Two Approaches to Modelling the Universe: Synthetic Differential Geometry and Frame-Valued Sets

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I describe two approaches to modelling the universe, the one having its origin in topos theory and differential geometry, the other in set theory. The first is *synthetic differential geometry.*

Traditionally, there have been two methods of deriving the theorems of geometry: the *analytic* and the *synthetic.* While the analytical method is based on the introduction of numerical coordinates, and so on the theory of real numbers, the idea behind the synthetic approach is to furnish the subject of geometry with a purely geometric foundation in which the theorems are then deduced by purely logical means from an initial body of postulates.

The most familiar examples of the synthetic geometry are classical Euclidean geometry and the synthetic projective geometry introduced by Desargues in the 17th century and revived and developed by Carnot, Poncelet, Steiner and others during the 19th century.

The power of analytic geometry derives very largely from the fact that it permits the methods of the calculus, and, more generally, of mathematical analysis, to be introduced into geometry, leading in particular to *differential geometry* (a term, by the way, introduced in 1894 by the Italian geometer *Luigi Bianchi*). That being the case, the idea of a "synthetic" differential geometry seems elusive: how can differential geometry be placed on a "purely geometric" or "axiomatic" foundation when the apparatus of the calculus seems inextricably involved?

To my knowledge there have been two attempts to develop a synthetic differential geometry. The first was initiated by Herbert Busemann in the 1940s, building on earlier work of Paul Finsler. Here the idea was to build a differential geometry that, in its author's words, "requires no derivatives": the basic objects in Busemann's approach are not differentiable manifolds, but metric spaces of a certain type in which the notion of a geodesic can be defined in an intrinsic manner. I shall not have anything more to say about this approach.

The second approach, that with which I shall be concerned here, was originally proposed in the 1960s by F. W. Lawvere, who was in fact striving to fashion a decisive axiomatic framework for continuum mechanics. His ideas have led to what I shall simply call *synthetic differential geometry* (SDG) (sometimes called *smooth infinitesimal analysis*). SDG is formulated within *category theory,* the branch of

mathematics created in 1945 by Eilenberg and Mac Lane which deals with mathematical form and structure in its most general manifestations. As in biology, the viewpoint of category theory is that mathematical structures fall naturally into species or *categories*. But a category is specified not just by identifying the species of mathematical structure which constitute its *objects*; one must also specify the transformations or *maps* linking these objects. Thus one has, for example, the category **Set** with objects all sets and maps all functions between sets; the category **Grp** with objects all groups and maps all group homomorphisms; the category **Top** with objects all topological spaces and maps all continuous functions; and **Man**, with objects all (Hausdorff, second countable) smooth manifolds and maps all smooth functions. Since differential geometry "lives" in **Man**, it might be supposed that in formulating a "synthetic differential geometry" the category-theorist's goal would be to find an axiomatic description of **Man** itself.

But in fact the category **Man** has a couple of "deficiencies" which make it unsuitable as the object of axiomatic description:

1. It lacks exponentials: that is, the "space of all smooth maps" from one manifold to another in general fails to be a manifold. And even if it did—
2. It also lacks "infinitesimal objects"; in particular, there is no "infinitesimal" or *incredible shrinking manifold* Δ for which the tangent bundle TM of an arbitrary manifold M can be identified as the exponential "manifold" M^{Δ} of all "infinitesimal paths" in M. (It may be remarked parenthetically that it is this deficiency that makes the construction of the tangent bundle in **Man** something of a headache.)

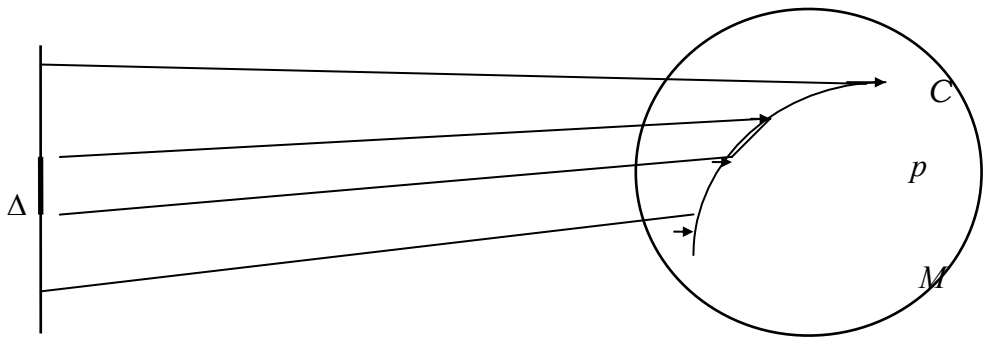
Lawvere's idea was to enlarge **Man** to a category **S**—a category of so-called *smooth spaces* or a *smooth category*—which avoids these two deficiencies, admits a simple axiomatic description, and at the same time is sufficiently similar to **Set** for mathematical construction and calculation to take place in the familiar way.

The essential features of a smooth category **S** are these:

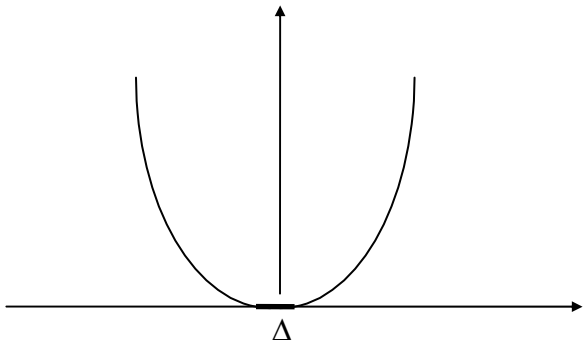
- In enlarging **Man** to **S** no "new" maps between manifolds are added, that is, all maps in **S** between objects of **Man** are smooth. (Notice that this is not the case when **Man** is enlarged to **Set.**)
- **S** is *Cartesian closed*, that is, contains products and exponentials of its objects in the appropriate sense.
- **S** satisfies the *principle of microstraightness.* Let $\mathbb{R}$ be the real line considered as a object of **Man,** and hence also of **S**. Then there is a nondegenerate segment Δ of $\mathbb{R}$ around 0 which remains *straight* and

unbroken under any map in **S.** In other words, Δ is subject in **S** to *Euclidean motions only.*

Δ may be thought of as a *generic tangent vector.* For consider any curve C in a space M—that is, the image of a segment of $\mathbb{R}$ (containing Δ) under a map f into M. Then the image of Δ under f may considered as a short straight line segment lying along C around the point $p = f(0)$ of C.



In fact, by considering the curve in $\mathbb{R} \times \mathbb{R}$ given by $f(x) = x^2$, we see that Δ is the intersection of the curve $y = x^2$ with the x-axis:



That is,

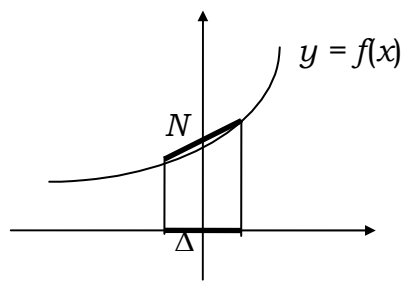
$$\Delta = \{x \in \mathbb{R}\colon\ x^2 = 0\}.$$

Thus Δ consists of *nilsquare infinitesimals,* or *microquantities.* We use the letter ε to denote an arbitrary microquantity.

Now classically Δ coincides with $\{0\}$, but a precise version of the principle of microstraightness—the *Principle of Microtaffineness*—ensures that this is not the case in **S.** The principle states that

- in **S**, any map $f\colon \Delta \to \mathbb{R}$ is (uniquely) *affine*, that is, for some *unique* $b \in \mathbb{R}$, we have, for all ε,
$$f(\varepsilon) = f(0) + b\varepsilon.$$

Here b is the *slope* of the segment N in the diagram:



Thus the principle of microaffineness asserts that each map $\Delta \to \mathbb{R}$ has a *unique* slope. This reduces the development of the *differential calculus* to simple algebra.

The principle of microaffineness asserts also that the map $\mathbb{R}^\Delta \to \mathbb{R} \times \mathbb{R}$ which assigns to each $f \in \mathbb{R}^\Delta$ the pair ($f(0)$, slope of f) is an isomorphism:
$$\mathbb{R}^\Delta \cong \mathbb{R} \times \mathbb{R}.$$

Since $\mathbb{R}^\Delta$ is the *tangent bundle* of $\mathbb{R}$, so is $\mathbb{R}^\Delta$.

This suggests that, for any space M in **S**, we take the tangent bundle TM of M to be the exponential M^Δ. Elements of M^Δ are called *tangent vectors* to M. Thus a *tangent vector* to M at a point $p \in M$ is just a map $t\colon \Delta \to M$ with $t(0) = x$ That is, a tangent vector at p is a *micropath in M with base point p*. The *base point map* $\pi\colon TM \to M$ is defined by $\pi(t) = t(0)$. For $p \in M$, the fibre $\pi^{-1}(p) = T_pM$ is the *tangent space to M at p*.

Observe that, if we identify each tangent vector with its image in M, then *each tangent space to M may be regarded as lying in M.* In this sense each space in **S** is "infinitesimally flat".

We check the compatibility of this definition of TM with the usual one in the case of Euclidean spaces:

$$T(R^n) = (R^n)^\Delta \cong (R^\Delta)^n \cong (R \times R)^n \cong R^n \times R^n.$$

The assignment $M \mapsto TM$ can be turned into a functor in the natural way—the *tangent bundle functor.* (For $f: M \to N$, $Tf: TM \to TN$ is defined by $(Tf)t = f \circ t$ for $t \in TM$.)

The whole point of synthetic differential geometry is to render *the tangent bundle functor representable*: TM becomes identified with the space of all maps from some fixed object—in this case Δ)—to M. (Classically, this is impossible.) This in turn simplifies a number of fundamental definitions in differential geometry.

For instance, a *vector field* on a space M is an assignment of a tangent vector to M at each point in it, that is, a map $\xi: M \to TM = M^{\Delta}$ such that $\xi(x)(0) = x$ for all $x \in M$. This means that $\pi \circ \xi$ is the identity on M, so that a *vector field is a section of the base point map.*

A *differential k- form* ((0, k) tensor field) on M may be considered as a map $M^{\Delta n} \to \mathbb{R}$.

Recall the condition that **S** be Cartesian closed. This means that for any pair S,T of spaces in **S**, **S** also contains their *product* $S \times T$ and their *exponential* T^S, the space of all (smooth) maps $S \to T$. These are connected in the following way: for any spaces S, T, U, there is a natural bijection of maps

$$\frac{S \to T^U}{S \times U \to T}$$

In the usual function-argument notation, this bijection is given by:

$$(f: S \times U \to T) \mapsto (f\hat{}: S \to T^U) \quad \text{with} \quad f\hat{}(s)(u) = f(s, u) \text{ for } s \in S,\ u \in U.$$

This gives rise to a bijective correspondence between vector fields on M and what we shall call *microflows* on M:

$$\frac{\xi: M \to M^{\Delta}}{\xi\hat{}: M \times \Delta \to M} \quad \begin{matrix}\text{(vector fields on } M\text{)} \\ \text{(microflows on } M\text{)},\end{matrix}$$

with

$$\xi\hat{}(x,\varepsilon) = \xi(x)(\varepsilon).$$

Notice that then $\xi\hat{}(x,0) = x$.

We also get, in turn, a bijective correspondence between microflows on M and *micropaths in M^M with the identity map as base point:*

$$\frac{\xi\hat{}: M \times \Delta \to M}{\xi^*: \Delta \to M^M} \quad \begin{matrix}\text{(microflows on } M\text{)} \\ \text{(micropaths in } M^M\text{)},\end{matrix}$$

with

$$\xi^*(\varepsilon)(x) = \xi\hat{}(x,\varepsilon) = \xi(x)(\varepsilon).$$

Thus, in particular,

$$\xi^*(0)(x) = \xi(x)(0) = x,$$

so that $\xi^*(0)$ is the identity map on M. Each $\xi^*(\varepsilon)$ is a microtransformation of M into itself which is "very close" to the identity map.

Accordingly, in **S**, *vector fields, microflows, and micropaths are equivalent.* Classically, this is a metaphor at best.

The notions of affine connection, geodesic, and the whole apparatus of Riemannian geometry can also be developed within *SDG*, as has been shown by Bunge, Kock and Reyes. Guts and Grinkevich have shown how Einstein's field equations can be formulated within *SDG*, resulting in a synthetic theory of relativity.

In a spacetime the metric can be written in the form

$$(*) \qquad ds^2 = \Sigma g_{\mu\nu} dx_\mu dx_\nu \qquad \mu,\nu = 1,2,3,4.$$

In the classical setting (*) is in fact an abbreviation for an equation involving derivatives and the "differentials" ds and dx_μ are not really quantities at all. What form does this equation take in SDG? Notice that the "differentials" cannot be taken as nilsquare infinitesimals since all the squared terms would vanish. But the equation does have a very natural form in terms of nilsquare infinitesimals. Here is an informal way of obtaining it.

We think of the dx_μ as being multiples $k_\mu e$ of some small quantity e. Then (*) becomes

$$ds^2 = e^2 \Sigma g_{\mu\nu} k_\mu k_\nu,$$

so that

$$ds = e[\Sigma g_{\mu\nu} k_\mu k_\nu]^{1/2}$$

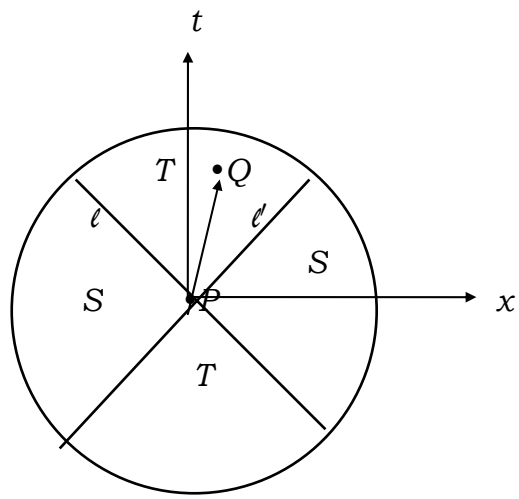
Now replace e by a nilsquare infinitesimal ε. Then we obtain the metric relation in SDG:

$$ds = \varepsilon[\Sigma g_{\mu\nu} k_\mu k_\nu]^{1/2}.$$

This tells us that the "infinitesimal distance" ds between a point P with coordinates (x_1, x_2, x_3, x_4) and an infinitesimally near point Q with coordinates $(x_1 + k_1\varepsilon, x_2 + k_2\varepsilon, x_3 + k_3\varepsilon, x_4 + k_4\varepsilon)$ is $\varepsilon[\Sigma g_{\mu\nu} k_\mu k_\nu]^{1/2}$. Here a curious situation arises. For when the "infinitesimal interval" ds between P and Q is timelike (or lightlike), the quantity $\Sigma g_{\mu\nu} k_\mu k_\nu$ is nonnegative, so

that its square root is a real number. In this case ds may be written as εd, where d is a real number. On the other hand, if ds is spacelike, then $\Sigma g_{\mu\nu}k_\mu k_\nu$ is negative, so that its square root is imaginary. In this case, then, ds assumes the form iεd, where d is a real number (and, of course $\mathrm{i} = \sqrt{-1}$). On comparing these we see that, if we take ε as the "infinitesimal unit" for measuring infinitesimal timelike distances, then iε serves as the "imaginary infinitesimal unit" for measuring infinitesimal spacelike distances.

For purposes of illustration, let us restrict the spacetime to two dimensions (x, t), and assume that the metric takes the simple form $\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}x^2$. The infinitesimal light cone at a point P divides the infinitesimal neighbourhood at P into a timelike region T and a spacelike region S,



bounded by the null lines ℓ and ℓ' respectively. If we take P as origin of coordinates, a typical point Q in this neighbourhood will have coordinates ($a\varepsilon$, $b\varepsilon$) with a and b real numbers: if $|b| > |a|$, Q lies in T; if $a = b$, P lies on ℓ or ℓ'; if $|a| < |b|$, P lies in S. If we write $d = |a^2 - b^2|^{1/2}$, then in the first case, the infinitesimal distance between P and Q is εd, in the second, it is 0, and in the third it is iεd.

Minkowski introduced "ict" to replace the "t" coordinate so as to make the metric of relativistic spacetime positive definite. This was, despite its daring, purely a matter of formal convenience, and was later rejected by (general) relativists (see, for example Box 2.1, *Farewell to* "ict", of Misner, Thorne and Wheeler *Gravitation* [1973]). In conventional physics one never works with nilpotent quantities so it is always possible to replace formal imaginaries by their (negative) squares. But spacetime theory in SDG *forces* one to use imaginary units, since, infinitesimally,

one can't "square oneself out of trouble". This being the case, it would seem that, infinitesimally, Wheeler *et al.'s* dictum needs to be replaced by

Vale "ic(t)", *ave* "iε" !

To quote once again from Misner, Thorne and Wheeler's massive work,

> *Another danger in curved spacetime is the temptation to regard ... the tangent space as lying in spacetime itself. This practice can be useful for heuristic purposes, but is incompatible with complete mathematical precision.*

The consistency of synthetic differential geometry shows that, on the contrary, yielding to this temptation *is* compatible with complete mathematical precision: there tangent spaces may indeed be regarded as lying in spacetime itself. If (as Hilbert said) set theory is "Cantor's paradise" then I would submit that SDG is nothing less than "Riemann's paradise"!

I turn now to *frame-valued set theory*. First I must define *frames.* A *lattice* is a partially ordered set L with partial ordering $\leq$ in which each two-element subset $\{x, y\}$ has a supremum or *join*—denoted by $x \vee y$—and an infimum or *meet*—denoted by $x \wedge y$. A lattice L is *complete* if every subset X (including $\varnothing$) has a supremum or *join*—denoted by $\bigvee X$—and an infimum or *meet*—denoted by $\bigwedge X$. Note that $\bigvee\varnothing = 0$, the least or *bottom* element of L, and $\bigwedge\varnothing = 1$, the largest or *top* element of L.

A *Heyting algebra* is a lattice L with top and bottom elements such that, for any elements $x, y \in L$, there is an element—denoted by $x \Rightarrow y$—of L such that, for any $z \in L$,

$$z \leq x \Rightarrow y \text{ iff } z \wedge x \leq y.$$

Thus $x \Rightarrow y$ is the *largest* element z such that $z \wedge x \leq y$. So in particular, if we write $\neg x$ for $x \Rightarrow 0$, then $\neg x$ is the largest element z such that $x \Rightarrow z = 0$: it is called the *pseudocomplement* of x.

A *Boolean algebra* is a Heyting algebra in which $\neg\neg x = x$ for all x, or equivalently, in which $x \vee \neg x = 1$ for all x.

Heyting algebras are related to intuitionistic propositional logic in precisely the same way as Boolean algebras are related to classical propositional logic. That is, suppose given a propositional language $\mathscr{L}$; let $\mathscr{P}$ be its set of propositional variables. Given a map $f\colon \mathscr{P} \to L$ to a Heyting algebra L, we extend f to a map $\alpha \mapsto [\![\alpha]\!]$ of the set of formulas of $\mathscr{L}$ to L à la Tarski:

$$[\![\alpha \wedge \beta]\!] = [\![\alpha]\!] \wedge [\![\beta]\!] \qquad [\![\alpha \vee \beta]\!] = [\![\alpha]\!] \vee [\![\beta]\!] \qquad [\![\neg\alpha]\!] = \neg[\![\alpha]\!]$$
$$[\![\alpha \to \beta]\!] = [\![\alpha]\!] \Rightarrow [\![\beta]\!]$$

A formula α is said to be (Heyting) *valid*—written $\vDash\alpha$—if $[\![\alpha]\!] = 1$ for any such map f. It can then be shown that α is valid iff α is deducible in the intuitionistic propositional calculus, A basic fact about *complete* Heyting algebras is that the following identity holds in them:

$$(*) \qquad x \wedge \bigvee_{i\in I} y_i = \bigvee_{i\in I}(x \wedge y_i)$$

And conversely, in any complete lattice satisfying (*), defining the operation $\Rightarrow$ by $x \Rightarrow y = \bigvee\{z\colon z \wedge x \leq y\}$ turns it into a Heyting algebra.

In view of this result a complete Heyting algebra is frequently defined to be a complete lattice satisfying (*). A complete Heyting algebra is often called a *frame.*

Frames are related to (free) intuitionistic first-order logic in the same way as complete Boolean algebras are related to classical first-order logic.

Let P be a preordered set. A *sieve* in P is a subset S satisfying $p \in S$ and $q \leq p \to q \in S$. The set $\widehat{P}$ of all sieves in P partially ordered by inclusion is then a frame—the *completion*[1] of P— in which joins and meets are just set-theoretic unions and intersections, and in which the operations $\Rightarrow$ and $\neg$ are given by

$$I \Rightarrow J = \{p : I \cap p{\downarrow} \subseteq J\} \qquad\qquad \neg I = \{p : I \cap p{\downarrow} = \varnothing\}.$$

Associated with each frame H is an H-valued model $V^{(H)}$ of (intuitionistic) set theory (see, e.g. [1] or [2]): here are some of its principal features.

- Each of the members of $V^{(H)}$—the *H-sets*—is a map from a subset of $V^{(H)}$ to H.
- Corresponding to each sentence σ of the language of set theory (with names for all elements of $V^{(H)}$) is an element $[\![\sigma]\!] = [\![\sigma]\!]^H \in H$ called its *truth value in* $V^{(H)}$. These "truth values" satisfy the following conditions. For $a, b \in V^{(H)}$,

[1] Writing **Lat** for the category of complete lattices and join preserving homomorphisms, $\widehat{P}$ is in fact the object in **Lat** freely generated by P.

$$[\![b \in a]\!] = \bigvee_{c \in dom(a)} [\![b = c]\!] \wedge a(c) \qquad [\![b = a]\!] = \bigvee_{c \in dom(a) \cup dom(b)} ([\![c \in b]\!] \Leftrightarrow [\![c \in a]\!])$$

$$[\![\sigma \wedge \tau]\!] = [\![\sigma]\!] \wedge [\![\tau]\!], \text{ etc.}$$

$$[\![\exists x \varphi(x)]\!] = \bigvee_{a \in V^{(H)}} [\![\varphi(a)]\!]$$

$$[\![\forall x \varphi(x)]\!] = \bigwedge_{a \in V^{(H)}} [\![\varphi(a)]\!]$$

A sentence σ is *valid* , or *holds,* in $V^{(H)}$, written $V^{(H)} \models \sigma$, if $[\![\sigma]\!] = 1$, the top element of H.

- The axioms of intuitionistic Zermelo-Fraenkel set theory are valid in $V^{(H)}$. Accordingly the category $\mathscr{S}et^{(H)}$ of sets constructed within $V^{(H)}$ is a topos: in fact $\mathscr{S}et^{(H)}$ can be shown to be equivalent to the topos of canonical sheaves on H.
- There is a canonical embedding $x \mapsto \hat{x}$ of the universe V of sets into $V^{(H)}$ satisfying

$$[\![u \in \hat{x}]\!] = \bigvee_{y \in x} [\![u = \hat{y}]\!] \text{ for } x \in V, u \in V^{(H)}$$

$$x \in y \leftrightarrow V^{(H)} \models \hat{x} \in \hat{y}, \quad x = y \leftrightarrow V^{(H)} \models \hat{x} = \hat{y} \text{ for } x, y \in V$$

$$\varphi(x_1, \ldots, x_n) \leftrightarrow V^{(H)} \models \varphi(x_1, \ldots, x_n) \text{ for } x_1, \ldots, x_n \in V \text{ and restricted } \varphi$$

(Here a formula φ is *restricted* if all its quantifiers are restricted, i.e. can be put in the form $\forall x \in y$ or $\exists x \in y$.)

It follows from the last of these assertions that the canonical representative $\widehat{H}$ of H is a Heyting algebra in $V^{(H)}$. The *canonical prime filter* in $\widehat{H}$ is the H-set Φ_H defined by

$$dom(\Phi_H) = \{\hat{a} : a \in H\}, \quad \Phi_H(\hat{a}) = a \text{ for } a \in H.$$

Clearly $V^{(H)} \models \Phi_H \subseteq \widehat{H}$, and it is easily verified that

$$V^{(H)} \models \Phi_H \text{ is a (proper) prime filter}^2 \text{ in } \widehat{H}.$$

It can also be shown that Φ_H is *V-generic* in the sense that, for any subset $A \subseteq H$,

$$V^{(H)} \models \widehat{\bigvee A} \in \Phi_H \leftrightarrow \Phi_H \cap \widehat{A} \neq \varnothing.$$

[2] We recall that a filter F in a lattice is *prime* if $x \vee y \in F$ implies $x \in F$ or $y \in F$.

Moreover, for any $a \in H$ we have $[\![\hat{a} \in \Phi_H]\!] = a$, and in particular, for any sentence σ, $[\![\sigma]\!] = [\![\widehat{[\![\sigma]\!]} \in \Phi_H]\!]$. Thus $V^{(H)} \models \sigma \leftrightarrow V^{(H)} \models \widehat{[\![\sigma]\!]} \in \Phi_H$—in this sense Φ_H is the filter of "true" sentences in $V^{(H)}$.

This suggests that we define a *truth set* in $V^{(H)}$ to be an H-set F for which

$$V^{(H)} \models F \text{ is a filter in } \widehat{H} \text{ such that } F \supseteq \Phi_H.$$

Consider now the special case in which H is the completion $\widehat{P}$ of a preordered set P The topos $\mathscr{S}et^{(\widehat{P})}$ of sets in $V^{(\widehat{P})}$ is equivalent to the topos of canonical sheaves on $\widehat{P}$, which is itself, as is well known, equivalent to the topos $\mathscr{S}et^{P^{op}}$ of presheaves on P. The *forcing* relation $\Vdash_P$ in $V^{(\widehat{P})}$ between sentences and elements of P is defined by

$$p \Vdash_P \sigma \leftrightarrow p \in [\![\sigma]\!]^{\widehat{P}}.$$

This satisfies the usual rules governing Kripke semantics for predicate sentences, viz.,

- $p \Vdash_P \varphi \wedge \psi \leftrightarrow p \Vdash_P \varphi$ & $p \Vdash_P \psi$
- $p \Vdash_P \varphi \vee \psi \leftrightarrow p \Vdash_P \varphi$ or $p \Vdash_P \psi$
- $p \Vdash_P \varphi \rightarrow \psi \leftrightarrow \forall q \leq p[\, q \Vdash_P \varphi \rightarrow q \Vdash_P \psi]$
- $p \Vdash_P \neg\varphi \leftrightarrow \forall q \leq p \;\; q \nVdash_K \varphi$
- $p \Vdash_P \forall x\, \varphi \leftrightarrow p \Vdash_P \varphi(a)$ for every $a \in V^{(\widehat{P})}$
- $p \Vdash_P \exists x\, \varphi \leftrightarrow p \Vdash_P \varphi(a)$ for some $a \in V^{(\widehat{P})}$.

In spacetime physics any set $\mathscr{C}$ of events—a *causal set*—is taken to be partially ordered by the relation ≤ of *possible causation*: for $p, q \in \mathscr{C}$, $p \leq q$ means that q is in p's future light cone. In her groundbreaking paper [5] Fotini Markopoulou proposes that the causal structure of spacetime itself be represented by "sets evolving over $\mathscr{C}$" —that is, in essence, by the topos $\mathscr{S}et^{\mathscr{C}}$ of presheaves on $\mathscr{C}^{op}$. To enable what she has done to be the more easily expressed within the framework presented here, we will reverse the causal ordering, that is, $\mathscr{C}$ will be replaced by $\mathscr{C}^{op}$, and the latter written as P—which will, moreover, be required to be no more than a *preordered* set. Specifically, then: P is a set of events preordered by the relation ≤, where $p \leq q$ is intended to mean that p is in q's future light cone—that q *could* be the cause of p. In requiring that ≤ be no more than a preordering—in dropping, that is, the antisymmetry of ≤—we are, in physical terms, allowing for the possibility that the universe is of Gödelian type, containing closed timelike lines.

Accordingly we fix a preordered set $(P, \leq)$, which we shall call the *universal causal set.* Markopoulou, in essence, suggests that viewing the universe "from the inside" amounts to placing oneself within the topos of presheaves $\mathscr{S}et^{P^{op}}$. Since, as we have already observed, $\mathscr{S}et^{P^{op}}$ is equivalent to the topos of sets in $V^{(\widehat{P})}$, Markopoulou's proposal may be effectively realized by working within $V^{(\widehat{P})}$. Let us do so, writing for simplicity H for $\widehat{P}$.

Define the set $K \in V^{(H)}$ by dom(K) = $\{\hat{p} : p \in P\}$ and $K(\hat{p}) = p\downarrow$. Then, in $V^{(H)}$, K is a subset of $\widehat{P}$ and for $p \in P$, $[\![\hat{p} \in K]\!] = p\downarrow$. K is the counterpart in $V^{(H)}$ of the evolving set *Past* Markopoulou defines by *Past*(p) = $p\downarrow$, with insertions as transition maps. ($\widehat{P}$, incidentally, is the $V^{(H)}$- counterpart of the constant presheaf on P with value P —which Markopoulou calls *World*.) Accordingly the "causal past" of any "event" p is represented by the truth value in $V^{(H)}$ of the statement $\hat{p} \in K$. The fact that, for any $p, q \in P$ we have

$$q \Vdash_P \widehat{p} \in K \leftrightarrow \quad q \leq p$$

may be construed as asserting that *the events in the causal future of an event p are precisely those forcing (the canonical representative of) p to be a member of K.* For this reason we shall call K the *causal set in* $V^{(H)}$.

If we identify each $p \in P$ with $p\downarrow \in H$, P may then be regarded as a subset of H so that, in $V^{(H)}$, $\widehat{P}$ is a subset of $\widehat{H}$. It is not hard to show that, in $V^{(H)}$, K generates the canonical prime filter Φ_H in $\widehat{H}$. Using the V-genericity of Φ_H, and the density of P in H, one can show that $[\![\sigma]\!] = [\![\exists p \in K. p \leq \widehat{[\![\sigma]\!]}]\!]$, so that, with moderate abuse of notation,

$$V^{(H)} \models [\sigma \leftrightarrow \exists p \in K.\ p \Vdash \sigma].$$

That is, in $V^{(H)}$, a sentence holds precisely when it is forced to do so at some "causal past stage" in K. This establishes the centrality of K—and, correspondingly, that of the "evolving" set *Past*— in determining the truth of sentences "from the inside", that is, inside the universe $V^{(H)}$.

Markopoulou also considers the *complement* of *Past*—i.e., in the present setting, the $V^{(H)}$-set $\neg K$ for which $[\![\widehat{p} \in \neg K]\!] = [\![p \notin K]\!] = \neg p\downarrow$. Markopoulou calls (*mutatis mutandis*) the events in $\neg p\downarrow$ those *beyond p's causal horizon*, in that no observer at p can ever receive "information" from any event in $\neg p\downarrow$. Since clearly we have

(†) $$q \Vdash_P \widehat{p} \in \neg K \quad \leftrightarrow \quad q \in \neg p\downarrow,$$

it follows that *the events beyond the causal horizon of an event p are precisely those forcing (the canonical representative of) p to be a member of* $\neg K$. In this sense $\neg K$ reflects, or "measures" the causal structure of P.

In this connection it is natural to call $\neg\neg p\downarrow$ = $\{q : \forall r \leq q \exists s \leq r. s \leq p]$ the *causal horizon* of p: it consists of those events q for which an observer placed at p could, in its future, receive information from any event in the future of an observer placed at q. Since

$$q \Vdash_P \hat{p} \in \neg\neg K \leftrightarrow q \in \neg\neg p\downarrow,$$

it follows that *the events within the causal horizon of an event are precisely those forcing (the canonical representative of) p to be a member of* $\neg\neg K$.

It is easily shown that $\neg K$ is *empty* (i.e. $V^{(H)} \models \neg K = \varnothing$) if and only if P is *directed downwards*, i.e., for any $p, q \in P$ there is $r \in P$ for which $r \leq p$ and $r \leq q$. This holds in the case, considered by Markopoulou, of *discrete Newtonian time evolution*—in the present setting, the case in which P is the opposite $\mathbb{N}^{op}$ of the totally ordered set $\mathbb{N}$ of natural numbers. Here the corresponding complete Heyting algebra H is the family of all downward-closed sets of natural numbers. In this case the H-valued set K representing *Past is neither finite nor actually infinite in* $V^{(H)}$.

To see this, first note that, for any natural number n, we have, $[\![\neg(\hat{n} \in \neg K)]\!] = \mathbb{N}$. It follows that $V^{(H)} \models \neg\neg\forall n \in \hat{\mathbb{N}}.\ n \in K$. But, working in $V^{(H)}$, if $\forall n \in \hat{\mathbb{N}}\ n \in K$, then K is not finite, so if K is finite, then $\neg\forall n \in \hat{\mathbb{N}}.\ n \in K$, and so $\neg\neg\forall n \in \hat{\mathbb{N}}.\ n \in K$ implies the non-finiteness of K.

But, in $V^{(H)}$, K is not actually infinite. For (again working in $V^{(H)}$), if K were actually infinite (i.e., if there existed an injection of $\hat{\mathbb{N}}$ into K), then the statement

$$\forall x \in K\ \exists y \in K.\ x > y$$

would also have to hold in $V^{(H)}$. But calculating that truth value gives:

$$\begin{aligned}
&[\![\forall x \in K \exists y \in K. x > y]\!] \\
&= \bigcap_{m \in \mathbb{N}^{op}} [m\downarrow \Rightarrow \bigcup_{n \in \mathbb{N}^{op}} n\downarrow \cap [\![\hat{m} > \hat{n}]\!]] \\
&= \bigcap_{m} [m\downarrow \Rightarrow \bigcup_{n<m} n\downarrow] \\
&= \bigcap_{m} [m\downarrow \Rightarrow (m+1)\downarrow] \\
&= \bigcap_{m} (m+1)\downarrow = \varnothing
\end{aligned}$$

So $\forall x \in K\ \exists y \in K.\ x > y$ is false in $V^{(H)}$ and therefore K is not actually infinite. In sum, the causal set K in is *potentially, but not actually infinite.*

In order to formulate an observable causal *quantum theory* Markopoulou considers the possibility of introducing a *causally evolving algebra of observables.* This amounts to specifying a presheaf of C^*-

algebras on P, which, in the present framework, corresponds to specifying a set $\mathscr{A}$ in $V^{(H)}$ satisfying

$$V^{(H)} \models \mathscr{A} \textit{ is a } C^*\textit{-algebra.}$$

The "internal" C^*-algebra $\mathscr{A}$ is then subject to the intuitionistic internal logic of $V^{(H)}$: *any* theorem concerning C^*-algebras—provided only that it be constructively proved—automatically applies to $\mathscr{A}$. Reasoning with $\mathscr{A}$ is more direct and simpler than reasoning with $\mathscr{A}$.

This same procedure of "internalization" can be performed with any causally evolving object: each such object of type $\mathscr{T}$ corresponds to a set S in $V^{(H)}$ satisfying

$$V^{(H)} \models S \textit{ is of type } \mathscr{T}.$$

Internalization may also be applied in the case of the presheaves *Antichains* and *Graphs* considered by Markopoulou. Here, for each event p, *Antichains*(p) consists of all sets of causally unrelated events in *Past*(p), while *Graphs*(p) is the set of all graphs supported by elements of *Antichains*(p). In the present framework *Antichains* is represented by the $V^{(H)}$–set $Anti = \{ X \subseteq \widehat{P} : X \textit{ is an antichain}\}$ and *Graphs* by the $V^{(H)}$–set $Grph = \{G\colon \exists X \in A\ .G \textit{ is a graph supported by } A\}$. Again, both *Anti* and *Grph* can be readily handled using the internal intuitionistic logic of $V^{(H)}$.

Cover schemes or *Grothendieck topologies* may be used to force certain conditions to prevail in the associated models. A *cover scheme* on P is a map $\mathbf{C}$ assigning to each $p \in P$ a family $\mathbf{C}(p)$ of subsets of $p{\downarrow} = \{q\colon q \leq p\}$, called ($\mathbf{C}$-)*covers of p,* such that, if $q \leq p$, any cover of p can be sharpened to a cover of q, i.e.,

$$S \in \mathbf{C}(p) \,\&\, q \leq p \rightarrow \exists T \in \mathbf{C}(q)[\forall t \in T \exists s \in S(t \leq s)].$$

There are three naturally defined cover schemes on P we shall consider. First, each sieve A in P determines two cover schemes $\mathbf{C}_A$ and $\mathbf{C}^A$ defined by

$$S \in \mathbf{C}_A(p) \leftrightarrow p \in A \cup S \qquad S \in \mathbf{C}^{\mathbf{A}}(p) \leftrightarrow p{\downarrow} \cap A \subseteq S$$

Notice that $\varnothing \in \mathbf{C}_A(p) \leftrightarrow p \in A$ and $\varnothing \in \mathbf{C}^A(p) \leftrightarrow p{\downarrow} \cap A = \varnothing$.

Next, we have the *dense cover scheme* **Den** given by:

$$S \in \mathbf{Den}(p) \leftrightarrow \forall q \leq p \exists s \in S \exists r \leq s(r \leq q):$$

When S is a sieve, the above condition is easily seen to be equivalent to the familiar condition of *density below p*: that is, $\forall q \leq p \exists s \in S(s \leq q)$.

Given a cover scheme **C** on P, a sieve I will be said to *cover* an element $p \in P$ if I includes a **C**-cover of p. Call I **C**-*closed* if it contains every element of P that it covers, i.e. if

$$\exists S \in \mathbf{C}(p)(S \subseteq I) \to p \in I\ .$$

The set $\widehat{\mathbf{C}}$ of all **C**-closed sieves in P, partially ordered by inclusion, can be shown to be a frame—the frame *induced* by **C**—in which the operations of meet and $\Rightarrow$ coincide with those of $\widehat{P}$. It can be shown that $\widehat{\mathbf{Den}}$ *is a Boolean algebra*—it is in fact the complete Boolean algebra of $\neg\neg$-stable elements of $\widehat{P}$.

Now let us return to regarding P as a universal causal set. The frame induced by the dense cover scheme **Den** in P is a complete Boolean algebra B. The corresponding causal set K_B in $V^{(B)}$ then has the property

$$[\![\widehat{p} \in K_B]\!] = \neg\neg p\downarrow;$$

so that,

$$q \Vdash_B \widehat{p} \in K_B \leftrightarrow q \in \neg\neg p\downarrow$$
$$\leftrightarrow \ q \textit{ is in } p\textit{'s causal horizon.}$$

Comparing this with (†) above, we see that moving to the universe $V^{(B)}$—"Booleanizing" it, so to speak—*amounts to replacing causal futures by causal horizons.* When P is linearly ordered, as for example in the case of Newtonian time, the causal horizon of any event coincides with the whole of P, B is the two-element Boolean algebra **2,** so that $V^{(B)}$ is just the universe V of "static" sets. In this case, then, the effect of "Booleanization" is to *render the universe timeless.*

The universes associated with the cover schemes $\mathbf{C}^A$ and $\mathbf{C}_A$ seem also to have a rather natural physical meaning. Consider, for instance the case in which A is the sieve $p\downarrow$—the causal future of p. In the associated universe $V^{(\widehat{\mathbf{C}^A})}$ the corresponding causal set K^A satisfies

$$[\![\widehat{q} \in K^A]\!] = \textit{least } \mathbf{C^A}\textit{-closed sieve containing } q$$

so that , in particular

$$[\![\widehat{p} \in K^A]\!] = \textit{least } \mathbf{C^A}\textit{-closed sieve containing } p$$
$$= P.$$

This means that, for every event q,

$$q \Vdash_{\widehat{\mathbf{C}^A}} \widehat{p} \in K^A\ .$$

Comparing this with (*), we see that in $V^{(\widehat{\mathbf{C}^A})}$ that every event has been "forced" into p's causal future: in short, that p now marks the "beginning" of the universe as viewed from inside $V^{(\widehat{\mathbf{C}^A})}$.

Similarly, we find that the causal set K_A in the universe $V^{(\widehat{\mathbf{C}_A})}$ has the property

$$q \leq p \rightarrow \forall r[r \Vdash_{\widehat{\mathbf{C}_A}} \hat{q} \in \neg K_A];$$

that is, p, together with all events in its causal future, are, in $V^{(\widehat{\mathbf{C}^A})}$, beyond the causal horizon of any event. In effect, p has become a *black hole.*

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