\}$
$Fun(X,Y)$	$\{u\colon u \subseteq X \times Y \wedge \forall x \in X\, \exists! y \in Y.\ <x,y> \in u\}$

The following facts concerning the set-theoretic operations and relations may now be established as straightforward consequences of their definitions:

(i) $\vdash X = Y \leftrightarrow \forall x(x \in X \leftrightarrow x \in Y)$

(ii) $\vdash X \subseteq X$, $\vdash (X \subseteq Y \wedge Y \subseteq X) \rightarrow X = Y$,

$\vdash (X \subseteq Y \wedge Y \subseteq Z) \rightarrow X \subseteq Z$

(iii) $\vdash Z \subseteq X \cap Y \leftrightarrow Z \subseteq X \wedge Z \subseteq Y$

(iv) $\vdash X \cup Y \subseteq Z \leftrightarrow X \subseteq Z \wedge Y \subseteq Z$

(v) $\vdash x_{\mathbf{A}} \in U_{\mathbf{A}}$

(vi) $\vdash \neg\, (x \in \varnothing_{\mathbf{A}})$

(vii) $\vdash X \in PY \leftrightarrow X \subseteq Y$

(viii) $\vdash X \subseteq \cap U \leftrightarrow \forall u \in U\,.\ X \subseteq u$

(ix) $\vdash \cup U \subseteq X \leftrightarrow \forall u \in U\,.\ u \subseteq X$

(x) $\vdash x \in \{y\} \leftrightarrow x = y$

(xi) $\vdash \alpha \rightarrow \tau \in \{\tau : \alpha\}$

Here (i) is the *axiom of extensionality*, (iv) the *axiom of binary union,* (vi) the *axiom of the empty set*, (vii) the *power set axiom*, (ix) the *axiom of unions* and (x) the *axiom of singletons.* These, together with the comprehension axiom, form the core axioms for set theory in $\mathscr{L}$. The set theory is *local* because some of the set theoretic operations, e.g., intersection and union, may be performed only on sets of the same type, that is, "locally". Moreover, variables are constrained to range only over given types—locally—in contrast with the situation in classical set theory where they are permitted to range globally over an all-embracing universe of discourse.

III

CATEGORIES AND TOPOSES

CATEGORIES

A *category* $\mathbf{C}$ is determined by first specifying two classes $Ob(\mathbf{C})$, $Arr(\mathbf{C})$—the collections of $\mathbf{C}$-*objects* and $\mathbf{C}$-*arrows*. These collections are subject to the following axioms:

- Each $\mathbf{C}$-arrow f is assigned a pair of $\mathbf{C}$-objects dom(f), cod(f) called the *domain* and *codomain* of f, respectively. To indicate the fact that $\mathbf{C}$-objects X and Y are respectively the domain and codomain of f we write f: $X \rightarrow Y$ or $X \xrightarrow{f} Y$. The collection of $\mathbf{C}$-arrows with domain X and codomain Y is written $\mathbf{C}(X, Y)$.

- Each $\mathbf{C}$-object X is assigned a $\mathbf{C}$-arrow 1_X: $X \rightarrow X$ called the *identity arrow* on X.

- Each pair f, g of $\mathbf{C}$-arrows such that cod(f) = dom(g) is assigned an arrow $g \circ f$: dom(f) $\rightarrow$ cod(g) called the *composite* of f and g. Thus if f: $X \rightarrow Y$ and g: $Y \rightarrow Z$ then $g \circ f$: $X \rightarrow Z$. We shall sometimes write $X \xrightarrow{f} Y \xrightarrow{g} Z$ or simply gf for $g \circ f$. Arrows f, g satisfying cod(f) = dom(g) are called *composable.*

- *Associativity law.* For composable arrows (f, g) and (g, h), we have $h \circ (g \circ f) = h \circ (g \circ f)$.

- *Identity law.* For any arrow f: $X \rightarrow Y$, we have $f \circ 1_X = f = 1_Y \circ f$.

A category is *small* if its collections of obects and arrows are both *sets.*

As a basic example of a category, we have the category Set of sets whose objects are all sets and whose arrows are all maps between sets (strictly, triples (f, A, B) with domain$(f) = A$ and range$(f) \subseteq B$.) Other examples of categories are the category of groups, with objects all groups and arrows all group homomorphisms and the category of topological spaces with objects all topological spaces and arrows all continuous maps. A category possessing exactly one object may be identified with a *monoid*, that is, an algebraic structure with an associative multiplication and an identity element, while a category having at most one arrow between any pair of objects may be identified with a *preordered class,* i.e. a class carrying a reflexive transitive relation—a *preordering*[7].

A *subcategory* C of a category D is any category whose class of objects and arrows is included in the class of objects and arrows of D, respectively, and which is closed under domain, codomain, identities, and composition. If, further, for any objects A, B of C, we have C(A, B) = D(A, B), we shall say that C is a *full* subcategory of D.

BASIC CATEGORY-THEORETIC DEFINITIONS

Commutative diagram (in category C)	Diagram of objects and arrows such that the arrow obtained by composing the arrows of any connected path depends only on the endpoints of the path.
Initial object	Object 0 such that, for any object

[7] Thus a *partial ordering* is an antisymmetric preordering, i.e. a preodering $\leq$ satisfying $x \leq y$ and $y \leq x \Rightarrow x = y$. A set carrying a partial ordering will be called a *poset.*

X, there is a unique arrow $0 \to X$ (e.g., $\varnothing$ in Set)

Terminal object — Object 1 such that, for any object X, there is a unique arrow $X \to 1$ (e.g. any singleton in Set)

(C-)*Element* of an object X — Arrow $1 \to X$

Monic arrow $X \rightarrowtail Y$ — Arrow f: $X \to Y$ such that, for any arrows g, h: $Z \to X$, $f \circ g = f \circ h \Rightarrow g = h$ (in Set, *one-one* map)

Epic arrow $X \twoheadrightarrow Y$ — Arrow f: $X \to Y$ such that, for any arrows g, h: $Y \to Z$, $g \circ f = h \circ f \Rightarrow g = h$ (in Set, *onto* map)

Isomorphism $X \cong Y$ — Arrow f: $X \to Y$ for which there is g: $Y \to X$ such that $g \circ f = 1_X$, $f \circ g = 1_Y$

Product of objects X, Y — Object $X \times Y$ with arrows (projections) $X \xleftarrow{\pi_1} X \times Y \xrightarrow{\pi_2} Y$ such that any diagram

$$\begin{tikzcd} & \bullet \arrow[dl, "f"'] \arrow[dr, "g"] & \\ {} & & {} \end{tikzcd}$$

can be uniquely completed to a commutative diagram

$$\begin{tikzcd} & \bullet \arrow[dl, "f"'] \arrow[d, "<f{,}g>"'] \arrow[dr, "g"] & \\ X & X \times Y \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & Y \end{tikzcd}$$

Product of arrows f_1: $X_1 \to Y_1$, f_2: $X_2 \to Y_2$ — Unique arrow $f_1 \times f_2$: $X_1 \times X_2 \to Y_1 \times Y_2$ making the diagram

$$\begin{tikzcd} & & X_1 \times X_2 \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] \arrow[dd, "f_1 \times f_2"] & & \\ & X_1 \arrow[dl, "f_1"'] & & X_2 \arrow[dr, "f_2"] & \\ Y_1 & & Y_1 \times Y_2 \arrow[ll, "\pi'_1"'] \arrow[rr, "\pi'_2"] & & Y_2 \end{tikzcd}$$

commute. I..e., $f_1 \times f_2 = < f_1 \circ \pi_1, f_2 \circ \pi_2 >$.

Diagonal arrow on object X — Unique arrow $\delta_X : X \to X \times X$ making the diagram

$$\begin{tikzcd} & X \arrow[dl, "1_X"'] \arrow[d] \arrow[dr, "1_X"] & \\ {} & {} & {} \end{tikzcd}$$

$$\begin{tikzcd} & \delta_X & \\ X & X \times Y \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & Y \end{tikzcd}$$

commute. I.e., $\delta_X = \langle 1_X, 1_X \rangle$.

Coproduct of objects X, Y

Object $X + Y$ together with a pair of arrows $X \xrightarrow{\sigma_1} X + Y \xleftarrow{\sigma_2} Y$ such that for any pair of arrows $X \xrightarrow{f} A \xleftarrow{g} B$, there is a unique arrow $X + Y \xrightarrow{h} A$ such that the diagram

$$\begin{tikzcd} X \arrow[r, "\sigma_1"] \arrow[dr, "f"'] & X + Y \arrow[d, "h"] & Y \arrow[l, "\sigma_2"'] \arrow[dl, "g"] \\ & A & \end{tikzcd}$$

commutes

h is usually written $f + g$.

Pullback diagram or *square*

Commutative diagram of the form

$$\begin{tikzcd} \blacksquare \arrow[r] \arrow[d] & \bullet \arrow[d, "f"] \\ \bullet \arrow[r, "g"] & \bullet \end{tikzcd}$$

such that for any commutative diagram

$$\begin{tikzcd} \blacklozenge \arrow[r] \arrow[d] & \bullet \arrow[d, "f"] \\ \bullet \arrow[r, "g"] & \bullet \end{tikzcd}$$

there is a unique $\blacklozenge \xrightarrow{!} \blacksquare$ such that

$$\begin{tikzcd} \blacklozenge \arrow[dr, "!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & \blacksquare \arrow[r] \arrow[d] & \bullet \arrow[d] \\ & \bullet \arrow[r] & \bullet \end{tikzcd}$$

commutes.

Equalizer of pair of arrows $\bullet \overset{f}{\underset{g}{\rightrightarrows}} \blacksquare$

Arrow $\blacklozenge \xrightarrow{e} \bullet$ such that $f \circ e = g \circ e$ and, for any arrow $\blacktriangle \xrightarrow{e'} \bullet$ such that $f \circ e' = g \circ e'$ there is a unique $\blacktriangle \xrightarrow{u} \blacklozenge$ such that

$$\begin{tikzcd} \blacktriangle \arrow[r, "u"] \arrow[dr, "e'"'] & \blacklozenge \arrow[d, "e"] \\ & \bullet \end{tikzcd}$$

commutes.

Subobject of an object X

Pair (m, Y), with m a monic arrow $Y \rightarrowtail X$

Truth value object or *subobject classifier*

Object Ω together with arrow $\top$: $1 \to \Omega$ such that every monic m: $\bullet \rightarrowtail \blacklozenge$ (i.e., subobject of $\blacklozenge$) can be uniquely extended to a pullback diagram of the form

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "m"'] & 1 \arrow[d, "\top"] \\ \blacklozenge \arrow[r, "\chi(m)"] & \Omega \end{tikzcd}$$

and conversely every diagram of the form $A \xrightarrow{u} \Omega \longleftarrow 1$ has a pullback. $\chi(m)$ is called the *characteristic arrow* of *m.*

The *maximal* characteristic arrow T_A, or simply T, on A, is defined to be the characteristic arrow of 1_A. The characteristic arrow of $0 \rightarrowtail 1$ is written $\perp$: $1 \to \Omega$.

Power object of an object *X.*

An object PX together with an arrow ("evaluation") e_X: $X \times PX \to \Omega$ such that, for any f: $X \times PX \to \Omega$, there is a unique arrow $f: Y \to PX$ such that

$$\begin{tikzcd} X \times Y \arrow[d, "1_X \times f"'] \arrow[dr, "f"] & \\ X \times PX \arrow[r, "e_X"] & \Omega \end{tikzcd}$$

commutes. (In Set, PX is the power set of X and e_X the characteristic function of the membership relation between X

and *PX*.)

Exponential object of objects Y, X

An object Y^X, together with an arrow ev: $X \times Y^X \to Y$ such that, for any arrow f: $X \times Z \to Y$ there is a unique arrow $f : Z \to Y^X$—the *exponential transpose* of f— such that the diagram

$$\begin{tikzcd} X \times Z \arrow[d, "1_X \times f"'] \arrow[dr, "f"] & \\ X \times Y^X \arrow[r, "ev"'] & Y \end{tikzcd}$$

commutes. In Set, Y^X is the set of all maps $X \to Y$ and ev is the map that sends (x, f) to $f(x)$.

Product of indexed set $\{A_i : i \in I\}$ *of objects*

Object $\prod_{i \in I} A_i$ together with arrows $\prod_{i \in I} A_i \xrightarrow{\pi_i} A_i \quad (i \in I)$ such that, for any arrows $f_i : B \to A_i \;(i \in I)$ there is a *unique* arrow $h : B \longrightarrow \prod_{i \in I} A_i$ such that, for each $i \in I$, the diagram

$$\begin{tikzcd} B \arrow[r, "h"] \arrow[dr, "f_i"'] & \prod_{i \in I} A_i \arrow[d, "\pi_i"] \\ & A_i \end{tikzcd}$$

commutes.

Coproduct of indexed set $\{A_i : i \in I\}$ *of objects*

Object $\coprod_{i \in I} A_i$ together with arrows $A_i \xrightarrow{\sigma_i} \coprod_{i \in I} A_i \quad (i \in I)$ such that, for any arrows $f_i : A_i \to B \;(i \in I)$ there is a *unique* arrow $h : \coprod_{i \in I} A_i \longrightarrow B$ such that, for each $i \in I$, the diagram

$$\begin{tikzcd} A_i \arrow[r, "\sigma_i"] \arrow[dr, "f_i"'] & \coprod_{i \in I} A_i \arrow[d, "h"] \\ & B \end{tikzcd}$$

commutes.

A category is *cartesian closed* if it has a terminal object, as well as products and exponentials of arbitrary pairs of its objects. It is *finitely complete* if it has a terminal object, products of arbitrary pairs of its objects, and equalizers. A *topos* is a category possessing a terminal object, products, a truth-value object, and power objects. It can be shown that every topos is cartesian closed and finitely complete (so that this notion of topos is equivalent to that originally given by Lawvere and Tierney).

More on products in a category. A *product* of objects $A_1, \ldots, A_n$ in a category C is an object $A_1 \times \ldots \times A_n$ together with arrows $\pi_i: A_1 \times \ldots \times A_n \to A_i$ for $i = 1, \ldots, n$, such that, for any arrows $f_i: B \to A_i$, $i = 1, \ldots, n$, there is a unique arrow, denoted by $\langle f_1, \ldots, f_n\rangle: B \to A_1 \times \ldots \times A_n$ such that $\pi_i \circ \langle f_1, \ldots, f_n\rangle = f_i$, $i = 1, \ldots, n$. Note that, when $n = 0$, $A_1 \times \ldots \times A_n$ is the terminal object 1. The category is said to *have finite products* if $A_1 \times \ldots \times A_n$ exists for all $A_1, \ldots, A_n$. If C has binary products, it has finite products, since we may take $A_1 \times \ldots \times A_n$ to be $A_1 \times (A_2 \times (\ldots \times A_n)\ldots)$. It is easily seen that the product operation is, up to isomorphism, commutative and associative. The relevant isomorphisms are called *canonical* isomorphisms.

A *functor* F: $\mathsf{C} \to \mathsf{D}$ between two categories C and D is a map that "preserves commutative diagrams", that is, assigns to each C-object A a D-object FA and to each C-arrow $f: A \to B$ a D-arrow Ff: $FA \to FB$ in such a way that:

$$\begin{tikzcd} A \arrow[d, "f"'] & & FA \arrow[d, "Ff"] \\ B \arrow[r, phantom, "\longmapsto"] & & FB \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[loop right, "1_A"] \arrow[rr, mapsto] & & FA \arrow[loop right, "1_{FA}"] \end{tikzcd} \qquad F(1_A) = 1_{FA}$$

$$\begin{tikzcd} \bullet \arrow[rr, "f"] \arrow[dr, "h"'] & & \bullet \arrow[dl, "g"] \\ & \bullet & \end{tikzcd} \longmapsto \begin{tikzcd} \bullet \arrow[rr, "Ff"] \arrow[dr, "Fh"'] & & \bullet \arrow[dl, "Fg"] \\ & \bullet & \end{tikzcd} \qquad F(g \circ f) = Fg \circ Ff$$

commutes $\qquad\qquad\qquad\qquad$ commutes

A functor F: $\mathsf{C} \to \mathsf{D}$ is an *equivalence* if it is "an isomorphism up to isomorphism", that is, if it is

- *faithful*: $Ff = Fg \Rightarrow f = g$.
- *full*: for any h: $FA \to FB$ there is f: $A \to B$ such that $h = Ff$.
- *dense*: for any D-object B there is a C-object A such that $B \cong FA$.

Two categories are *equivalent,* written $\simeq$, if there is an equivalence between them. Equivalence is the appropriate notion of "identity of form" for categories.

Given functors F, G: C $\to$ D, a *natural transformation* between F and G is a map η from the objects of C to the arrows of D satisfying the following conditions:

- For each object A of C, ηA is an arrow $FA \to GA$ in D
- For each arrow f: $A \to A'$ in C,

$$\begin{array}{ccc} FA & \xrightarrow{\eta A} & GA \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ FA' & \xrightarrow{\eta A'} & GA' \end{array}$$ commutes.

Finally, two functors F: C $\to$ D and G: D $\to$C and are said to be *adjoint* to one another if, for any objects A of C, B of D, there is a "natural" bijection between arrows $A \to GB$ in C and arrows $FA \to B$ in D. To be precise, for each such pair A, B we must be given a bijection φ_{AB}: C$(A, GB) \to$ D(FA, B) satisfying the "naturality" conditions

- for each f: $A \to A'$ and h: $A' \to GB$, $\varphi_{AB}(h \circ f) = \varphi_{A'B}(h) \circ Ff$
- for each g: $B \to B'$ and h: $A \to GB'$, $\varphi_{AB}(Gg \circ h) = g \circ \varphi_{AB'}(h)$.

Under these conditions F is said to be *left adjoint* to G, and G *right adjoint* to F.

THE TOPOS OF SETS DETERMINED BY A LOCAL SET THEORY

Let S be a local set theory in a local language $\mathscr{L}$. Define the relation $\sim_S$ on the collection of all $\mathscr{L}$-sets by

$$X \sim_S Y \equiv \vdash_S X = Y.$$

This is an equivalence relation. An *S-set* is an equivalence class $[X]_S$—which we normally identify with X—of $\mathscr{L}$-sets under the relation $\sim_S$. An S-

map $f\colon X \to Y$ or $X \xrightarrow{f} Y$ is a triple (f, X, Y)—normally identified with f—of S-sets such that $\vdash_S f \in Fun(X, Y)$. X and Y are, respectively, the *domain* dom(f) and the *codomain* cod(f) of f.

We now claim that the collection of all S-sets and maps forms a category C(S), the *category of S-sets.* This is proved by showing

(1) if $f, g\colon X \to Y$, then

$$f = g \;\equiv\; x \in X \vdash_S \langle x, y\rangle \in f \leftrightarrow \langle x, y\rangle \in g.$$

Given $X \xrightarrow{f} Y \xrightarrow{g} Z$, define

$$g \circ f = \{\langle x, z\rangle : \exists y(\langle x, y\rangle \in f \wedge \langle y, z\rangle \in g)\}.$$

Then

(2) $g \circ f\colon X \to Z$ is associative.

Given an S-set X, define

$$\Delta_X = \{\langle x, x\rangle : x \in X\} \qquad\qquad 1_X = (\Delta_X, X, X).$$

Then

(3) $1_X\colon X \to X$; for any $f : X \to Y$, $f \circ 1_X = f = 1_Y \circ f$.

Now suppose we are given a term τ such that

$$\langle x_1, \ldots, x_n\rangle \in X \vdash_S \tau \in Y.$$

We write $\langle x_1, \ldots, x_n\rangle \mapsto \tau$ or simply $\boldsymbol{x} \mapsto \tau$ for

$$\{\langle\langle x_1, \ldots, x_n\rangle, \tau\rangle : \langle x_1, \ldots, x_n\rangle \in X\}.$$

If $x_1, \ldots, x_n$ includes all the free variables of τ and X, Y are S-sets, then $\langle x_1, \ldots, x_n\rangle \mapsto \tau$ is an S-map $X \to Y$, which we denote by $\tau\colon X \to Y$ or $X \xrightarrow{\tau} Y$. If $\mathbf{f}$ is a function symbol, we write f for $x \mapsto \mathbf{f}(x)$. It is not hard to show that, if, for all i, σ_i is free for y_i in τ, then

$$(\langle y_1, \ldots, y_n\rangle \mapsto \tau) \circ \pi\ (\langle x_1, \ldots, x_n\rangle \mapsto \langle \sigma_1, \ldots, \sigma_n\rangle) = (\langle x_1, \ldots, x_n\rangle \mapsto \tau(\boldsymbol{y}/\boldsymbol{\sigma})).$$

We next claim that C(S) *is a topos.* To prove this, observe that, first, U_1 *is a terminal object in*C (S). For, writing 1 for U_1, given a set X, we have the S-map $(x \mapsto \star)$: $X \to 1$. If f: $X \to 1$, then $x \in X \ \vdash_S \langle x, \star\rangle \in f$, so $f = \ (x \mapsto \star)$.

Products in C (S). Given S-sets X, Y, let

$$\pi_1 = (\langle x, y\rangle \mapsto x): X \times Y \to X, \qquad \pi\eta_2 = (\langle x, y\rangle \mapsto x): X \times Y \to Y.$$

Then $X \times Y$ is a product object in C(S) with projections π_1, π_2.

It is easily shown that an *S-map is monic iff*

$$\langle x, z\rangle \in f\,,\ \langle y, z\rangle \in f\ {}_S\ x = y.$$

Given an S-map f: $X \to U_\Omega$, write $f^*(x)$ for $\langle x, \top\rangle \in f$. Since $\exists!\omega\ \langle x, \omega\rangle \in f$, it follows from eliminability of descriptions that $x \in X \ \vdash_S \langle x, f^*(x)\rangle \in f$. So $f^*(x)$ is the *value* of f at x. And

$$(x \mapsto f^*(x)) = f, \qquad (x \mapsto \alpha)^*(\langle x_1, \ldots, x_n\rangle) \Leftrightarrow \alpha.$$

Write Ω for U_Ω and $\top$: $1 \to \Omega$ for $x \mapsto \top$. Then it can be shown that, *if* m*:* $Y \rightarrowtail X$ *is monic, the* C(S)*- diagram*

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "m"'] & 1 \arrow[d, "\top"] \\ X \arrow[r, "h"'] & \Omega \end{tikzcd}$$

is a pullback iff

$$h = (x \mapsto \exists y.\langle y, x\rangle \in m).$$

Using these facts we can show that $(\Omega, \top)$ *is a truth value object in* C(S). For suppose given a monic m: $Y \rightarrowtail X$. Define $\chi(m)$: $X \to \Omega$ by

$$\chi(m) = x \mapsto \exists y.\langle y, x\rangle \in m.$$

Then by the previous result $\chi(m)$ is the unique arrow $X \to \Omega$ such that the diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "m"'] & 1 \arrow[d, "\top"] \\ X \arrow[r, "\chi(m)"'] & \Omega \end{tikzcd}$$

is a pullback. Conversely, given h: $X \to \Omega$, let $Z = \{x\colon h^*(x)\}$ and $h^\# = (x \mapsto x)\colon Z \to X$. Then

$$\chi(h^\#) = (x \mapsto \exists y.\langle y, x\rangle \in h^\#) = (x \mapsto h^*(x)) = h.$$

So $(\Omega, \top)$ satisfies the conditions imposed on a truth value object.

We remark parenthetically here that it follows from the eliminability of descriptions for propositions that C(S)-*elements* of Ω *may be identified with sentences of the language of S.* To see this, associate with each sentence σ the C(S)-element of Ω $\star \mapsto \sigma\colon 1 \to \Omega$. Reciprocally, given a C(S)-element $f\colon 1 \to \Omega$, then $\vdash_S \exists!\omega\langle\star, \omega\rangle \in f$ and so, by Eliminability of Descriptions for Propositions, there is a sentence σ for which $\vdash_S \langle\star, \sigma\rangle \in f$. We associate σ with f. Provided we identify sentences when they are S-provably equal, these associations are mutually inverse.

Finally, C (S) *has power objects*. For, given an S-set X, define e_X: $X \times PX \to \Omega$ by $e_X = (\langle x, z\rangle \mapsto x \in z)$, and if $X \times Y \to \Omega$ define f : $Y \to PX$ by $f = (y \mapsto \{x\colon f^*(\langle x, y\rangle)\})$. It is now not hard to verify that f is the unique arrow $Y \to PX$ making the diagram

$$\begin{tikzcd} X \times Y \arrow[d, "1_X \times f"'] \arrow[dr, "f"] & \\ X \times PX \arrow[r, "e_X"'] & \Omega \end{tikzcd}$$

commute. So (PX, e_X) is a power object for X in $\mathsf{C}(S)$.

EXAMPLES OF TOPOSES

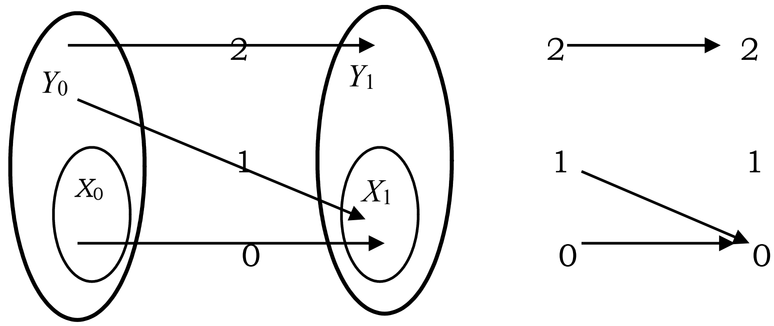
One of F. W. Lawvere's most penetrating insights was to conceive of a topos as a universe of *variable sets*. Here are some examples.

To begin with, consider the topos $\mathsf{Set}^{\rightarrow}$ of sets varying over two possible *states* 0 ('then"), 1 ("now"), with $0 \leqslant 1$. An *object* X here is a pair of sets X_0, X_1 together with a "transition" map p: $X_0 \rightarrow X_1$. An *arrow* f: $X \rightarrow Y$ is a pair of maps f_0: $X_0 \rightarrow Y_0$, f_1: $X_1 \rightarrow Y_1$ compatible with the transition maps in the sense that the diagram

$$\begin{tikzcd} X_0 \arrow[r, "p"] \arrow[d, "f_0"'] & X_1 \arrow[d, "f_1"] \\ Y_0 \arrow[r, "q"] & Y_1 \end{tikzcd}$$

commutes.

The truth value object Ω in $\mathsf{Set}^{\rightarrow}$ has 3 (rather than 2) elements. For if (m, X) is a subobject of Y in $\mathsf{Set}^{\rightarrow}$, then we may take $X_0 \subseteq Y_0$, $X_1 \subseteq Y_1$, f_0 and f_1 identity maps, and p to be the restriction of q to X_0. Then for any $y \in Y$ there are three possibilities, as depicted below: (0) $y \in X_0$, (1) $q(y) \in X_1$ and $y \notin X_0$, and (2) $q(y) \notin X_1$.

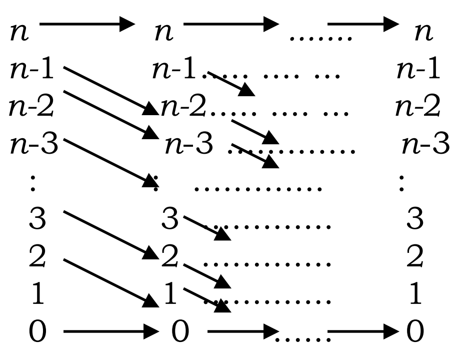


So if 2 = {0, 1} and 3 = {0, 1, 2} we take Ω to be the variable set 3 → 3 with 0 ↦ 1, 1 ↦ 1, 2 ↦ 2.

More generally, we may consider sets varying over *n*, or T, or any totally ordered set of stages. Objects in $\mathsf{Set}^{\boldsymbol{n}}$ are "sets through *n* successive stages", that is, (*n* – 1)-tuples of maps

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \ldots X_{n-2} \xrightarrow{f_{n-2}} X_{n-1}.$$

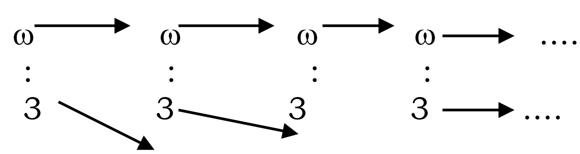
The truth value object in $\mathsf{Set}^{\boldsymbol{n}}$ looks like



Objects in Set^{ω} are "sets through discrete time", that is, infinite sequences of maps

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \ldots.$$

The truth value object in Set^{ω} looks like:



$$\begin{tikzcd}
2 \arrow[dr] & 2 \arrow[dr] & 2 \arrow[dr] & 2 \arrow[r] & \cdots \\
1 \arrow[dr] & 1 \arrow[dr] & 1 \arrow[dr] & 1 \arrow[r] & \cdots \\
0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

In each case there is "one more" truth value than stages: "truth" = "time" + 1.

Still more generally, we may consider the category $\mathrm{Set}^{\boldsymbol{P}}$ of sets varying over a poset P. As objects this category has *functors*[8] $P \to \mathrm{Set}$, i.e., maps F which assign to each $p \in P$ a set $F(p)$ and to each $p, q \in P$ such that $p \leqslant q$ a map F_{pq}: $F(p) \to F(q)$ satisfying:

$p \leqslant q \leqslant r$ implies that

$$\begin{tikzcd}
F(p) \arrow[r, "F_{pq}"] \arrow[dr, "F_{pr}"'] & F(q) \arrow[d, "F_{qr}"] \\
& F(r)
\end{tikzcd}$$

commutes

and

F_{pp} is the identity map on $F(p)$.

An *arrow* η: $F \to G$ in $\mathrm{Set}^{\boldsymbol{P}}$ is a *natural transformation* between F and G, which in this case is an assignment of a map η_p: $F(p) \to G(p)$ to each $p \in P$ in such a way that, whenever $p \leqslant q$, the diagram

$$\begin{tikzcd}
F(p) \arrow[r, "F_{pq}"] \arrow[d, "\eta_p"'] & F(q) \arrow[d, "\eta_q"] \\
G(p) \arrow[r, "G_{pq}"] & G(q)
\end{tikzcd}$$

commutes.

---

[8] Recall that any preordered set, and in particular any poset, may be regarded as a category.

To determine Ω in $\mathsf{Set}^{\boldsymbol{P}}$ we define a (pre)*filter over* $p \in P$ to be a subset U of $O_p = \{q \in P : p \leqslant q\}$ such that $q \in U, r \geq q \Rightarrow r \in U$. Then

$$\Omega(p) = \text{set of all filters over } p,$$

$$\Omega_{pq}(U) = U \cap O_q \ \text{ for } p \leqslant q, \ U \in \Omega(p).$$

The *terminal object* 1 in $\mathsf{Set}^{\boldsymbol{P}}$ is the functor on P with constant value $1 = \{0\}$ and t: $1 \to \Omega$ has $t_p(0) = O_p$ for each $p \in P$.

Objects in $\mathsf{Set}^{\boldsymbol{P}^*}$ —where P^* is the poset obtained by reversing the order on P— are called *presheaves* on P. In particular, when P is the partially ordered set $\mathsf{O}(X)$ of open sets in a topological space X, objects in $\mathsf{Set}^{\mathsf{O}(X)}$ called *presheaves on X*. So a presheaf on X is an assignment to each $U \in \mathsf{O}(X)$ of a set $F(U)$ and to each pair of open sets U, V such that $V \subseteq U$ of a map $F_{UV} : F(U) \to F(V)$ such that, whenever $W \subseteq U \subseteq V$, the diagram

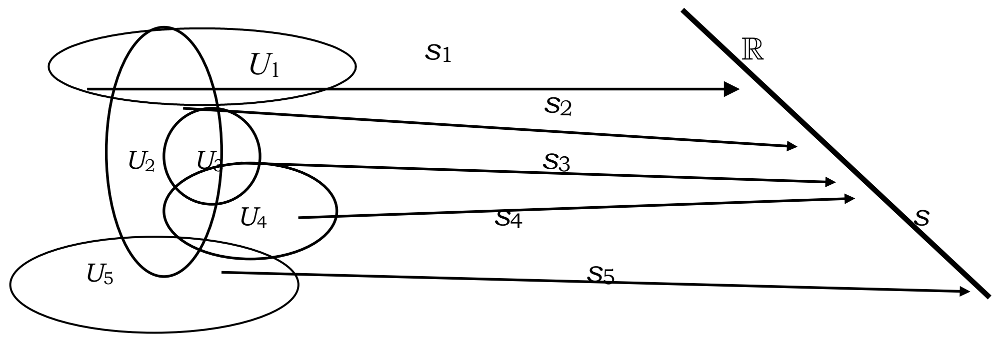
$$\begin{tikzcd} F(U) \arrow[r, "F_{UV}"] \arrow[dr, "F_{UW}"'] & F(V) \arrow[d, "F_{VW}"] \\ & F(W) \end{tikzcd}$$

commutes;

and

F_{UU} is the identity map on $F(U)$.

If $s \in F(U)$, write $s|_V$ for $F_{UV}(s)$—the *restriction* of s to V. A presheaf F is a *sheaf* if whenever $U = \bigcup_{i \in I} U_i$ and we are given a set $\{s_i : i \in I\}$ such that $s_i \in F(U_i)$ for all $i \in I$ and $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all $i, j \in I$, then there is a *unique* $s \in F(U)$ such that $s|_{U_i} = s_i$ for all $i \in I$.



For example, $C(U)$ = set of continuous real-valued functions on U, and $s|_V$ = restriction of s to V defines the sheaf of continuous real-valued functions on X.

It can be shown that the category of sheaves on a topological space is a topos: in fact the topos concept was originally devised by Grothendieck as a generalization of this idea.

The idea of a set varying over a poset can be naturally extended to that of a set varying over an arbitrary *small category.* Given a small category $\mathbf{C}$, we introduce the category $\mathsf{Set}^{\mathbf{C}}$ of *sets varying over* $\mathbf{C}$. Its objects are all functors $\mathbf{C} \to \mathsf{Set}$, and its arrows all natural transformations between such functors. Again, it can be shown that $\mathsf{Set}^{\mathbf{C}}$ is a topos.

An important special case arises when $\mathbf{C}$ is a one-object category, that is, a *monoid.* To be precise, a monoid is a pair $\mathbf{M} = (M, \cdot)$ with M a set and $\cdot$ a binary operation on M satisfying the associative law $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$ and possessing an identity element 1 satisfying $1 \cdot \alpha = \alpha \cdot 1 = \alpha$. (Note that a *group* is just a monoid with inverses, that is, for each α there is β for which $\alpha \cdot \beta = \beta \cdot \alpha = 1$.) Any object in $\mathsf{Set}^{\mathbf{M}}$ may be identified with a *set acted on by* $\mathbf{M}$, or $\mathbf{M}$-*set,* that is, a pair $(X, \bullet)$ with $\bullet$ a map $M \times X \to X$ satisfying $(\alpha \cdot \beta) \bullet x = \alpha \bullet (\beta \bullet x)$ and $1 \bullet x = x$. An arrow f: $(X, \bullet) \to (Y, \bullet)$ is an *equivariant* map f: $X \to Y$, i.e, such that $f(\alpha \bullet x) = \alpha \bullet f(x)$. The subobject classifier Ω in $\mathsf{Set}^{\mathbf{M}}$ is the collection of all *left ideals* of **M**, i.e. those $I \subseteq M$ for which $\alpha \in I, \beta \in M \Rightarrow \beta \cdot \alpha \in I$. The action of M on Ω is *division*, viz. $\alpha \bullet I = \{\beta \in M: \beta \cdot \alpha \in I\}$[9]. The truth arrow t: $1 \to \Omega$ is the map with value M.

Toposes can also arise as categories of "sets with a generalized equality relation", with arrows preserving that relation in an appropriate sense. Some of the most important examples in this regard are the *categories of Heyting algebra-valued sets.* Given a complete Heyting algebra H, an *H-valued set* is a pair (I, δ) consisting of a set I and a map δ: $I \times I \to H$ (the "generalized equality relation" on I) satisfying the following conditions, in which we write $\delta_{ii'}$ for $\delta(i, i')$ (and similarly below):

$$\delta_{ii'} = \delta_{i'i} \quad \text{(symmetry)}$$

$$\delta_{ii'} \wedge \delta_{i'i''} \leqslant \delta_{ii''} \text{ (substitutivity)}$$

The category Set_H of H-valued sets has as objects all H-valued sets. A Set_H -arrow f: $(I, \delta) \to (J, \varepsilon)$ is a map f: $I \times J \to H$ such that

$$\delta_{ii'} \wedge f_{ij} \leqslant f_{i'j} \qquad f_{ij} \wedge \varepsilon_{jj'} \leqslant f_{ij'} \quad \text{(preservation of identity)}$$

$$f_{ij} \wedge f_{ij'} \leqslant \varepsilon_{i'j} \quad \text{(single-valuedness)}$$

$$\bigvee_{j \in J} f_{ij} = \delta_{ii} \quad \text{(defined on } I\text{)}$$

The composite $g \circ f$ of two arrows f: $(I, \delta) \to (J, \varepsilon)$ and g: $(J, \varepsilon) \to (K, \eta)$ is given by

[9] This is because if X is a sub-**M**-set of Y, each $y \in Y$ is naturally classified by the left ideal $\{\alpha \in M: \alpha \bullet y \in X\}$.

$$(g \circ f)_{ik} = \bigvee_{j \in J} f_{ij} \wedge g_{jk}\,.$$

Then Set_H is a topos in which the subobject classifier is the H-valued set $(H, \Leftrightarrow)$.

BASIC PROPERTIES OF TOPOSES

Given a topos E, and an E-arrow u: $A \to \Omega$, we choose $\bar{u} : B \to A$, the *kernel* of u, so that

$$\begin{tikzcd} B \arrow[r] \arrow[d, "\bar{u}"'] & 1 \arrow[d, "\top"] \\ A \arrow[r, "u"'] & \Sigma \end{tikzcd}$$

is a pullback and $\Pi(1_A) = 1_A$. Note that then $\chi(\bar{u}) = u$.

Now given monics m, n with common codomain A, write $m \subseteq n$ if there is a commutative diagram of the form

$$\begin{tikzcd} \bullet \arrow[rr] \arrow[dr, "m"'] & & \bullet \arrow[dl, "n"] \\ & A & \end{tikzcd}.$$

Write $m \sim n$ if $m \subseteq n$ and $n \subseteq m$. Then $\sim$ is an equivalence relation and $m \sim n$ iff there is an isomorphism such that

$$\begin{tikzcd} \bullet \arrow[rr, phantom, "\cong"] \arrow[dr, "m"'] & & \bullet \arrow[dl, "n"] \\ & A & \end{tikzcd}$$

commutes

Equivalence classes under $\cong$ are called *subobjects* of A. Write $[m]$ for the equivalence class of m: for u: $A \to \Omega$, $[\bar{u}]$ is called the subobject of A *classified* by u. We define $[m] \subseteq [n] \equiv m \subseteq n$. The relation $\subseteq$—*inclusion*—is a partial ordering on the collection **Sub**(A) of subobjects of A. It is easily shown that $[m] = [n] \equiv \chi(m) = \chi(n)$, so we get a bijection between **Sub**(A)

and E(A, Ω), the collection of E-arrows $A \to \Omega$. Define, for $u, v \in$ E(A, Σ), $u \leqslant v \equiv \bar{u} \subseteq \bar{v}$. This transfers the partial ordering $\subseteq$ on **Sub**(A) to E(A, Ω).

It can be shown by an elementary argument that, in a topos, any diagram of the form

$$\begin{tikzcd} & \bullet \arrow[d, "m"] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

with m monic can be completed to a pullback

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "f^{-1}(m)"'] & \bullet \arrow[d, "m"] \\ \bullet \arrow[r, "f"] & \bullet \end{tikzcd}$$

The arrow $f^{-1}(m)$ is called the *inverse image* of m under f. We may in fact take $f^{-1}(m)$ to be $\overline{\chi(m) \circ f}$.

Now define $\delta_A = \langle 1_A, 1_A \rangle: A \rightarrowtail A \times A$, $eq_A = \chi(\delta_A)$, $T = T_A = \chi(1_A)$. Then $\overline{T_A} = 1_A$, so $u \leqslant T_A$ for all $u \in$ E(A, Ω).

Given a pair of monics m, n with common codomain A, we obtain their *intersection* $m \cap n$ by first forming the pullback

$$\begin{tikzcd} \bullet \arrow[r, "n^{-1}(m)"] \arrow[d, "m^{-1}(n)"'] & \bullet \arrow[d, "m"] \\ \bullet \arrow[r, "n"] & \bullet \end{tikzcd}$$

and then defining $m \cap n = n \circ m^{-1}(n) = m \circ n^{-1}(m)$. This turns (**Sub**$(A)$, $\subseteq$) into a *lower semilattice,* that is, a partially ordered set with meets. We transfer $\cap$ to E(A, Ω) by defining $u \wedge v = \chi(\bar{u} \wedge \bar{v})$. This has the effect of turning E(A, Ω) into a lower semilattice as well.

IV

INTERPRETING A LOCAL LANGUAGE IN A TOPOS: SOUNDNESS AND COMPLETENESS

INTERPRETATIONS AND SOUNDNESS

Let $\mathscr{L}$ be a local language and E a topos. An *interpretation* I of $\mathscr{L}$ in E is an assignment:

- to each type $\mathbf{A}$, of an E-object $\mathbf{A}_I$ such that:

  $(\mathbf{A}_1 \times \ldots \times \mathbf{A}_n)_I = (\mathbf{A}_1)_I \times \ldots \times (\mathbf{A}_n)_I,$

  $(\mathbf{PA})_I = P\mathbf{A}_I,$

  $\mathbf{1}_I = 1$, the terminal object of E,

  $\Omega_I = \Omega$, the truth-value object of E.

- to each function symbol $\boldsymbol{f}$**:** $\mathbf{A} \to \mathbf{B}$, an E-arrow

  $\boldsymbol{f}_I\mathbf{:}\ \mathbf{A}_I \to \mathbf{B}_I.$

We shall sometimes write A_{E} or just A for $\mathbf{A}_I$.

We extend I to terms of $\mathscr{L}$ as follows. If $\tau : \mathbf{B}$, write $\boldsymbol{x}$ for $(x_1, \ldots, x_n)$, any sequence of variables containing all variables of τ (and call such sequences *adequate* for τ). Define the E-arrow

$$[\![\tau]\!]_{\boldsymbol{x}}\colon A_1 \times \ldots \times A_n \to B$$

recursively as follows:

$$[\![\star]\!]_{\boldsymbol{x}} = A_1 \times \ldots \times A_n \to 1,$$

$$[\![x_i]\!]_{\boldsymbol{x}} = \pi_i\colon A_1 \times \ldots \times A_n \to A_i,$$

$$[\![\boldsymbol{f}(\tau)]\!]_{\boldsymbol{x}} = \boldsymbol{f}_I \circ [\![\tau]\!]_{\boldsymbol{x}}$$

$$[\![\tau_1, \ldots, \tau_n]\!]_{\boldsymbol{x}} = \langle [\![\tau_1]\!]_{\boldsymbol{x}}, \ldots, [\![\tau_n]\!]_{\boldsymbol{x}} \rangle,$$

$$\llbracket (\tau)_i \rrbracket_{\boldsymbol{x}} = \pi_i \circ \llbracket \tau \rrbracket_{\boldsymbol{x}},$$

$$\llbracket \{y\colon \alpha\} \rrbracket_{\boldsymbol{x}} = \ (\llbracket \alpha(y/u) \rrbracket_{u\boldsymbol{x}} \circ can)\hat{}\,,$$

where in this last clause u differs from $x_1, \ldots, x_n$, is free for y in α, y is of type **C**, (so that B is of type **PC**), can is the canonical isomorphism $C \times (A_1 \times \ldots \times A_n) \cong C \times A_1 \times \ldots \times A_n$, and f is as defined for power objects. (To see why, consider the diagrams

$$\begin{tikzcd}
C \times A_1 \times \ldots \times A_n \arrow[rr, "(\llbracket \alpha(y/u) \rrbracket_{u\boldsymbol{x}}"] & & \Omega \\
C \times (A_1 \times \ldots \times A_n) \arrow[u, "can"] \arrow[urr, "f"] & &
\end{tikzcd}
\qquad
A_1 \times \ldots \times A_n \xrightarrow{f} PC$$

In set theory, $f(a_1, \ldots, a_n) = \{y \in C\colon \alpha(y, a_1, \ldots, a_n)\}$, so we take $\llbracket \{y\colon \alpha\} \rrbracket_{\boldsymbol{x}}$ to be f .)

Finally,

$\llbracket \sigma = \tau \rrbracket_{\boldsymbol{x}} = eq_C \circ \llbracket <\sigma, \tau> \rrbracket_{\boldsymbol{x}}$ (with σ, τ **: C**)

$\llbracket \sigma \in \tau \rrbracket_{\boldsymbol{x}} = e_C \circ \llbracket <\sigma, \tau> \rrbracket_{\boldsymbol{x}}$ (with σ **: C**, τ **: PC** and where e_C is as defined for power objects.)

If τ **: B** is closed, then $\boldsymbol{x}$ may be taken to be the empty sequence $\varnothing$. In this case we write $\llbracket \tau \rrbracket$ for $\llbracket \tau \rrbracket_\varnothing$; this is an arrow $1 \to B$. In particular, if τ is $\{y : \alpha\}$ of type **PC**, then $\llbracket \{y\colon \alpha\} \rrbracket$ is an arrow $1 \to PC$ which corresponds to the subobject of C classified by $\llbracket \alpha \rrbracket_y\colon C \to \Omega$.

We note that

$$\llbracket \top \rrbracket_{\boldsymbol{x}} = \llbracket \bigstar = \bigstar \rrbracket_{\boldsymbol{x}} = eq \circ <\llbracket \bigstar \rrbracket_{\boldsymbol{x}}, \llbracket \bigstar \rrbracket_{\boldsymbol{x}}> = T.$$

For any finite set $\Gamma = \{\alpha_1, \ldots, \alpha_m\}$ of formulas write

$$[\![\Gamma]\!]_{I,\boldsymbol{x}} \text{ for } \begin{cases} [\![\alpha_1]\!]_{I,\boldsymbol{x}} \wedge \ldots \wedge [\![\alpha_m]\!]_{I,\boldsymbol{x}} & \text{if } m \geq 1 \\ T & \text{if } m = 0. \end{cases}$$

Given a formula α, let $\boldsymbol{x} = (x_1, \ldots, x_n)$ list all free variables of $\Gamma \cup \{\alpha\}$; write

$$\Gamma \vDash_I \alpha \ \text{ or } \Gamma \vDash_{\mathsf{E}} \alpha \ \text{ for } \ [\![\Gamma]\!]_{I,\boldsymbol{x}} \leqslant [\![\alpha]\!]_{I,\boldsymbol{x}}$$

$\Gamma \vDash_I \alpha$ is read "$\Gamma : \alpha$ is *valid* under the interpretation I in E." If S is a local set theory, we say that I is a *model* of S if every member of S is valid under I. Notice that

$$\vDash_I \beta \ \equiv \ [\![\beta]\!]_{\boldsymbol{x}} = T.$$

So if I is an interpretation in a *degenerate* topos, i.e., a topos possessing just one object up to isomorphism, then $\vDash_I \alpha$ for all α, so that I is a model of the collection of all formulas.

We write:

$\Gamma \vDash \alpha$ for $\Gamma \vDash_I \alpha$ for all I

$\Gamma \vDash_S \alpha$ for $\Gamma \vDash_I \alpha$ for every model I of S.

It can be shown (tediously) that the basic axioms and rules of inference of any local language are valid under any interpretation. This yields the

Soundness Theorem.

$\Gamma \vdash \alpha \ \Rightarrow \ \Gamma \vDash \alpha \qquad \Gamma \vdash_S \alpha \ \Rightarrow \ \Gamma \vDash_S \alpha.$

A local set theory S is said to be *consistent* if it is not the case that $\vdash_S \bot$. The Soundness Theorem yields the

Corollary. *Any pure local set theory is consistent.*

Proof. Set up an interpretation I of $\mathscr{L}$ in the topos Finset of finite sets as follows: $\mathbf{1}_I = 1$, $\Omega_I = \{0, 1\} = 2$, for any ground type **A,** $\mathbf{A}_I$ is any nonempty finite set. Extend I to arbitrary types in the obvious way. Finally $\mathbf{f}_I$: $\mathbf{A}_I \to \mathbf{B}_I$ is to be any map from $\mathbf{A}_I$ to $\mathbf{B}_I$.

If $\vdash \bot$, then $\vdash \alpha$, so $\vDash_I \alpha$ for any formula α. In particular $\vDash_I \quad u = v$ where u, v are variables of type **P1.** Hence $[\![u]\!]_{I,\,uv} = [\![v]\!]_{I,\,uv}$, that is, the two projections $P1 \times P1 \to P1$ would have to be identical, a contradiction.

THE COMPLETENESS THEOREM

Given a local set theory S in a language $\mathscr{L}$, define the *canonical interpretation* $C(S)$ of $\mathscr{L}$ in $\mathrm{C}(S)$ by:

$$\mathbf{A}_{C(S)} = U_{\mathbf{A}} \qquad \mathbf{f}_{C(S)} = (x \mapsto \mathbf{f}(x)) : U_{\mathbf{A}} \to U_{\mathbf{B}} \qquad \text{for } \mathbf{f} : \mathbf{A} \to \mathbf{B}$$

A straightforward induction establishes

$$[\![\tau]\!]_{C(S)\,\boldsymbol{x}} = (\boldsymbol{x} \mapsto \tau).$$

This yields

(*) $$\Gamma \vDash_{\mathrm{C}(S)} \alpha \equiv \Gamma \vdash_S \alpha\ .$$

For:

$$\begin{aligned} \vDash_{\mathrm{C}(S)} \alpha \quad &\equiv \quad [\![\alpha]\!]_{C(S)\,\boldsymbol{x}} = T \\ &\equiv \quad (\boldsymbol{x} \mapsto \alpha) = (\boldsymbol{x} \mapsto \top) \\ &\equiv \quad \vdash_S \alpha = \top \\ &\equiv \quad \vdash_S \alpha. \end{aligned}$$

Since $\Gamma \vdash_S \alpha \ \equiv\ \vdash_S \gamma \to \alpha$, where γ is the conjunction of all the formulas in Γ, the special case yields the general one.

Equivalence (*) may be read as asserting that $C(S)$ is a *canonical model* of S. This fact yields the

Completeness Theorem.

$\Gamma \vDash \alpha \;\Rrightarrow\; \Gamma \vdash \alpha \qquad\qquad \Gamma \vDash_S \alpha \;\Rrightarrow\; \Gamma \vdash_S \alpha$

Proof. We know that $C(S)$ is a model of S. Therefore, using (*),

$$\Gamma \vDash_S \alpha \;\Rrightarrow\; \Gamma \vDash_{C(S)} \alpha \;\Rrightarrow\; \Gamma \vdash_S \alpha.$$

EVERY TOPOS IS LINGUISTIC

A topos of the form $\mathrm{C}(\mathcal{S})$ is called a *linguistic* topos. We sketch a proof that every topos is equivalent to a linguistic one.

Given a topos E, we shall exhibit a theory $Th(\mathrm{E})$ and an equivalence $\mathrm{E} \simeq \mathrm{C}(Th(\mathrm{E}))$.

We define the local language $\mathscr{L}_\mathrm{E}$ associated with E—also called the *internal langue* of E—as follows. The ground type symbols of $\mathscr{L}_\mathrm{E}$ are taken to match the objects of E other than its terminal and truth-value objects, that is, for each E-object A (other than 1, Ω) we assume given a ground type $\mathbf{A}$ in $\mathscr{L}_\mathrm{E}$. Next, we define for each type symbol $\mathbf{A}$ an E-object $\mathbf{A}_\mathrm{E}$ by

$\mathbf{A}_\mathrm{E} = A$ for ground types $\mathbf{A}$,

$(\mathbf{A} \times \mathbf{B})_\mathrm{E} = \mathbf{A}_\mathrm{E} \times \mathbf{B}_\mathrm{E}$ [10]

$(\mathbf{PA})_\mathrm{E} = P(\mathbf{A})_\mathrm{E}$.

The function symbols of $\mathscr{L}$ are triples $(f, \mathbf{A}, \mathbf{B}) = \boldsymbol{f}$ with $f: \mathbf{A}_\mathrm{E} \to \mathbf{B}_\mathrm{E}$ in E. The signature of $\boldsymbol{f}$ is $\mathbf{A} \to \mathbf{B}$.[11]

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[10] Note that, if we write C for $A \times B$, then while $\mathbf{C}$ is a ground type, $\mathbf{A} \times \mathbf{B}$ is a product type. Nevertheless $\mathbf{C}_\mathrm{E} = (\mathbf{A} \times \mathbf{B})_\mathrm{E}$.

The *natural interpretation*—denoted by E—of $\mathscr{L}_{\mathsf{E}}$ in E is determined by the assignments

$$\mathbf{A}_{\mathsf{E}} = \mathbf{A} \ \text{ for each ground type } \mathbf{A} \qquad\qquad (f, \mathbf{A}, \mathbf{B})_{\mathsf{E}} = \boldsymbol{f}.$$

The local set theory $Th(\mathsf{E})$, the *theory* (or *internal logic*) of E, is the theory in $\mathscr{L}_{\mathsf{E}}$ generated by the collection of all sequents $\Gamma : \alpha$ such that $\Gamma \vDash_{\mathsf{E}} \alpha$ under the natural interpretation of $\mathscr{L}_{\mathsf{E}}$ in E. Then we have

$$\Gamma \vdash_{Th(\mathsf{E})} \alpha \quad \equiv \; \Gamma \vDash_{\mathsf{E}} \alpha \ .$$

For if $\Gamma \vdash_{Th(\mathsf{E})} \alpha$ then by Soundness $\Gamma \vDash_{Th(\mathsf{E})} \alpha$ i.e., $\Gamma : \alpha$ is valid in every model of $Th(\mathsf{E})$. But by definition E is a model of $Th(\mathsf{E})$.

It can now be shown that the canonical functor $F\colon \mathsf{E} \to \mathsf{C}(\mathrm{Th}(\mathsf{E}))$ defined by

$FA = U_{\mathbf{A}}$ for each E-object A

$Ff = (x \mapsto \boldsymbol{f}(x))\colon U_{\mathbf{A}} \to U_{\mathbf{B}}$ for each E-arrow $f\colon A \to B$

is an equivalence of categories. This is the **Equivalence Theorem.**

Here is another fact about $Th(\mathsf{E})$.

A local set theory S in a language $\mathscr{L}$ is said to be *well-termed* if:

- whenever $\vdash_S \exists! x\alpha$, there is a term τ of $\mathscr{L}$ whose free variables are those of α with x deleted such that $\vdash_S \alpha(x/\tau)$,

---

[11] Note the following: if $f\colon A \times B \to D$, in E, then, writing C for $A \times B$ as in the footnote above, $(f, \mathbf{C}, \mathbf{D})$ and $(f, \mathbf{A} \times \mathbf{B}, \mathbf{D})$ are both function symbols of $\mathscr{L}_{\mathsf{E}}$ associated with f. But the former has signature $\mathbf{C} \to \mathbf{D}$, while the latter has the different signature $\mathbf{A} \times \mathbf{B} \to \mathbf{D}$.

and *well-typed* if

- for any S-set X there is a type symbol $\mathbf{A}$ of $\mathscr{L}$ such that $U_{\mathbf{A}} \cong X$ in $\mathsf{C}(S)$.

A local set theory which is both well-termed and well-typed is said to be *well-endowed.* It follows from the Equivalence Theorem that, *for any topos* E, $Th(\mathsf{E})$ *is well-endowed.*

The property of being well-endowed can also be expressed category-theoretically. For local set-theory S, let $\mathsf{T}(S)$—the *category of S-types and terms* —be the subcategory of $\mathsf{C}(S)$ whose objects are all S-sets of the form $U_{\mathbf{A}}$ and whose arrows are all S-maps of the form $(\boldsymbol{x} \mapsto \tau)$. Then S is well-endowed exactly when the insertion functor $\mathsf{T}(S) \to \mathsf{C}(S)$ is an equivalence of categories.

We remark finally that, for well-termed S, and any S-set X of type $\mathbf{PA}$, $\mathsf{C}(S)$-arrows $1 \to X$—the $\mathsf{C}(S)$-elements of X—may be identified with closed terms τ of type $\mathbf{A}$ for which $\vdash_S \tau \in X$.

V

TRANSLATIONS OF LOCAL LANGUAGES

TRANSLATIONS

A *translation* $\mathbf{K}$: $\mathscr{L} \to \mathscr{L}'$ of a local language $\mathscr{L}$ into a local language $\mathscr{L}'$ is a map which assigns to each type $\mathbf{A}$ of $\mathscr{L}$ a type $\mathbf{KA}$ of $\mathscr{L}'$ and to each function symbol $\boldsymbol{f}$: $\mathbf{A} \to \mathbf{B}$ of signature $\mathbf{KA} \to \mathbf{KB}$ in such a way that

$$\mathbf{K1} = \mathbf{1},\ \ \mathbf{K}\Omega = \Omega,\ \ \mathbf{K}(\mathbf{A}_1 \times \ldots \times \mathbf{A}_n) = \mathbf{KA}_1 \times \ldots \times \mathbf{KA}_n,\ \ \mathbf{K}(\mathbf{PA}) = \mathbf{PKA}.$$

Any translation $\mathbf{K}$: $\mathscr{L} \to \mathscr{L}'$ may be extended to the terms of $\mathscr{L}$ in the evident recursive way—i.e., by defining $\mathbf{K}\star = \star$, $\mathbf{K}(\boldsymbol{f}(\tau)) = \mathbf{K}\boldsymbol{f}(\mathbf{K}\tau)$, $\mathbf{K}(\sigma \in \tau) = \mathbf{K}\sigma \in \mathbf{K}\tau$, etc.—so that if $\tau : \mathbf{A}$, then $\mathbf{K}\tau : \mathbf{KA}$. We shall sometimes write $\alpha_{\mathbf{K}}$ for $\mathbf{K}\alpha$.

If S, S' are local set theories in $\mathscr{L}$, $\mathscr{L}'$ respectively, a translation $\mathbf{K}$: $\mathscr{L} \to \mathscr{L}'$ is a *translation of S into S'*, and is written $\mathbf{K}$: $S \to S'$ if, for any sequent $\Gamma : \alpha$ of $\mathscr{L}$,

(*) $$\Gamma \vdash_S \alpha \quad \Rightarrow \quad \mathbf{K}\Gamma \vdash_{S'} \mathbf{K}\alpha,$$

where if $\Gamma = \{\alpha_1, \ldots, \alpha_n\}$, $\mathbf{K}\Gamma = \{\mathbf{K}\alpha_1, \ldots, \mathbf{K}\alpha_n\}$. If the reverse implication to (*) also holds, $\mathbf{K}$ is called a *conservative translation* of S into S'. If S' is an extension of S and the identity translation of S into S' is conservative, S' is called a *conservative extension of S.*

There is a natural correspondence between models of S in a topos E and translations of S into *Th*(E): in particular the *identity translation Th*(E) $\to$ *Th*(E) corresponds to the *natural interpretation* of *Th*(E) in E.

Now let E, E' be toposes with specified terminal objects, products, projection arrows, truth-value objects, power objects and evaluation arrows: a functor F: $\mathsf{E} \to \mathsf{E}'$ which preserves all these is called a *logical functor.* It is easily seen that the canonical functor $\mathsf{E} \to \mathsf{C}(Th(\mathsf{E}))$ is logical.

If $\mathbf{K}$: $S \to S'$ is a translation, then for terms σ, τ of $\mathscr{L}$, $\vdash_S \sigma = \tau$ implies $\vdash_{S'} \mathbf{K}\sigma = \mathbf{K}\tau$, so that $\mathbf{K}$ induces a map $C_{\mathbf{K}}$ from the class of S-sets to the class of S'-sets via

$$\mathfrak{C}_{\mathbf{K}}([\sigma]_S) = [\mathbf{K}\sigma]_{S'}.$$

$\mathfrak{C}_{\mathbf{K}}$ is actually a *logical functor* $\mathsf{C}(S) \to \mathsf{C}(S')$. Writing Loc for the category of local set theories and translations, and Top for the category of toposes and logical functors, $\mathfrak{C}$ is a functor $\mathsf{Loc} \to \mathsf{Top}$. And reciprocally any logical functor F: $\mathsf{E} \to \mathsf{E}'$ induces a translation $Th(F)$: $\mathrm{Th}(\mathsf{E}) \to Th(\mathsf{E}')$ in the natural way, so yielding a functor Th: $\mathsf{Loc} \to \mathsf{Top}$. C and Th are "almost" inverse, making Loc and Top "almost" equivalent.

Given a local set theory S in a language $\mathscr{L}$, define a translation $\mathbf{K}$: $\mathscr{L} \to \mathscr{L}_{\mathsf{C}(S)}$ by

$$\mathbf{KA} = \mathbf{U}_{\boldsymbol{A}}, \qquad \mathbf{K}\boldsymbol{f} = (f, \mathbf{A}, \mathbf{B}) \ \text{ if } \boldsymbol{f}\mathbf{:}\ \mathbf{A} \to \mathbf{B}.$$

An easy induction on the formation of terms shows that, for any term τ of $\mathscr{L}$,

$$[\![\tau]\!]_{C(S)\ \boldsymbol{x}} = [\![\mathbf{K}\tau]\!]_{C(S)\ \mathbf{K}\boldsymbol{x}}$$

It follows from this that $\mathbf{K}$ is a *conservative* translation of S into $Th(\mathsf{C}(S))$. For

$$\Gamma \vdash_S \alpha \ \equiv \Gamma \vdash_{\mathsf{C}(S)} \alpha \ \equiv \ \mathbf{K}\Gamma \vdash_{\mathsf{C}(S)} \mathbf{K}\alpha \ \equiv \ \mathbf{K}\Gamma \vdash_{Th(\mathsf{C}(S))} \mathbf{K}\alpha,$$

Accordingly *any local set theory can be conservatively embedded in one which is well-endowed.*

ADJOINING INDETERMINATES

A *constant* of type $\mathbf{A}$ in a local language $\mathscr{L}$ is a term of the form $\boldsymbol{f}(\star)$, where $\boldsymbol{f}\colon \mathbf{1} \to \mathbf{A}$. Write $\mathscr{L}(\boldsymbol{c})$ for the language obtained from $\mathscr{L}$ by adding a new function symbol $\boldsymbol{c}\colon \mathbf{1} \to \mathbf{A}$ and write c for $\boldsymbol{c}(\star)$. Given a local set theory S in $\mathscr{L}$, and a formula α of $\mathscr{L}$ with exactly one free variable x of type $\mathbf{A}$, write $S(\alpha)$ for the theory in $\mathscr{L}(\mathbf{c})$ generated by S together with all sequents of the form $:\beta(x/c)$ where $\alpha \vdash_S \beta$. Since clearly $\vdash_{S(\alpha)} \alpha(x/c)$, it follows that $\vdash_{S(\alpha)} \exists x\alpha$.

In $S(\alpha)$, c behaves as an *indeterminate,* or *generic* of sort α in the sense that *it can be arbitrarily assigned any value satisfying* α. To be precise, one can prove the

Theorem. *Let S' be a local set theory in a local language $\mathscr{L}'$ and let* $\mathbf{K}$: $S \to S'$. *Then for any constant c' of $\mathscr{L}'$ of type* $\mathbf{KA}$ *such that* $\vdash_{S'} \alpha_{\mathbf{K}}(c')$, *there is a unique translation* $\mathbf{K}'\colon S(\alpha) \to S$ *extending* $\mathbf{K}$ *such that* $\mathbf{K}'(c) = c'$.

The proof, which is omitted here, uses a

Lemma. *For any sequent* $\Gamma : \gamma$ *of* $\mathscr{L}$ *we have*

$$\Gamma(x/c) \vdash_{\mathbf{S(\alpha)}} \gamma(x/c) \;\equiv\; \alpha, \Gamma \vdash_{\mathbf{S}} \gamma\ .$$

If I is an S-set and α the formula $x \in_\gamma I$, we write S_I or $S(I)$ for $S(\alpha)$ and call it the theory obtained from S by *adjoining an indeterminate* (or *generic*) element *of I.* It follows from the Lemma above that, for any formula γ of $\mathscr{L}$ in which i is free for x,

$$\vdash_{\mathrm{S}(I)} \gamma(x/c) \quad\equiv\quad \vdash_S \forall i{\in}I\, \gamma(x/i).$$

If α is the formula $x = x$ with x **: A,** then $S(\alpha)$ is written $S(\mathbf{A})$ and called the theory obtained from S by *adjoining an indeterminate of type* **A.** In particular, let S_0 be the pure local set theory in the local language $\mathscr{L}_0$ with no ground types or function symbols. Evidently S_0 is an *initial object* in the category **Loc**: there is a unique translation of L_0 into any given local set theory S. (Similarly, the topos $\mathbf{C}(S_0)$ is an initial object in the category **top**.) Now consider the theory $S_0(\mathbf{A})$, where **A** is a type symbol of $\mathscr{L}_0$: **A** may be considered a type symbol of *any* local language $\mathscr{L}$. If d is a constant of type **A** in $\mathscr{L}$, and S a local set theory in $\mathscr{L}$, there is then a unique translation **K**: $S_0(\mathbf{A}) \rightarrow S$ mapping c to d. So $S_0(\mathbf{A})$ may be considered *the universal theory of an indeterminate of type* **A.**

It is not hard to show that:

$$\vdash_S \exists x\, \alpha \;\equiv\; S(\alpha) \text{ is a conservative extension of } S$$

$$\vdash_S \neg\exists x\, \alpha \;\equiv\; S(\alpha) \text{ is inconsistent.}$$

In set theory it is customary to introduce the function value $f(x)$ or fx when x is in the domain of a function f. This device can also be employed legitimately within a local set theory. Let S be a local set theory in a local language $\mathscr{L}$, and let f: $X \rightarrow Y$ be an S-map with X : **PA** and Y: **PB.** Let $\mathscr{L}^*$ be obtained from $\mathscr{L}$ by adding a new function symbol $\boldsymbol{f^*}$**: A → B** and let S^* be the theory generated by S together with the sequent

$$x \in X : \langle x, \boldsymbol{f^*}(x)\rangle \in f.$$

In S^*, $\boldsymbol{f^*}(x)$ is the *value of f at x.* It can be shown that S^* is a *conservative extension* of S: this means that we can add function values to any local set theory without materially altering it.

VI
USES OF THE EQUIVALENCE THEOREM

BASIC APPLICATIONS

Whenever a given property P preserved under equivalence of categories can be shown to hold in any linguistic topos, it follows from the Equivalence Theorem that P holds in any topos whatsoever. For many (but not all) properties P holding in Set this is usually a straightforward matter. For example, it is easy to see that $\varnothing_{\mathbf{1}}$ is an initial object in $\mathsf{C}(S)$, so that *any topos has an initial object. We write* 0 *for* $\varnothing_{\mathbf{1}}$.

Similarly any linguistic topos has coproducts: if X and Y are S-sets, the S-set $X + Y$ is a coproduct and the arrows σ_1 and σ_2 are the S-maps $x \mapsto <\{x\}, \varnothing>$ and $y \mapsto <\varnothing, \{y\}>$ respectively.

In the same sort of way the Equivalence Theorem can be used to show that any topos E possesses the following properties:

- E has exponentials.
- E is *balanced,* that is, any arrow which is simultaneously epic and monic is an isomorphism.
- E has *epic-monic factorization*, i.e., each arrow f is the composite of an epic and a monic: this monic is called the *image* of f.
- In E, pullbacks of epic arrows are epic.

THE STRUCTURE OF Ω AND **SUB**(A)

Let S be a local set theory. We define the *entailment relation* on Ω to be the S-set

$$\blacktriangleright = \{<\omega, \omega'>: \omega \rightarrow \omega'\}.$$

Given an S-set X, we define the *inclusion relation* on PX to be the S-set

$$\preceq_X = \{<u, v> \in PX \times PX: u \subseteq v\}.$$

It follows from facts concerning $\rightarrow$, $\wedge$, $\vee$ already established that

$\vdash_S <\Omega, \blacktriangleright>$ *is a Heyting algebra with top element* $\top$ *and bottom element* $\bot$.

Similarly,

$\vdash_S <PX, \preceq_X>$ *is a Heyting algebra with top element* X *and bottom element* $\varnothing$.

Let $\Omega(S)$ be the collection of *sentences* (closed formulas) of $\mathscr{L}$, where we identify two sentences α, β whenever $\vdash_S \alpha \leftrightarrow \beta$. Define the relation $\leqslant$ on $\Omega(S)$ by

$$\alpha \leqslant \beta \equiv \vdash_S \alpha \rightarrow \beta.$$

Then $<\Omega(S), \leqslant>$ is a Heyting algebra, called the (external) *algebra of truth values of S.* Its top element is T_Ω and its bottom element is the characteristic arrow of $\varnothing \rightarrowtail 1$.

If X is an S-*set,* write $Pow(X)$ for the collection of all S-sets U such that $\vdash_S U \subseteq V$ and define the relation $\sqsubseteq$ on $Pow(X)$ by $U \sqsubseteq V \equiv \vdash_S U \subseteq V$.

Then $(Pow(X), \sqsubseteq)$ is a Heyting algebra, called the (external) *algebra of subsets of X.*

Given a topos E, we can apply all this to the theory $Th(\mathsf{E})$; invoking the fact that $\vdash_{Th(\mathsf{E})} \alpha \equiv \models_{\mathsf{E}} \alpha$ then gives

$\models_{\mathsf{E}} <\Omega, \preceq>$ *and* $<PA, \preceq_A>$ *are Heyting algebras,*

where A is any E-object. These facts are sometimes expressed by saying that Ω and PA are *internal* Heyting algebras in E.

What are the "internal" logical operations on Ω in E? That is, which arrows represent $\wedge$, $\vee$, $\neg$, $\rightarrow$? Working in a linguistic topos and then transferring the result to an arbitrary topos via the Equivalence Theorem shows that, in E,

$\wedge$: $\Omega \times \Omega \rightarrow \Omega$ is the characteristic arrow of the monic

$<\top, \top>: 1 \rightarrow \Omega \times \Omega$

$\vee$: $\Omega \times \Omega \rightarrow \Omega$ is the characteristic arrow of the image of

$$\Omega + \Omega \xrightarrow{<T_\Omega, 1_\Omega> + <1_\Omega, T_\Omega>} \Omega \times \Omega$$

$\neg$: $\Omega \rightarrow \Omega$ is the characteristic arrow of $\perp : 1 \rightarrow \Omega$.

$\rightarrow$: $\Omega \times \Omega \rightarrow \Omega$ is the characteristic arrow of the equalizer of the pair of arrows $\pi_1, \wedge$: $\Omega \times \Omega \rightarrow \Omega$. (Here we recall that the *equalizer* of a pair of arrows with a common domain is the largest subobject of the domain on which they both agree.)

It can then be shown that these "logical arrows" are the natural interpretations of the logical operations in any topos E, in the sense that, for any interpretation of a language $\mathscr{L}$ in E,

$$[\![\alpha \wedge \beta]\!]_{\boldsymbol{x}} = \wedge \circ [\![<\alpha, \beta>]\!]_{\boldsymbol{x}}$$

$$[\![\alpha \vee \beta]\!]_{\boldsymbol{x}} = \vee \circ [\![<\alpha, \beta>]\!]_{\boldsymbol{x}}$$

$$[\![\neg\alpha]\!]_{\boldsymbol{x}} = \neg \circ [\![\alpha]\!]_{\boldsymbol{x}}$$

$$[\![\alpha \rightarrow \beta]\!]_{\boldsymbol{x}} = \rightarrow \circ [\![<\alpha, \beta>]\!]_{\boldsymbol{x}} .$$

We now turn to the "external" formulation of these ideas. First, for any topos E and any E-object A, ($\mathbf{Sub}(A)$, $\subseteq$) is a Heyting algebra. For when E is of the form $\mathsf{C}(S)$, and A an S-set X, we have a natural isomorphism ($Pow(X)$, $\sqsubseteq$) $\cong$ ($\mathbf{Sub}(X)$, $\subseteq$) given by

$$U \mapsto [(x \mapsto x): U \rightarrowtail X]$$

for $U \in Pow(X)$. Since we already know that ($Pow(X)$, $\sqsubseteq$) is a Heyting algebra, so is ($\mathbf{Sub}(X)$, $\subseteq$). Thus the result holds in any linguistic topos, and hence in any topos.

Since $\mathbf{Sub}(A) \cong \mathsf{E}(1, PA)$, it follows that $\mathsf{E}(1, PA)$ (with the induced ordering) is a Heyting algebra. And since ($\mathsf{E}(A, \Omega)$, $\leqslant$) $\cong$ ($\mathbf{Sub}(A)$, $\subseteq$), it follows that the former is a Heyting algebra as well. Taking $A = 1$, we see that the ordered set $\mathsf{E}(1, \Omega)$ of E-elements of Ω is also a Heyting algebra.

Recall that a partially ordered set is *complete* if every subset has a supremum (join) and an infimum (meet). We claim that, for any local set theory S, and any S-set X,

$$\vdash_S \langle \Omega, \preceq \rangle \textit{ and } \langle PX, \subseteq \rangle \textit{ are complete.}$$

For we have

$u \subseteq \Omega \vdash_S (\top \in u)$ is the $\preceq$-join of u,

$u \subseteq \Omega \vdash_S (\forall \omega \in u.\ \omega)$ is the $\preceq$-inf of u,
$v \subseteq X \vdash_S \bigcup v$ is the $\subseteq$-join of v,
$v \subseteq X \vdash_S \bigcap v$ is the $\subseteq$-meet of v.

To prove, e.g., the first assertion, observe that, first,

$$u \subseteq \Omega, \omega \in u, \omega \ \vdash_S \ \omega \in u \wedge \omega = \top \ \vdash_S \top \in u,$$

so

$$u \subseteq \Omega, \omega \in u \ \vdash_S \ \omega \rightarrow (\top \in u) \wedge \omega = \top \ \vdash_S \omega \preceq (\top \in u)$$

whence

$$u \subseteq \Omega \vdash_S \omega \in u \rightarrow \omega \preceq (\top \in u),$$

and thus

$$u \subseteq \Omega \vdash_S \top \in u \textit{ is an } \preceq\textit{-upper bound for } u.$$

Also

$$u \subseteq \Omega, \forall\omega\in u(\omega \rightarrow \alpha), (\top \in u) \vdash_S \top \rightarrow \alpha \vdash_S \alpha,$$

whence

$$u \subseteq \Omega, \forall\omega\in u(\omega \rightarrow \alpha), (\top \in u) \vdash_S \alpha,$$

i.e.,

$$u \subseteq \Omega, \alpha \textit{ is an } \preceq\textit{-upper bound for } u \vdash_S (\top \in u) \preceq \alpha,$$

which establishes the first assertion.

As a consequence, for any topos $\mathbf{E}$,

$$\vDash_{\mathbf{E}} <\Omega, \preceq> \textit{ and } <PA, \subseteq> \textit{ are complete.}$$

That is, Ω and PA are *internally complete* in $\mathbf{E}$.

MORE ON INDETERMINATES

Let S be a local set theory and I an S-set. We define the category $\mathbf{C}(S)^I$ of *I-indexed S-sets* as follows. An *object* of $\mathbf{C}(S)^I$ is an S-set of the form

$$M = \{<i, M_i>: i \in I\}$$

with M_i a term of power type having at most the free variable i: thus M is an "I-indexed S-set of S-sets." An *arrow* f: $M \rightarrow N$ between $\mathbf{C}(S)^I$-objects $M = \{<i, M_i>: i \in I\}$ and $N = \{<i, N_i>: i \in I\}$ is an S-set of the form

$$<i, f_i>: i \in I\}$$

such that

$$\vdash_S \forall i{\in}I.\ f_i \in Fun(M_i, N_i).$$

Thus f: $M \to N$ in $\mathsf{C}(S)^I$ is an I-indexed S-set of maps $M_i \to N_i$. Composites and identity arrows in are defined $\mathsf{C}(S)^I$ in the obvious way.

Recall that S_I is the theory obtained from S by adjoining an indeterminate I-element c. It is easily shown that there is an isomorphism of categories

$$\mathsf{C}(S_I) \cong \mathsf{C}(S)^I.$$

The isomorphism G: $\mathsf{C}(S_I) \cong \mathsf{C}(S)^I$ is defined on objects as follows. Given an S_I-set $X = \{x\colon \alpha(x, z/c)\}$, define the $\mathsf{C}(S)^I$-object GX by

$$GX = \{<i, X_i>: i \in I\},$$

where

$$X_i = \{x\colon \alpha(x, z/i)\}.$$

Given an arbitrary category C, and a C-object A, we define the *slice category* (or *category of objects over* A) C/A to have as objects all C-arrows of the form $X \xrightarrow{f} A$ and as arrows
$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ A \arrow[r] & A \end{tikzcd}$$
all triples (f, h, g) where $X \xrightarrow{h} Y$ is a C-arrow such that the triangle
$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & A & \end{tikzcd}$$
commutes.

It can be shown that there is an equivalence of categories

$$\mathsf{C}(S)/I \simeq \mathsf{C}(S)^I.$$

The equivalence F: $\mathsf{C}(S)/I \to \mathsf{C}(S)^I$ is defined as follows:

$$F(X \xrightarrow{f} I) = \{<i, f^{-1}(i)>: i \in I\},$$

where $f^{-1}(i) = \{x\colon <x, i> \in f\}$. Given

$$h : (X \xrightarrow{f} I) \longrightarrow (Y \xrightarrow{g} I)$$

in $\mathsf{C}(S)/I$,

$$Fh = \{<i, h_i>: i \in I\},$$

with

$$h_i = \{\langle x, y\rangle : x \in f^{-1}(i) \wedge \langle x, y\rangle \in h\}.$$

We conclude that $\mathsf{C}(S)/I$ and $\mathsf{C}(S)^I$ are equivalent categories, so that $\mathsf{C}(S)/I$ is a topos. The Equivalence Theorem now implies what was at one time regarded as the fundamental theorem of topos theory, namely: *if* E *is a topos, then so is* E/A *for any* E*-object* A*.*

VII

NUMBER SYSTEMS IN LOCAL SET THEORIES

NATURAL NUMBERS

Let S be a local set theory in a language $\mathscr{L}$. A *natural number system in S* is a triple ($\mathbf{N}$, $\boldsymbol{s}$. $\underline{0}$), consisting of a type symbol $\mathbf{N}$, a function symbol $\boldsymbol{s}\colon \mathbf{N} \to \mathbf{N}$ and a closed term $\underline{0}:\mathbf{N}$, satisfying the following *Peano axioms.*

(P1) $\vdash_S \boldsymbol{s}n \neq \underline{0}$

(P2) $\boldsymbol{s}m = \boldsymbol{s}n \vdash_S m = n$

(P3) $\underline{0} \in u, \forall n(n \in u \to \boldsymbol{s}n \in u) \vdash_S \forall n.\ n \in u$

Here m, n are variables of type $\mathbf{N}$, u is a variable of type $\mathbf{PN}$, and we have written $\boldsymbol{s}n$ for $\boldsymbol{s}(n)$. (P3) is the *axiom of induction.*

A local set theory with a natural number system will be called *naturalized.*

In any naturalized local set theory S, $\underline{0}$ is called the *zeroth numeral.* For each natural number $n \geq 1$, the *nth numeral* $\underline{n}$ in S is defined recursively by putting $\underline{n} = \boldsymbol{s}(\underline{n-1})$. Numerals are closed terms of type $\mathbf{N}$ which may be regarded as *formal representatives* in S of the natural numbers.

It is readily shown that (P3) is equivalent to the following *induction scheme*[12]*:*

For any formula α *with exactly one free variable of type* $\mathbf{N}$*, if* $\vdash_S \alpha(\underline{0})$ *and* $\alpha(n) \vdash_S \alpha(\boldsymbol{s}n)$, then $\vdash_S \forall n\alpha(n)$.

[12] While the induction principle holds for $\mathbf{N}$, the least number principle can be shown to hold in $\mathbf{N}$ (with the usual linear ordering) if and only if S is classical, i.e. the law of excuded middle holds.

It can also be shown that functions may be defined on N by the usual process of simple recursion. In fact we have, for any naturalized local set theory S, the following *simple recursion principle SRP:*

For any S-set X: **PA,** *any closed term a* **:** **A**, *and any S-map g: X → X, there is a unique S-map f: N → X such that*

$$\vdash_S f(\underline{0}) = a \wedge \forall n[f(\boldsymbol{s}n = g(f(n))].$$

It follows from this that a natural number system on a local set theory is determined uniquely up to isomorphism in the evident sense.

Conversely, it can be shown that *SRP yields the Peano axioms,* so that they are equivalent ways of characterizing a natural number system.

Given a naturalized local set theory S, if we denote the map $n \mapsto \boldsymbol{s}(n)$: $N \to N$ by s and the map $\star \mapsto \underline{0}$: $1 \to \mathbf{N}$ by o, it is easy to see, using P! and P2, that the map $s + o$: $N + 1 \to N$ is an isomorphism in $\mathsf{C}(S)$. Conversely, it can be shown that the presence of an S-set X with an isomorphism f: $X + 1 \cong X$ yields a natural number system. For if we define

$$U = \bigcap\{u \subseteq X : f\star \in u \wedge \forall x{\in}u.f(x) \in u\},$$

it is straightforward to show that the triple $(U, f, f\star)$ is a natural number system.

An important feature of a natural number system ($\mathbf{N}$, $\boldsymbol{s}$, $\underline{0}$) is that *the equality relation on N is decidable,* or *N is discrete,* that is,

$$\vdash_S m = n \vee m \neq n.$$

Frege's construction of the natural numbers can also be carried out in a local set theory and the result shown to be equivalent to the satisfaction of the Peano axioms. Thus suppose given a local set theory S.

We shall work entirely within S, so that all the assertions we make will be understood as being demonstrable in S. In particular, by "set", "family", etc. we shall mean "S-set", "S-family", etc.

A family $\mathscr{F}$ of subsets of a set E is *inductive* if $\varnothing \in \mathscr{F}$ and $\mathscr{F}$ is *closed under unions with disjoint unit sets,* that is, if

$$\forall X \in \mathscr{F}\, \forall x \in E - X\, (X \cup \{x\} \in \mathscr{F})^{13}.$$

A *Frege structure* is a pair (E, ν) with ν a map to E whose domain is an inductive family of subsets of E such that, for all $X, Y \in \mathrm{dom}(\nu)$,

$$\nu(X) = \nu(Y) \equiv X \approx Y,$$

where we have written $X \approx Y$ for *there is a bijection between X and Y.*

It can be shown that, for any Frege structure (E, ν), there is a subset N of E which is the domain of a natural number system. In fact, for $X \in \mathrm{dom}(\nu)$ write X^+ for $X \cup \{\nu(X)\}$ and call a subfamily $\mathscr{E}$ of $\mathrm{dom}(\nu)$ *weakly inductive* if $\varnothing \in \mathscr{E}$ and $X^+ \in \mathscr{E}$ whenever $X \in \mathscr{E}$ and $\nu(X) \notin X$. Let $\mathcal{N}$ be the intersection of the collection of all weakly inductive families, and define $\underline{0} = \nu(\varnothing)$, $N = \{\nu(X){:}\ X \in \mathcal{N}\}$, and $s{:}\ N \to N$ by $s(\nu(X)) = \nu(X^+)$ for $X \in \mathcal{N}$. Then $(N, s, \underline{0})$ is a natural number system.

Conversely, each natural number system $(N, s, \underline{0})$ yields a Frege structure. For one can define the map $g{:}\ N \to PN$ recursively by

$$g(\underline{0}) = \varnothing \qquad g(sn) = g(n) \cup \{n\},$$

and the map ν by

$$\nu = \{(X, n) \in PN \times N{:}\ X \approx g(n)\}.$$

The domain of ν is the family of finite subsets of N and ν assigns to each such subset the number of its elements. (N, ν) is a Frege structure.

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[13] The members of the *least* inductive family of subsets of X are precisely the finite discrete subsets of X (where discreteness is defined on p.). These are the subsets of X which are bijective with an initial segment of the natural numbers. Calling a family of subsets of X *strongly inductive* if contains $\varnothing$ and is closed under unions with arbitrary, i.e. not necessarily disjoint, unit sets, the members of the least such family coincides with the Kuratowski finite subsets of X. While every finite discrete subset is Kuratowski finite, the converse does not necessarily hold.

We next describe the interpretation of the concept of natural number system in a topos E. Let (N, s, o) be a triple consisting of an E-object N and E-arrows s: $N \to N$, o: $1 \to N$. Let ***s*, *o*** be the function symbols in $\mathscr{L}_{\mathrm{E}}$ corresponding to s, o respectively and let $\underline{0}$ be the closed term ***o***(★). The clearly (**N**, ***s***, $\underline{0}$) satisfies the simple recursion principle in *Th*(E) iff the following condition, known as the *Peano-Lawvere axiom,* holds:

For any diagram $1 \xrightarrow{a} X \xrightarrow{g} X$ *in* E, *there exists a unique* $N \xrightarrow{f} X$ *such that the diagram*

$$\begin{tikzcd} 1 \arrow[r, "o"] \arrow[dr, "a"'] & N \arrow[r, "s"] \arrow[d, "f"] & N \arrow[d, "f"] \\ & X \arrow[r, "g"'] & X \end{tikzcd}$$

commutes.

A triple (N, s, o) satisfying this condition is called a *natural number system,* and N a *natural number object*, in E. From previous observations it follows that a topos has a natural number object if and only if it contains an *infinite* object, that is, an object A which is isomorphic to $A + 1$.

REMARK ON THE REAL NUMBERS

The familiar set-theoretic constructions of the ring of integers and thence the fields of rational numbers and real numbers can be carried out in any local set theory with a natural number system (or any topos with a natural number object). For the integers and rational numbers, the results are independent of the method of construction and yield essentially the same structures as in the classical case. But while in the classical situation all the various constructions of the real numbers (e.g. via Dedekind cuts or Cauchy sequences) yield isomorphic results, this is no longer true in the non-classical logic of a local set theory or a topos. Even certain basic properties of the real numbers which hold classically, for example order-completeness, the property that a bounded set of reals has a supremum and an infimum, can fail. In fact, it can be shown that the system of Dedekind real numbers constructed in a local set theory S is order complete if and only if the intuitionistically invalid instance of DeMorgan's law $\neg(\alpha \wedge \beta) \rightarrow \neg\alpha \vee \neg\beta$ holds in S. And while in general the Cauchy reals can be considered a subset of the Dedekind reals, they rarely coincide. A sufficient condition for them to do so is the validity of the countable axiom of choice.

THE FREE TOPOS

Let $\mathscr{L}_\mathbf{N}$ be the language with just one ground type symbol $\mathbf{N}$, one function symbol $\boldsymbol{s}\colon \mathbf{N} \rightarrow \mathbf{N}$ and one function symbol $\boldsymbol{O}\colon \mathbf{1} \rightarrow \mathbf{N}$. Write $\underline{0}$ for $\boldsymbol{O}(\star)$. Let P be the local set theory in $\mathscr{L}_\mathbf{N}$ generated by the sequents

$$: \boldsymbol{s}n \neq \underline{0}$$

$$\boldsymbol{s}m = \boldsymbol{s}n : m = n$$

$$\underline{0} \in u, \forall n(n \in u \rightarrow \boldsymbol{s}n \in u) : \forall n.\ n \in u$$

where m, n are variables of type $\mathbf{N}$ and u is a variable of type $\mathbf{PN}$. The triple $(\mathbf{N}, \boldsymbol{s}, \underline{0})$ is then a natural number system in P, so that P is a naturalized local set theory: it is called the *free* naturalized local set theory.

P is particularly important because it is *an initial object* in the category of naturalized local set theories. Given two such theories S, S', a *natural translation* of S into S' is a translation K: $S \rightarrow S'$ which preserves $\mathbf{N}$, $\boldsymbol{s}$ and $\underline{0}$. Write Natloc for the category of naturalized local set theories and natural translations. It should be clear that P is an initial object in Natloc. The associated topos $\mathrm{C}(P)$ is called the *free topos*.

P has some features which make it attractive from a constructive standpoint: for instance it is witnessed in the sense of the next chapter and has the *disjunction property*, namely, for sentences α, β,

$$\vdash_P \alpha \vee \beta \;\equiv\; \vdash_P \alpha \;\text{ or }\; \vdash_P \beta.$$

These facts have led some to suggest that *P* is the *ideal theory* and its model the free topos the *ideal universe*, for the constructively minded mathematician.

If to the axioms of *P* we add the *law of excluded middle*

$$: \forall\omega(\omega \vee \neg\omega),$$

we get the theory P^c—the *free classical naturalized local set theory*—which is the classical counterpart of *P*. The associated topos $\mathrm{C}(P^c)$ is called the *free Boolean topos*. It would seem natural to regard this topos as the ideal universe for the classically minded mathematician; however, the incompleteness of first-order set theory implies that P^c is not complete, so that there are more than two "truth values" in $\mathrm{C}(P^c)$, an evident drawback from the classical standpoint.

VII

SYNTACTIC PROPERTIES OF LOCAL SET THEORIES AND THEIR TOPOS COUNTERPARTS

SYNTACTIC PROPERTIES OF LOCAL SET THEORIES

Let S be a local set theory in a language $\mathscr{L}$. We make the following definitions.

- S is *classical* if $\vdash_S \forall\omega(\omega \vee \neg\omega)$. This is the full law of excluded middle for S.
- S is *sententially classical* if $\vdash_S \sigma \vee \neg\sigma$ for any sentence σ. This is a weakened form of the law of excluded middle.
- S is *complete* if $\vdash_S \sigma$ or $\vdash_S \neg\sigma$ for any sentence σ.
- For each S-set A **: PB** let $\Delta(A)$ be the set of closed terms τ such that $\vdash_S \tau \in A$.

  A is *standard* if for any formula α with at most the variable x **: B** free the following is valid:

$$\frac{\vdash_S \alpha(x/\tau) \quad \text{for all } \tau \text{ in } \Delta(A)}{\vdash_S \forall x{\in}A\ \alpha}$$

  S is *standard* if every S-set is so.
- If A is an S-set of type **PB,** an A**-***singleton* is a closed term U of type **PB** such that $\vdash_S U \subseteq A$ and $\vdash_S \forall x{\in}U\forall y{\in}U.\ x = y$. X is said to be *near-standard* if for any formula α with at most the variable x **: B** free the following is valid

$$\frac{\vdash_S \forall x{\in}U\alpha(x) \quad \text{for all } A\text{-singletons } U}{\vdash_S \forall x{\in}A\ \alpha}$$

  S is *near-standard* if every S- set is so.
- S is *witnessed* if for any type symbol **B** of $\mathscr{L}$ and any formula α with at most the variable x **: B** free the following rule is valid:

$$\frac{\vdash_S \exists x\, \alpha}{\vdash_S \alpha(x/\tau) \quad \text{for some closed term } \tau : \mathbf{B}.}$$

- S is *choice* if, for any S-sets X, Y and any formula α with at most the variables x, y free the following rule (the *choice rule*) is valid:

$$\frac{\vdash_S \forall x \in X\, \exists y \in Y\, \alpha(x, y)}{\vdash_S \forall x \in X\, \alpha(x, fx) \ \textit{ for some } f\!: X \to Y}$$

- S is *internally choice* if under the conditions of the previous definition

$\forall x \in X\, \exists y \in Y\, \alpha(x, y) \vdash_S \exists f \in Fun(X, Y)\ \forall x \in X\, \exists y \in Y\, [\alpha\, (x, y) \wedge \langle x, y\rangle \in f]$.

- An S-set X is *discrete* if

$$\vdash_S \forall x \in X\, \forall y \in X.\, x = y \vee x \neq y.$$

- A *complement* for an S-set X **: PA** is an S-set Y **: PA** such that $\vdash_S X \cup Y = A \ \wedge\ X \cap Y = \varnothing$. An S-set that has a complement is said to be *complemented.*

- S is *full* if for each set I there is a type symbol $\hat{\mathbf{I}}$ of the language $\mathscr{L}$ of S together with a collection $\{\hat{i} : i \in I\}$ of closed terms each of type $\hat{\mathbf{I}}$ satisfying the following:

(i) $\vdash_S \hat{i} = j \ \Rrightarrow\ i = j$.

(ii) For any I - indexed family $\{\tau_i\!: i \in I\}$ of closed terms of common type **A**,

there is a term $\tau(x)$ **: A,** x **:** $\hat{\mathbf{I}}$ such that $\vdash_S \tau_i = \tau(\hat{i})$ for all $i \in I$,

and, for any term $\sigma(x)$ **: A,** x **:** $\hat{\mathbf{I}}$, if $\vdash_S \tau_i = \sigma(\hat{i})$ for all $i \in I$, then $\vdash_S \tau = \sigma$.

$\hat{\mathbf{I}}$ may be thought of as the representative in S of the set I.

We prove the ***Generalization Principle for hatted type symbols:***

Suppose S is full. Then the following rule is valid for any formula $\alpha(x)$ *with* x **:** $\hat{\mathbf{I}}$:

$$\frac{\vdash_S \alpha(\hat{i}) \quad \textit{for all } i \in I}{\vdash_S \forall x\, \alpha}$$

and similarly for more free variables. In particular, $\hat{1}$ *is standard.*

Proof. Assume the premises. Then for any $i \in I$ we have $\vdash_S \alpha(\hat{i}) = \top$ and it follows from the uniqueness condition that $\vdash_S \alpha(x) = \top$, whence have $\vdash_S \forall x\, \alpha$. ■

We next establish some facts concerning these notions. In formulating our arguments we shall assume that our background metatheory is constructive, in that no use of the metalogical law of excluded middle will be made.

Proposition 1. *Any of the following conditions is equivalent to the classicality of S:*

(i) $\vdash_S \Omega = \{\top, \bot\}$

(ii) $\vdash_S \neg\neg\omega \Rightarrow \omega$

(iii) $\vdash_S \Omega$ *is a Boolean algebra*

(iv) *any S-set is complemented,*

(v) *any S-set is discrete,*

(vi) Ω *is discrete,*

(vii) $\vdash_S 2 = \{0, 1\}$ *is well-ordered under the usual ordering,.*

Proof. (iv) If S is classical, clearly $\{x: x \notin X\}$ is a complement for X. Conversely, if $\{\top\}$ has a complement U, then

$$\vdash_S \omega \in U \Rightarrow \neg(\omega = \top) \Rightarrow \neg\omega \Rightarrow \omega = \bot.$$

Hence $\vdash_S U = \{\bot\}$, whence $\vdash_S \Omega = \{\top\} \cup U = \{\top, \bot\}$.

(vi) If Ω is discrete, then $\vdash_S \omega = \top \vee \neg(\omega = \top)$, so $\vdash_S \omega \vee \neg\omega$.

(vii) If S is classical, then 2 is trivially well-ordered under the usual well-ordering. Conversely, if 2 is well-ordered, take any formula α, and define $X = \{x \in 2: x = 1 \vee \alpha\}$.

Then X has a least element, a, say. Clearly $\vdash_S a = 0 \Leftrightarrow \alpha$, so, since $\vdash_S a = 0 \vee a = 1$, we get $\vdash_S a = 1 \Leftrightarrow \neg\alpha$, and hence $\vdash_S \alpha \vee \neg\alpha$. ■

Proposition 2. *For well-termed S, S choice ≡ S internally choice and witnessed.*

Proof. Suppose S is choice. If $\vdash_S \exists x\alpha$, let u **: 1** and define $\beta(u, x) \equiv \alpha(x)$. Then $\vdash_S \forall u{\in}1\exists x{\in}X\beta(u,x)$. Now choice yields an S-map f: $1 \to X$ such that $\vdash_S \forall u{\in}1\beta(u,f(u))$ i.e., $\vdash_S \beta(\star,f\star)$ or $\vdash_S \alpha(f\star)$. By well-termedness, $f\star$ may be taken to be a closed term τ, and we then have $\vdash_S \alpha(\tau)$. So S is witnessed.

To derive internal choice from choice, we argue as follows: let

$$X^* = \{x \in X: \exists y{\in}Y\alpha(x, y)\}.$$

Then $\vdash_S \forall x{\in}X^*\exists y{\in}Y\alpha(x,y)$. Accordingly choice yields a map f: $X^* \to Y$ such that $\vdash_S \forall x{\in}X^*\alpha(x, fx)$, i.e. $\vdash_S \forall x{\in}X^*\exists y{\in}Y[\langle x, y\rangle \in f \wedge \alpha(x, y)]$. Now

$$\forall x{\in}X\exists y{\in}Y\, \alpha(x, y) \vdash_S X = X^* \;\vdash_S f \in Fun(X, Y)$$

so

$$\forall x{\in}X\exists y{\in}Y\, \alpha(x, y) \vdash_S \forall x{\in}X\exists y{\in}Y[\langle x, y\rangle \in f \wedge \alpha(x, y)].$$

Hence

$$\forall x \in X\, \exists y \in Y\, \alpha(x, y) \vdash_S \exists f \in Fun(X,Y)\; \forall x \in X\, \exists y \in Y\, [\alpha\, (x, y) \wedge \langle x, y\rangle \in f],$$

as required. The converse is easy. ■

Proposition 3. *If S is well-endowed, then S is choice ≡ S(X) is witnessed for every S-set X.*

Proof. Suppose S is choice and $\vdash_{S(X)} \exists y\alpha(y)$. We may assume that X is of the form $U_{\mathbf{A}}$, in which case α is of the form $\beta(x/c, y)$ with x **: A.** From $\vdash_{S(X)} \exists y\beta(x/c, y)$ we infer $\vdash_S \forall x\exists y\beta(x/c, y)$. So using choice in S and the well-termedness of S we obtain a term

$\tau(x)$ such that $\vdash_S \forall x\beta(x, \tau(x))$. Hence $\vdash_{S(X)} \beta(c, \tau(c))$, i.e., $\vdash_{S(X)} \alpha(\tau(c))$. Therefore S_X is witnessed.

Conversely, suppose S_X is witnessed for every S-set X, and that $\vdash_S \forall x{\in}X\exists y{\in}Y\, \alpha(x, y)]$. Then $\vdash_{S(X)} \exists y{\in}Y\, \alpha(c, y)]$, so there is a closed $\mathscr{L}_X$-term τ such that $\vdash_{S(X)} \tau \in Y \wedge \alpha(c, \tau)$. But τ is $\tau'(x/c)$ for some $\mathscr{L}$-term $\tau'(x)$. Thus $\vdash_{S(X)} \tau'(c){\in}Y \wedge \alpha(c,\tau'(c))$, whence $\vdash_S \forall x{\in}X\, [\tau'(x) \in Y \wedge \alpha(x, \tau')]$. Defining $f = (x \mapsto \tau')\colon X \to Y$ then gives $\vdash_S \forall x \in X\, \alpha(x, fx)]$ as required. ■

Proposition 4 (Diaconescu's Theorem). *S choice ⇒ S classical.*

Proof. Step 1. *S choice ⇒ S_I choice for any S-set I.*

Proof of step 1. Suppose that S is choice, and

$$\vdash_{S(I)} \forall x \in X(c)\, \exists y \in Y(c)\, \alpha\, (x, y, c).$$

Then

$$\vdash_S \forall x \in X(i)\, \exists y \in Y(i)\, \alpha\, (x, y, i).$$

Define

$$X^* = \{<x, i>: x \in X(i) \wedge i \in I\}, \quad Y^* = \bigcup_{i\in I} Y(i),$$

$$\beta(u, i) \equiv \exists x \in X(i) \exists i \in I[u = <x, i> \wedge \alpha(x, y, i) \wedge y \in Y(i)].$$

Then $\vdash_S \forall u \in X^*\, \exists y \in Y^*\, \beta(u, y)$. So choice yields $f^*\colon X^* \to Y^*$ such that $\vdash_S \forall u \in X^*\, \beta(u, f^*u)$, i.e.

$$\vdash_S \forall i \in I\, \forall x \in X(i)\, \alpha(x, f^*(<x ,i>, i) \wedge f^*(<x ,i>)\in Y(i)],$$

whence

$$\vdash_S \forall x \in X(c)\, \alpha(x, f^*(<x\ c>, c) \wedge f^*(<x ,c>)\in Y(c)],$$

Now define $f = (x \mapsto f^*(<x, c>))$. Then $f\colon X(c) \to Y(c)$ in S_I and

$$\vdash_{S(I)} \forall x \in X(c)\, \alpha(x, fx, c).$$

This completes the proof of step 1.

Step 2. *S choice* $\Rightarrow$ *S sententially classical.*

Proof of step 2. Define $2 = \{0, 1\}$ and let $X = \{u \subseteq 2 \colon \exists y . y \in u\}$. Then

$$\vdash_S \forall u \in X \, \exists y \in 2 \, . \, y \in u.$$

So by choice there is $f \colon X \to 2$ such that

$$\vdash_S \forall u \in X \, . \, fu \in u.$$

Now let σ be any sentence; define

$$U = \{x \in 2 \colon x = 0 \vee \sigma\},\ V = \{x \in 2 \colon x = 1 \vee \sigma\},$$

Then $\vdash_S U \in X \wedge V \in X$, so, writing $a = fU$, $b = fV$, we have

$$\vdash_S [a = 0 \vee \sigma] \wedge [b = 1 \vee \sigma],$$

whence

$$\vdash_S [a = 0 \wedge b = 1] \vee \sigma,$$

so that

(*) $$\vdash_S a \neq b \vee \sigma.$$

But $\sigma \vdash_S U = V \vdash_S a = b$, so that $a \neq b \vdash_S \neg\sigma$. It follows from this and (*) that

$$\vdash_S \sigma \vee \neg\sigma,$$

as claimed. This establishes step 2.

Moral of step 2: if pair sets have choice functions, then logic is classical.

Step 3. *S classical* $\equiv$ *S*(Ω) *sententially classical.* This follows from the fact that, if ϖ is the generic element of Ω introduced in $S(\Omega)$, then $\vdash_S \forall\omega(\omega \vee \neg\omega) \equiv \vdash_{S(\Omega)} \varpi \vee \neg\varpi$.

To complete the proof of Diaconescu's theorem, we now have only to observe that *S choice* $\Rightarrow$ S_Ω *choice* $\Rightarrow$ S_Ω *sententially classical* $\Rightarrow$ *S classical.* ■

It follows immediately from Diaconescu's theorem that, since not every local set theory is classical, **AC** is *independent* of pure local set theory.

Proposition 5. *If S is well-termed, then S choice* $\Rightarrow$ *S near-standard.*

Proof. Assume that S is choice. To show that S is near-standard, we first obtain, for any S-set A of type **PB** and any formula $\alpha(x)$ with x : **B**, an A-singleton

V for which (1) $\vdash_S \forall x{\in}V\neg\alpha$ and (2) $\vdash_S \exists x{\in}A\neg\alpha \Rightarrow \exists x.\ x \in V$. Let $X = \{u\colon \exists x{\in}A\ \neg\alpha\}$ with $u : \mathbf{1}$ and $Y = \{x{\in} A\colon \alpha\}$. Then $\vdash_S \forall u{\in}X\exists x{\in}Y\neg\alpha$, so by choice there is a map $f\colon X \to Y$ such that $\vdash_S \forall u{\in}X\neg\alpha(x/fu)$. If we define $V = \{x\colon \langle\bigstar, x\rangle \in f\}$, it is easily checked that V is an A-singleton satisfying conditions (1) and (2).

Now to show that S is near-standard, suppose that $\vdash_S \forall x{\in}U\ \alpha$ for any A-singleton U. Then in particular $\vdash_S \forall x{\in}V\ \alpha$, which with (1) gives $\vdash_S \neg\exists x.\ x \in V$. We then deduce, using (2), that $\vdash_S \neg\exists x{\in}A\neg\alpha$. Since S, being choice, is also classical (Prop. 4), it follows that $\vdash_S \forall x{\in}A\ \alpha$. Hence S is near-standard.

Proposition 6. *If S is well-termed, then S choice and complete* $\Rightarrow$ *S standard.*

Proof. Assume the premises. Then by Prop. 5, S is near-standard. We use completeness to show that S is standard. Suppose then that $\vdash_S \alpha(x/\tau)$ for all $\tau \in \Delta(A)$. If U is an A-singleton, then, assuming S is complete, either $\vdash_S \exists x.\ x \in U$ or $\vdash_S \neg\exists x.\ x \in U$. In the former case, the well-termedness of S yields a closed term τ such that $U = \{\tau\}$ and from $\vdash_S \alpha(x/\tau)$ it then follows that $\vdash_S \forall x{\in}U\ \alpha$. If, on the other hand, $\vdash_S \neg\exists x.\ x \in U$, then clearly $\vdash_S \forall x{\in}U\ \alpha$. So $\vdash_S \forall x{\in}U\ \alpha$ for any A-singleton U, and the near-standardness of S yields $\vdash_S \forall x{\in}A\ \alpha$, showing that S is standard. ■

THE FOREGOING PRINCIPLES INTERPRETED IN TOPOSES

When S is the theory Th(E) of a topos E, the conditions on S formulated in the previous section are correlated with certain properties of E, which we now proceed to determine.

E is said to be *extensional* provided that, for any objects A, B of E and any pair of arrows $A \xrightarrow{f} B, A \xrightarrow{g} B$, if $fh = gh$ for every arrow $1 \xrightarrow{h} A$, then $f = g$. We recall that this says that each object of E satisfies the axiom of extensionality in the sense that its identity as a domain is entirely determined by its "elements".

A weaker version of extensionality is obtained by replacing 1 with subobjects of 1, that is, objects U for which the unique arrow $U \rightarrow 1$ is monic. Thus E is said to be *subextensional* provided that for any objects A, B of E and any pair of arrows $A \xrightarrow{f} B, A \xrightarrow{g} B$, if $fh = gh$ for every $U \xrightarrow{h} A$ with $U \rightarrowtail 1$, then $f = g$.

We say that a category is said to satisfy the Axiom of Choice (**AC)** if , for any epic $f: A \twoheadrightarrow B$, there is a (necessarily monic) $g: B \rightarrow A$ such that $fg = 1_B$, or equivalently, if each of its objects is projective.

E is *Boolean* if the arrow $1+1 \xrightarrow{\top+\bot} \Omega$ is an isomorphism, and *bivalent* if $\top$ and $\bot$ are the only arrows $1 \rightarrow \Omega$, or equivalently, 1 has only the two subobjects 0 and 1.

Let A be an object of E, and let m: $B \rightarrowtail A$ be a subobject of A. A *complement* for B is a subobject n: $C \rightarrowtail A$ such that the arrow $m + n$: $B + C \rightarrow A$ is an isomorphism. Then it is easy to show that E *is Boolean if and only if every object in* E *has a complement.*

Notice that, even if we only assume intuitionistic logic in our metatheory, **Set** is extensional. If full classical logic is assumed, **Set** is both Boolean and bivalent.

If S is a well-endowed local set theory, and E is a topos, we have the following concordance between properties of S (respectively Th(E)) and properties of C(S) (respectively E):

S , Th(E)	C(S), E
CLASSICAL	BOOLEAN
COMPLETE	BIVALENT
STANDARD	EXTENSIONAL
NEAR-STANDARD	SUBEXTENSIONAL

WITNESSED	1 IS PROJECTIVE
CHOICE	SATISFIES **AC**
FULL	ALL SET-INDEXED COPOWERS OF 1 EXIST

We prove a couple of these equivalences.

If S is well-endowed, then S standard $\equiv$ $\mathsf{C}(S)$ *extensional.* If S is well-endowed, then $\mathsf{C}(S)$ is equivalent to the category $\mathsf{T}(S)$ of S-types and terms, so to establish the extensionality of $\mathsf{C}(S)$ it is enough to establish that of $\mathsf{T}(S)$. Accordingly let **A, B** be type symbols and suppose that f, g: $A \to B$ are $\mathsf{T}(S)$-arrows such that, for any $\mathsf{T}(S)$-arrow $1 \xrightarrow{h} A$, we have $fh = gh$. Now f is $(x \mapsto \xi)$ and g is $(x \mapsto \eta)$ for some terms ξ, η, and the condition just stated becomes: for any closed term τ of type **A**, we have $\vdash_S \xi(\tau) = \eta(\tau)$. Supposing that S is standard, it follows that $\vdash_S \forall x(\xi(x) = \eta(x))$, whence $f = g$. So $\mathsf{T}(S)$, and hence also $\mathsf{C}(S)$, is extensional.

Conversely, suppose $\mathsf{C}(S)$ is extensional. Let **A** be a type symbol and $\alpha(x)$ a formula with a free variable of type **A**. Let f be the S-map $(x \mapsto \alpha)$: $A \to \Omega$. If $\vdash_S \alpha(\tau)$ for all closed terms τ of type **A**, it follows that the diagram

$$\begin{tikzcd} 1 \arrow[r, "\tau"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "T_A"'] & \Omega \end{tikzcd}$$

commutes for all such τ. Since $\mathsf{C}(S)$ is extensional (and well-termed), we deduce that $f = T_A$, in other words that $\vdash_S \forall x(\alpha(x) = \top)$, i.e. $\vdash_S \forall x\alpha(x)$. So S is standard.

S is choice $\equiv$ $\mathcal{C}(S)$ *satisfies* **AC.** Given $g\colon Y \twoheadrightarrow X$ in $\mathcal{C}(S)$, let α be the formula $\langle y, x\rangle \in g$. Then $\vdash_S \forall x{\in}X\exists y{\in}Y\alpha(x,y)$. If S is choice there is $f\colon X \to Y$ such that $\vdash_S \forall x{\in}X\alpha(x,fx)$, from which it follows easily that $gf = 1_X$. So $\mathcal{C}(S)$ satisfies **AC**.

Conversely, suppose $\mathcal{C}(S)$ satisfies **AC** and $\vdash_S \forall x{\in}X\exists y{\in}Y\alpha(x,y)$ for a given formula α. Define $Z = \{\langle x,y\rangle \in X \times Y\colon \alpha\}$ and $g = (\langle x,y\rangle \mapsto x)\colon Z \to X$, $k = (\langle x,y\rangle \mapsto y)\colon Z \to Y$. Then g is epic, and so by **AC** there is $h\colon X \to Z$ such that $gh = 1_X$. If we now define $f = kh\colon X \to Y$, it is easy to see that $\vdash_S \forall x{\in}X\alpha(x,fx)$. So S is choice.

It follows from this that *any topos satisfying* **AC** *is Boolean*, so that subobjects always possess complements.

Remark. The original proof that any topos satisfying **AC** is Boolean is based on the idea of constructing a complement for any subobject. Here is a highly informal version of the argument.

Suppose that the topos satisfies **AC**, and let X be a subobject of an object A. Form the coproduct $A + A$, and think of it as the union of two disjoint copies of A. Regard the elements of the first copy as being coloured black and those of the second as being coloured white. Thus each element of A has been 'split' into a 'black' copy and a 'white' copy. Next, identify each copy of an element of X in the first (black) copy with its mate in the second (white) copy; the elements thus arising we agree to colour grey, say. In this way we obtain a set Y consisting of black, white and grey elements[14], together with an epic map $A + A \twoheadrightarrow Y$. Now we use **AC** to assign each element $y \in Y$ an element $y' \in A + A$ in such a way that y' is sent to y by the map $A + A \twoheadrightarrow Y$ above. The whole process—call it P, say—accordingly transforms each element of $A + A$ into an element (possibly the same) of $A + A$. Now, for $n = 0, 1, 2$, define

[14] One should not be misled into thinking that *at this stage* the 'grey' elements of Y can be clearly distinguished from the 'black' and 'white' ones: since the former are correlated with the elements of X, such distinguishability would be tantamount to assuming that X already possesses a complement!

$$A_n = \{a \in A: P \text{ effects a change in colour in exactly } n \text{ copies of } a\}.$$

Then clearly $A = A_0 \cup A_1 \cup A_2$, $A_1 = X$ and $A_2 = \varnothing$. It follows that A_0 is a complement for X.

Some examples[15].

(i) Set *is extensional, satisfies* **AC**§**,** *and is both Boolean*§ *and bivalent*§ .

(ii) *For any partially ordered set* ***P*****,** Set$^{\boldsymbol{P}}$ *is subextensional . It satisfies* ***AC*** *if*§**,** *and only if,* ***P*** *is trivially ordered, that is, if the partial ordering in* ***P*** *coincides with the identity relation.* To show that Set$^{\boldsymbol{P}}$ is subextensional, given α, β: $F \to G$ in Set$^{\boldsymbol{P}}$**,** $p_0 \in P$ and $a \in F(p_0)$, define $U \in$ Set$^{\boldsymbol{P}}$ by $U(p) = \{x: x = 0 \wedge p_0 \leq p\}$ with the U_{pq} the obvious maps. Then U is a subobject of 1 in Set$^{\boldsymbol{P}}$. Define φ: $U \to F$ by $\varphi_p = U(p) \times \{a\}$. If $\alpha\varphi = \beta\varphi$, then $\alpha_{p_0} \circ \varphi_{p_0}(0) = \beta_{p_0} \circ \varphi_{p_0}(0)$, whence $\alpha_{p_0}(a) = \beta_{p_0}(a)$. Since p_0 and a were arbitrary, $\alpha = \beta$. So Set$^{\boldsymbol{P}}$ is subextensional.

To show that **AC** holds in Set$^{\boldsymbol{P}}$ only if **P** is trivially ordered, suppose that $p_0 < q_0$ in **P** and define A, B in Set$^{\boldsymbol{P}}$ by $A(p) = \{0, 1\}$ for all $p \in P$, and each A_{pq} the identity map; $B(p) = \{0\}$ if $p_0 < p$, $B(p) = \{0,1\}$ if $p_0 \nless p$, each B_{pq} either the identity map on $\{0,1\}$ or the map $\{0,1\} \to \{0\}$ as appropriate. Then it is easy to show that the map f: $A \to B$ in Set$^{\boldsymbol{P}}$—with each f_p either the identity map on $\{0,1\}$ or the map $\{0,1\} \to \{0\}$ as appropriate—has no section.

(iii) *For any complete Heyting algebra H,* Set$_H$ *is subextensional. It satisfies* ***AC*** *if*§**,** *and only if, H is a Boolean algebra*[16]. To show that Set$_H$ is

[15] In presenting these examples we indicate by appending the symbol **§** when we need to assume that Set satisfies **AC**, or at least that its internal logic is classical and bivalent.

subextensional, suppose given $f, g: (I, \delta) \to (J, \varepsilon)$ in $\mathbf{Set}_H$. For $i_0 \in I, j_0 \in J$, let $\eta_i = g_{i_0 j_0} \wedge \delta_{ii_0}$ and $a = \bigvee_{i \in I} \eta_i$. Then $(\{0\}, \lambda)$ with $\lambda_{00} = a$ is a subobject of 1 in $\mathbf{Set}_H$ and the η_i define an arrow $\eta: (\{0\}, \lambda) \to (I, \delta)$. If $f\eta = g\eta$, then a calculation shows that $f_{i_0 j_0} = g_{i_0 j_0}$. Since i_0 and j_0 were arbitrary, $f = g$.

As for the second contention, if $\mathbf{Set}_H$ satisfies **AC**, it is Boolean, and so H must be a Boolean algebra. Conversely, If H is a Boolean algebra, then $\mathbf{Set}_H$ is Boolean, so $\mathrm{Th}(\mathbf{Set}_H)$ is classical. It is not hard to show that $\mathbf{Set}_H$ has all set-indexed copowers of 1, so that $\mathrm{Th}(\mathbf{Set}_H)$ is full. We also know that $\mathbf{Set}_H$ is subextensional, so that $\mathrm{Th}(\mathbf{Set}_H)$ is near-standard. It follows from the Corollary to Prop. 8. that $\mathrm{Th}(\mathbf{Set}_H)$ is choice, so that $\mathbf{Set}_H$ satisfies **AC.**

(iv)§ *For a monoid* **M**, *the topos* $\mathbf{Set}^{\mathbf{M}}$ *of* **M**-*sets is bivalent*. For the terminal object in $\mathbf{Set}^{\mathbf{M}}$ is the one-point set 1 with trivial **M-**action and evidently this has only the two subobjects 0, 1.

(v) For *a monoid* **M**, *if the topos* $\mathbf{Set}^{\mathbf{M}}$ *is Boolean, then* **M** *is a group*[17], *and conversely*§. For suppose that $\mathbf{Set}^{\mathbf{M}}$ is Boolean. Regard **M** as an **M**-set with the natural multiplication on the left by elements of **M.** For $a \in M$, $U = \{xa: x \in M\}$ is a sub-**M**-set of **M**, and so has a complement V in $\mathbf{Set}^{\mathbf{M}}$ which must itself be an sub-**M**-set of **M.** Now $1 \notin V$, since otherwise $V = M$ which would make U empty. It follows that $1 \in U$ and so a has a left inverse. Since any monoid with left inverses is a group, **M** is a group. Conversely, if **M** is a group (and **Set** is Boolean), then the set-theoretical complement of any sub-***M***-set Y of an **M**- set X is itself a sub-**M**-set and therefore the complement in $\mathbf{Set}^{\mathbf{M}}$ of Y.

[16] If B is a complete Boolean algebra, $\mathbf{Fuz}_B$ is equivalent to $\mathbf{Set}_B$, so **AC** also holds in $\mathbf{Fuz}_B$.
[17] It follows that if **M** is not a group, then $\mathbf{Set}^{\mathbf{M}}$ is bivalent§ but not Boolean.

(vi) *If* **G** *is a nontrivial group, then 1 is not projective in* $\mathrm{Set}^{\mathbf{G}}$. For $G \to 1$ in $\mathrm{Set}^{\mathbf{G}}$ is epic, but an arrow $1 \to G$ in $\mathrm{Set}^{\mathbf{G}}$ corresponds to an element $e \in G$ such that $ge = e$ for all $g \in G$, which cannot exist unless G has just one element.

(**vii**). *For a monoid* **M**, $\mathrm{Set}^{\mathbf{M}}$ *satisfies* **AC** *if*[§]**,** *and only if,* **M** *is trivial.* If $\mathrm{Set}^{\mathbf{M}}$ *satisfies* **AC,** then $\mathrm{Set}^{\mathbf{M}}$ is Boolean and so by (**v**) **M** is a group. But by **(vi)** if **M** is nontrivial, 1 is not projective in $\mathrm{Set}^{\mathbf{M}}$**,** and so $\mathrm{Set}^{\mathbf{M}}$ does not satisfy **AC.** It follows that **M** is trivial.

IX

CHARACTERIZATION OF **Set**

We remind the reader that we are assuming that our background metatheory is constructive. For definiteness we will take that metatheory to be intuitionistic set theory **IST.** Now consider the category **Set** of sets in **IST**. Its objects are all sets and its arrows all maps between sets. **Set** is a topos in with truth-value object *P1.* We seek a characterization of **Set** in terms of its associated local set theory, that is, *a characterization of the category of sets in type-theoretic terms.* We shall carry this out in a constructive manner.

Theorem. *The following conditions on a well-endowed consistent local set theory S are equivalent:*

(i) *S is full and standard,*

(ii) $\mathrm{C}(S) \simeq$ **Set**.

Proof. For (ii) $\Rrightarrow$ (i), follows from the fact that Set. is extensional and clearly has all set-indexed copowers of 1. Now assume that S is full. Since S is well-termed, for any S-map $f: X \to Y$ we can write $f(\tau)$ for each closed term τ such that $\vdash_S \tau \in X.$

We define functors Δ: $\mathrm{C}(S) \to$ **Set**, ^: **Set** $\to \mathrm{C}(S)$, which, under the specified conditions, we show to define an equivalence.

First, $\Delta(X)$ is the set of closed terms τ such that $\vdash_S \tau \in X,$ where we identify σ, τ if that $\vdash_S \sigma = \tau$. Given $f: X \to Y$, we define $\Delta(f)$ to be the map $(\tau \mapsto f(\tau)): \Delta(X) \to \Delta(Y)$.

Next, given I in **Set**, we define $\hat{I}$ to be the S-set $U_{\hat{\mathbf{I}}}$. Given $f: I \to J$, there is a term $f(x)$ **:** $\mathbf{J}$ with x **:** $\hat{\mathbf{I}}$ such that that $\vdash_S f(\hat{i}) = f(i)$ for all $i \in I$. We define $f : \hat{I} \to J$ to be the S-map $x \mapsto f(x)$.

For any set I and any S-set X, we have natural maps η_I: $I \to \Delta(\hat{I})$ and ε: $\Delta(X) \to X$ defined as follows:

$$\eta_I(i) = \hat{i} \text{ for } i \in I; \quad \vdash_S \varepsilon(\hat{\tau}) = \tau \text{ for all } \tau \in \Delta(X).$$

Clearly η is monic. The same is true of ε since for $\sigma, \tau \in \Delta(X)$,

$$\vdash_S \varepsilon(\sigma) = \varepsilon(\hat{\tau}) \to \sigma = \tau,$$

whence

$$\vdash_S \forall x \forall y[\varepsilon(x) = \varepsilon(y) \to x = y]$$

by generalization for hatted type symbols.

Now suppose that S is *standard.* We claim that then ε is epic and hence an isomorphism. For we have, for all $\tau \in \Delta(X)$, $\vdash_S \varepsilon(\hat{\tau}) = \tau$, whence $\vdash_S \exists y \varepsilon(y) = \tau$. Since X is standard, we infer that

$$\vdash_S \forall x {\in} X \exists y\ \varepsilon(y) = x,$$

so that ε is onto, hence epic.

Using the fact that ε is an isomorphism we can now show that η is epic, and hence also an isomorphism. To do this we require the readily established fact that, for $f: I \to J$, $f : \hat{I} \to J$ is a map in S, and if f is epic, then so is f.

Now consider $\eta : \hat{I} \to \Delta\hat{I}$. We note that

(*) $$\varepsilon \circ \eta = 1_{\hat{I}}.$$

For if $i \in I$, then

$$\vdash_S \ \varepsilon(\eta(\hat{i})) = \varepsilon(\eta i) = \eta i = \hat{i}$$

It follows by generalization that

$$\vdash_S \ \forall x \in \hat{I}\ \varepsilon(\eta x) = x\,,$$

whence (*).

Since ε is an isomorphism, it follows easily from (*) that η is an isomorphism, hence also epic. Accordingly η is itself epic, and hence also an isomorphism.

We conclude that (Δ, ^) define an equivalence between $\mathsf{C}(S) \to \mathsf{Set}$, as required. ■

Remark. It is also possible to formulate similar characterizations of other toposes, for example categories of presheaves over partially ordered sets, sheaves over topological spaces, and H-sets. For example, $\mathsf{C}(S) \simeq \mathsf{Set}_H$ for some complete Heyting algebra H if and only if S is full and near-stansard.

X

TARSKI'S AND GÖDEL'S THEOREMS IN LOCAL SET THEORIES

Let $\mathcal{L}$ be a local language containing a type symbol **C,** which we shall call the *type of codes of formulas:* letters u, v will be used as variables of type **C** and the letter **u** will denote a closed term of type **C.** We shall also suppose that $\mathcal{L}$ contains formulas $\tau(u)$**,** $\delta(u.v)$, and for each formula $\alpha(u)$, containing at most the free variable u, a closed term $\ulcorner\alpha\urcorner|\mathbf{C}$ called the *code* of α. The assignment $\alpha \rightarrow \ulcorner\alpha\urcorner$ is called the *coding map.* A local language satisfying these conditions will be called *codable.*

Let S be a theory in a codable language $\mathcal{L}$. We say that

- is a *diagonal relation* in S if

  $\vdash_S \ \forall v\delta(\ulcorner\alpha\urcorner, v) \leftrightarrow v = \ulcorner\alpha(\ulcorner\alpha\urcorner)\urcorner$ for any formula $\alpha(u)$.

- is a *truth definition* for S if

  $\vdash_S \ \tau(\ulcorner\alpha\urcorner) \leftrightarrow \alpha$ for any sentence α.

Since $\leftrightarrow$ is the same as =, a truth definition thus amounts to a (sentence-by-sentence) *left inverse* to the coding map.

- τ is a *demonstration predicate* for S if

  $\vdash_S \ \alpha \ \Leftrightarrow \ \vdash_S \ \tau(\ulcorner\alpha\urcorner)$ for any sentence α.

We first prove the

Fixed Point Lemma. Suppose that S is a theory in a codable language with a diagonal operator. Then any formula $\alpha(u)$ has a "fixed point", i.e., there is a sentence β such that

$$\vdash_S \ \beta \leftrightarrow \alpha(\ulcorner \beta \urcorner).$$

Proof. Let δ be a diagonal relation in S; given $\alpha(u)$, write $\gamma(u)$ for the formula $\exists v[\delta(u,v) \wedge \alpha(v)]$, let $\mathbf{u}$ be the term $\ulcorner\gamma\urcorner$, and define β to be the sentence $\gamma(\mathbf{u})$, i.e. $\gamma(\ulcorner\gamma\urcorner)$. Because δ is a diagonal relation we have

(1) $$\forall v\delta(\ulcorner\gamma\urcorner, v) \leftrightarrow v = \ulcorner\gamma(\ulcorner\gamma\urcorner)\urcorner .$$

Then, using (1), we have

$$\begin{aligned} \vdash_S \ \beta &\leftrightarrow \gamma(\ulcorner\gamma\urcorner) \\ &\leftrightarrow \exists v[\delta(\ulcorner\gamma\urcorner, v) \wedge \alpha(v)] \\ &\leftrightarrow \exists v\,[v = \ulcorner\gamma(\ulcorner\gamma\urcorner)\urcorner \wedge \alpha(v)] \quad \text{(by (1)} \\ &\leftrightarrow \alpha(\ulcorner\gamma(\ulcorner\gamma\urcorner)\urcorner) \\ &\leftrightarrow \alpha(\ulcorner\beta\urcorner), \end{aligned}$$

as required.

We use this to prove

Tarski's Theorem. Let S be a theory in a codable language with a diagonal relation. Then if S has a truth definition, it is inconsistent, that is, $\vdash_S \perp$.

Proof. Let δ be a diagonal relation in S and τ a truth definition for S. Let β be a fixed point for the formula $\neg\tau(u)$; thus

(1) $$\vdash_S \ \beta \leftrightarrow \neg\tau(\ulcorner\beta\urcorner).$$

Since τ is a truth definition for S we have

(2) $$\vdash_S \ \beta \leftrightarrow \tau(\ulcorner\beta\urcorner).$$

Now (1) and (2) give $\vdash_S \perp$, that is, S is inconsistent.

It follows that a consistent codable typed intuitionistic theory with a diagonal relation cannot have a truth definition.

We next prove

Gödel's First Incompleteness Theorem. Let *S* be a theory in a codable language with a diagonal relation. Then if *S* is consistent and has a demonstration predicate, it is incomplete.

Proof. The proof is similar to that of Tarski's theorem. Let δ be a diagonal relation in *S* and τ a demonstration predicate for *S*. Let β be a fixed point for the formula $\neg\tau(u)$; thus

(1) $$\vdash_S \beta \leftrightarrow \neg\tau(\ulcorner\beta\urcorner).$$

Since τ is a demonstration predicate for *S* we have

(2) $$\vdash_S \beta \Leftrightarrow \vdash_S \tau(\ulcorner\beta\urcorner).$$

From (1) and (2) it follows that

$$\vdash_S \tau(\ulcorner\beta\urcorner) \Leftrightarrow \vdash_S \beta \Leftrightarrow \vdash_S \neg\tau(\ulcorner\beta\urcorner).$$

So if *S* is consistent, the sentence $\tau(\ulcorner\beta\urcorner)$ is neither provable nor refutable in *S*, that is, *S* is incomplete.

In order to formulate Gödel's Second Incompleteness Theorem in intuitionistic type theories we require the following definition:

- τ is a *proof predicate* for *S* if it satisfies, for any sentences α, β,
  - (a) $\vdash_S \alpha \Rightarrow \vdash_S \tau(\ulcorner\alpha\urcorner)$,
  - (b) $\tau(\ulcorner\alpha \to \beta\urcorner) \vdash_S \tau(\ulcorner\alpha\urcorner) \to \tau(\ulcorner\beta\urcorner)$,
  - (c) $\tau(\ulcorner\tau(\ulcorner\alpha\urcorner)\urcorner) \vdash_S \tau(\ulcorner\alpha\urcorner)$.

If τ is a proof predicate, we shall write $\Box\alpha$ for $\tau(\ulcorner\alpha\urcorner)$. Then $\Box$ is a *provability* *operator,* i.e. satisfies

$$\text{(a}'\text{)}\quad \vdash_S \alpha \;\Rightarrow\; \vdash_S \Box\alpha$$

$$\text{(b}'\text{)}\quad \Box(\alpha \to \beta) \vdash_S \Box\alpha \to \Box\beta,$$

$$\text{(c}'\text{)}\quad \Box\alpha \vdash_S \Box\Box\alpha.$$

It is readily shown that

$$\text{(d}'\text{)}\quad \alpha \vdash_S \beta \Rightarrow \Box\alpha \vdash_S \Box\beta$$

$$\text{(e}'\text{)}\quad \vdash_S \alpha \leftrightarrow \beta \Rightarrow \vdash_S \Box\alpha \leftrightarrow \Box\beta.$$

Let us call a theory in a codable language *adequate* if it has both a diagonal relation and a proof predicate. We prove

Löb's Theorem[18]**.** Suppose that S is an adequate theory. Then for any sentence α,

(i) $\quad \Box(\Box\alpha \to \alpha) \vdash_S \Box\alpha.$

(ii) $\quad \neg\Box\alpha \vdash_S \neg\Box\neg\Box\alpha.$

(iii) $\quad \Box\alpha \vdash_S \alpha \;\Rightarrow\; \vdash_S \alpha.$

Proof. **(i)** Applying the Fixed Point Lemma to the formula $\tau(u) \to \alpha$ yields a sentence β for which

$$\vdash_S \beta \leftrightarrow (\tau(\ulcorner\beta\urcorner) \to \alpha),$$

i.e.

(1) $\quad \vdash_S \beta \leftrightarrow (\Box\beta \to \alpha).$

It follows that

$$\vdash_S \beta \to (\Box\beta \to \alpha),$$

whence by (b')

$$\vdash_S \Box\beta \to \Box(\Box\beta \to \alpha).$$

Hence, again using (b')

(2) $\quad \Box\beta \vdash_S \Box(\Box\beta \to \alpha) \vdash_S \Box\Box\beta \to \Box\alpha.$

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[18] Theorem 4.1.1 of [6].

Then, since by (c') $\Box\beta \vdash_S \Box\Box\beta$, it follows from (2) that

(3) $$\Box\beta \vdash_S \Box\alpha,$$

whence $\Box\alpha \to \alpha \vdash_S \Box\beta \to \alpha$. Hence by (d')

(4) $$\Box(\Box\alpha \to \alpha) \vdash_S \Box(\Box\beta \to \alpha).$$

Now it follows from (1) and (e') that $\vdash_S \Box\beta \leftrightarrow \Box(\Box\beta \to \alpha)$, and from this and (4) we obtain $\Box(\Box\alpha \to \alpha) \vdash_S \Box\beta$. This, together with (3), gives $\Box(\Box\alpha \to \alpha) \vdash_S \Box\alpha$, i.e. (i).

(ii) Using (i),

$$\Box\neg\Box\alpha \vdash_S \Box(\Box\alpha \to \bot) \vdash_S \Box(\Box\alpha \to \alpha) \vdash_S \Box\alpha.$$

Hence $$\neg\Box\alpha \vdash_S \neg\Box\neg\Box\alpha.$$

(iii) Suppose $\Box\alpha \vdash_S \alpha$. Then $\vdash_S \Box\alpha \to \alpha$ and so, by (a') $\vdash_S \Box(\Box\alpha \to \alpha)$. From this and (i) it follows that $\vdash_S \Box\alpha$.

From (i) of Löb's Theorem we see that $\Box$ provably satisfies the so-called *GL* (Gödel-Löb) *axiom*[19] for a normal modal logic, i.e. the scheme

$$\Box(\Box A \to A) \to \Box A.$$

Corollary . The following conditions are equivalent for an adequate theory *S:*

(i) For any sentence α, $\alpha \vdash_S \neg\neg\Box\alpha$

(ii) For any sentence α, $\vdash_S \neg\neg\Box\alpha$

(iii) $\vdash_S \neg\neg\Box\bot$.

A fortiori $\alpha \vdash_S \neg\neg\Box\alpha$ for all sentences α implies $\vdash_S \neg\neg\Box\bot$.

[19] See [5], p. 5.

Proof. (i) $\Rightarrow$ (ii). Suppose $\alpha \vdash_S \neg\neg\Box\, \alpha$ for any sentence α. Then in particular

(1) $$\neg\Box\alpha \vdash_S \neg\neg\Box\, \neg\Box\alpha.$$

By (ii) of Löb's theorem,

(2) $$\neg\,\Box\alpha\ \vdash_S \neg\Box\neg\Box\alpha.$$

From (1) and (2) it follows that $\neg\,\Box\alpha\ \vdash_S \bot$, whence $\vdash_S \neg\neg\,\Box\alpha$.

(ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (i). Suppose $\vdash_S \neg\neg\Box\,\bot$. Now for any sentence α, $\vdash_S \neg\neg\Box\,\bot \to \neg\neg\Box\alpha$. It follows that $\vdash_S \neg\neg\Box\alpha$ so that $\alpha \vdash_S \neg\neg\Box\alpha$.

The sentence $\neg\Box\,\bot$, i.e. $\neg\tau(\ulcorner\bot\urcorner)$, expresses, with respect to the proof predicate τ, the unprovability within S of $\bot$, that is, the *internal consistency* of S. From Löb's Theorem one derives:

Gödel's Second Incompleteness Theorem. Suppose that S is consistent and adequate. Then, for any sentence α, it is not the case that $\vdash_S \neg\Box\alpha$. In particular, the sentence expressing the internal consistency of S is not provable in S.

Proof. Suppose that $\vdash_S \neg\Box\alpha$, i.e. $\vdash_S \Box\alpha \to \bot$. Since $\vdash_S \bot \to \alpha$, it follows that $\vdash_S \Box\bot \to \Box\alpha$, and hence $\vdash_S \Box\bot \to \bot$. It follows from Löb's theorem that $\vdash_S \bot$, i.e., S is inconsistent.

The second incompleteness theorem may be taken to assert that in an adequate consistent theory *there is no proposition whose unprovability is provable.*

The idea of internal consistency can be extended to the following concordance:

Proposition	Meaning
$\Box\bot$	*S is internally inconsistent*
$\neg\Box\bot$	*S is internally consistent*
$\neg\neg\Box\bot$	*S is weakly internally inconsistent*
$\Box\neg\Box\bot$	*S is provably internally consistent*
$\neg\Box\neg\Box\bot$	*S is not provably internally consistent*

In each case, the claim that the proposition is provable in *S* is correlated with an assertion about S : for example, $\neg\Box\bot$ with the assertion "*S* is internally inconsistent" and similarly for the others.

In this spirit, consider a special case of (ii) of Löb's theorem, namely the inequality $\neg\Box\bot \vdash_S \neg\Box\neg\Box\bot$. This may be paraphrased: in *S*, *internal consistency implies the unprovability of internal consistency* . This is an *internal version* of Gödel's Second Incompleteness Theorem.

In this same spirit, the above Corollary translates as: if, in S, every proposition implies its own provability, then S is weakly internally inconsistent.

Notice that *consistency and internal inconsistency are compatible.* This follows from the fact that the provability predicate $\tau(u)$ can be taken to be the formula $u = u$, so that every proposition can be taken to satisfy the internal condition "__ *is provable*" . All this shows is that internal consistency need have little to do with consistency, or, more generally, that provability maps need have little to do with provability[20].

[20] An observation also made in [6].

Finally, consider the local set theory P in the language $\mathscr{L}_{\mathbf{N}}$. This language can be made codable as follows: take $\mathbf{N}$ to be the type of codes, and for each formula α, take $\ulcorner\alpha\urcorner$ to be the numeral $\underline{n}$, where n is the Gödel number of α under some standard Gödel numbering of the expressions of $\mathscr{L}_{\mathbf{N}}$. It is known[21] that all primitive recursive functions and relations are representable in P, so that , in particular, the diagonal map $\alpha(u) \mapsto \ulcorner\alpha(\ulcorner\alpha\urcorner)\urcorner$ and the proof relation of formulas in P are both representable therein. Take $\delta(m,n)$ to be the formula of $\mathscr{L}_{\mathbf{N}}$ representing in P the diagonal map and let $Prf(m, n)$ be a formula in $\mathscr{L}_{\mathbf{N}}$ representing in P the provability relation, namely, " *m is the Gödel number of a proof in FITT of the formula with Gödel number n"* . Finally take $\tau(n)$ to be the formula $\exists m Prf(m, n)$.

It follows from this prescription that δ is a diagonal relation in P. Also $\tau(n)$ is a demonstration predicate. To see this, suppose that α is a sentence with Gödel number n and $\vdash_P \alpha$. Then there is a proof of α in P. Let m be the Gödel number of such a proof. Then since Prf represents the provability relation in *FITT,* it follows that $\vdash_P Prf(\underline{m}, \underline{n})$, i.e., $\vdash_P Prf(\underline{m}, \ulcorner\alpha\urcorner)$. It follows that $\vdash_P \exists m Prf(m, \ulcorner\alpha\urcorner)$, i.e. $\vdash_{FITT} \tau(\ulcorner\alpha\urcorner))$. Conversely, suppose that $\vdash_P \tau(\ulcorner\alpha\urcorner))$, i.e. $\vdash_P \exists m Prf(m, \ulcorner\alpha\urcorner)$. Then for some numeral $\underline{m}$, $\vdash_P Prf(\underline{m}, \ulcorner\alpha\urcorner)$. Since Prf represents the provability relation in P, it follows that m is the Gödel number of a proof in P of α, whence $\vdash_P \alpha$.

From all this we deduce that P is subject to Gödel's first incompleteness theorem, namely, that if P is consistent, it is incomplete.

It can also be shown that the formula $\tau(n)$ as just defined is a proof predicate in P, so that P is adequate. Accordingly, P satisfies Gödel's 2nd incompleteness theorem, that is, if P is consistent, the sentence expressing its internal consistency is not provable therein.

Similar remarks apply to P^c.

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[21] This is proved in [7].

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