CENTRAL H-SPACES AND BANDED TYPES

ULRIK BUCHHOLTZ, J. DANIEL CHRISTENSEN, JARL G. TAXERÅS FLATEN, AND EGBERT RIJKE

Abstract. We introduce and study *central* types, which are generalizations of Eilenberg–Mac Lane spaces. A type is central when it is equivalent to the component of the identity among its own self-equivalences. From centrality alone we construct an infinite delooping in terms of a tensor product of *banded types*, which are the appropriate notion of torsor for a central type. Our constructions are carried out in homotopy type theory, and therefore hold in any ∞-topos.

Even when interpreted into the ∞-topos of spaces, our main results and constructions are new. In particular, we give a description of the moduli space of H-space structures on an H-space which generalizes a formula of Arkowitz–Curjel and Copeland which counts the number of path components of this moduli space.

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1. Introduction

In this paper we study H-spaces and their deloopings. We work in homotopy type theory, so our results apply to any ∞-topos. Many of our results are new, even for the ∞-topos of spaces.

A key concept is that of a central type. A pointed type A is **central** if the map $(A \to A)_{(\mathsf{id})} \to A$ sending a function f to $f(\mathsf{pt})$ is an equivalence. Here $(A \to A)_{(\mathsf{id})}$ denotes the identity component of the type of all self-maps of A, and pt denotes the base point of A. Every central type is a connected H-space, and a connected H-space is central precisely when the type $A \to_* A$ of pointed self-maps is a set. We prove this and other characterizations of central types in Proposition 3.6. It follows, for example, that every Eilenberg–Mac Lane space $\mathrm{K}(G, n)$, with G abelian and $n \geq 1$, is central. We show in Section 5.3 that some, but not all, products of Eilenberg–Mac Lane spaces are central. We don't know whether every central type is a product of Eilenberg–Mac Lane spaces.

Our first result is:

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Theorem 4.6. *Let A be a central type. Then A has a unique delooping.*

The key ingredient of this result and much of the paper is that we have a concrete description of the delooping of A. It is given by the type $\mathsf{BAut}_1(A) :\equiv \Sigma_{X:\mathcal{U}}\|A = X\|_0$ of types *banded* by A, which is the 1-connected cover of $\mathsf{BAut}(A)$. As an example, since $\mathrm{K}(G, n)$ is central for G abelian and $n \geq 1$, this gives an alternative way to define $\mathrm{K}(G, n+1)$ in terms of $\mathrm{K}(G, n)$, as previously indicated by the first author [Buc19].

We also show:

Theorem 4.10. *Let A be a central type. Then every pointed map $f : A \to_* A$ is uniquely deloopable to a map $Bf : \mathsf{BAut}_1(A) \to_* \mathsf{BAut}_1(A)$.*

It follows that the type of pointed self-maps of $\mathsf{BAut}_1(A)$ is a set, since it is equivalent to $A \to_* A$.

One of the motivations for studying $\mathsf{BAut}_1(A)$ is that one can define a *tensoring* operation. Given two banded types X and Y in $\mathsf{BAut}_1(A)$, the type $X^* = Y$ has a natural banding, where X^* is a certain dual of X. We write $X \otimes Y$ for this banded type, and show in Theorem 4.19 that it makes $\mathsf{BAut}_1(A)$ into an abelian H-space. Combined with Theorem 4.6, Theorem 4.10, and the characterization of central types mentioned earlier, we therefore deduce:

Corollary 4.20. *For a central type A, the type $\mathsf{BAut}_1(A)$ is again central. Therefore, A is an infinite loop space, in a unique way. Moreover, every pointed map $A \to_* A$ is infinitely deloopable, in a unique way.*

Our tensoring operation gives a new description of the H-space structure on $\mathrm{K}(G, n)$, which will be helpful for calculations of Euler classes in work in progress and is what originally motivated this work.

We also give an alternate description of the delooping of a central type A as a certain type of A-torsors, and give an analogous description of $\mathrm{K}(G, 1)$ for any group G.

To prove the above results, we first need to further develop the theory of H-spaces. One difference between our work and classical work in topology is that we emphasize the moduli space $\mathrm{HSpace}(A)$ of H-space structures on a pointed type A, rather than just the set of components. For example, we prove:

Theorem 2.27. *Let A be an H-space such that for all $a : A$, the map $a \cdot -$ is an equivalence. Then the type $\mathrm{HSpace}(A)$ of H-space structures on A is equivalent to the type $A \wedge A \to_* A$ of pointed maps.*

This generalizes a classical formula of Arkowitz–Curjel and Copeland, which plays a key role in classical results on the number of H-space structures on various spaces. The classical formula only determines the path components of the type of H-space structures, while our formula gives an equivalence of types. From our formula it immediately follows, for example, that the type of H-space structures on the 3-sphere is $\Omega^6\mathbb{S}^3$. The proof of Theorem 2.27 uses *evaluation fibrations*, which generalize the map appearing in the definition of "central." In fact, these evaluation fibrations play an important role in much of the paper. For example, we include results relating the existence of sections of an evaluation fibration to the vanishing of Whitehead products, and use this to show that no even spheres besides $\mathbb{S}^0$ admit H-space structures.

In Proposition 3.3 we show that every central type has a unique H-space structure, in the strong sense that the type $\mathrm{HSpace}(A)$ is contractible. We prove several results about types with unique H-space structures. For example, we show that such H-space structures are associative and coherently abelian, and that every pointed self-map is an H-space map, a weaker version of the delooping above. We also give an example showing that not every type with a unique H-space structure is central.

We note that these results rely on us defining "H-space" to include a coherence between the two unit laws (see Definition 2.1).

Outline. In Section 2, we give results about H-spaces which do not depend on centrality, including a description of the moduli space of H-space structures, results about Whitehead products and H-space structures on spheres, and results about unique H-space structures. In Section 3, we define central types, show that central types have a unique H-space structure, give a characterization of which H-spaces are central, and prove other results needed in the next section. Section 4 is the heart of the paper. It defines the type $\mathsf{BAut}_1(A)$ of bands for a central type A, shows that it is a unique delooping of A, proves that it is an H-space under a tensoring operation, and shows that central types and their self-maps are uniquely infinitely deloopable. We also give the alternate description of the delooping in terms of A-torsors. Finally, Section 5 gives examples and non-examples of central types, mostly related to Eilenberg–Mac Lane spaces and their products.

Notation and conventions. In general, we follow the notation used in [Uni13]. For example, we write path composition in diagrammatic order: given paths $p : x = y$ and $q : y = z$, their composite is $p \cdot q$. The reflexivity path is written refl.

Given a type A and an element $a : A$, we write (A, a) for the type A pointed at a. If A is already a pointed type with unspecified base point, then we write pt for the base point. If A and B are pointed types, and $f, g : A \to_* B$ are pointed maps, then $f =_* g$ is the type of pointed homotopies between f and g. If A is an H-space, then we write $x \cdot y$ for the product of two elements $x, y : A$ (unless another notation for the multiplication is given). For a pointed type A, we write $\mathrm{HSpace}(A)$ for the type of H-space structures on A with the basepoint as the identity element (Definition 2.1). We write $\mathbb{S}^n$ for the n-sphere, and $\mathcal{U}$ for a fixed universe of types.

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2. H-spaces and evaluation fibrations

In Section 2.1, we begin by recalling the notion of a (coherent) *H-space structure* on a pointed type A. We discuss the type of pointed extensions of a map $B \vee C \to_* A$ to $B \times C$, and show that the type of H-space structures on A is equivalent to the type of pointed extensions of the fold map. We relate the existence of extensions to the vanishing of Whitehead products, and use this to show that there are no H-space structures on even spheres except $\mathbb{S}^0$. In addition, we show that for any n-connected H-space A, the Freudenthal map $\pi_{2n+1}(A) \to \pi_{2n+2}(\Sigma A)$ is an isomorphism, not just a surjection.

In Section 2.2, we study *evaluation fibrations*. We show that the type of H-space structures is equivalent to a type of sections of an evaluation fibration, and use this to show that the type of H-space structures on a left-invertible H-space A is equivalent to $A \wedge A \to_* A$, generalizing a classical formula of Arkowitz–Curjel and Copeland. It immediately follows, for example, that the type of H-space structures on the 3-sphere is $\Omega^6\mathbb{S}^3$. We end with a result relating the existence of sections of an evaluation fibration to the vanishing of Whitehead products.

Section 2.3 is a short section which studies the case when the type of H-space structures is contractible. We stress that this is not the same as $\mathrm{HSpace}(A)$ having a single component, which is what is classically meant by "A has a unique H-space structure." This situation is interesting in its own right. We show that such H-space structures are associative and coherently abelian, and we prove that all pointed self-maps of A are automatically H-space maps.

2.1. H-space structures.

We begin by giving the notion of H-space structure that we will consider in this paper.

Definition 2.1. Let A be a pointed type.

(1) A **non-coherent H-space structure** on A consists of a binary operation $\mu : A \to A \to A$, along with two homotopies $\mu_l : \mu(\mathsf{pt}, -) = \mathsf{id}_A$ and $\mu_r : \mu(-, \mathsf{pt}) = \mathsf{id}_A$;

(2) A **(coherent) H-space structure** on A consists of a non-coherent H-space structure μ on A along with a **coherence** $\mu_{lr} : \mu_l(\mathsf{pt}) =_{\mu(\mathsf{pt},\mathsf{pt})=\mathsf{pt}} \mu_r(\mathsf{pt})$.
(3) We write **HSpace(A)** for the type of (coherent) H-space structures on A.

When the H-space structure is clear from the context we may write $x \cdot y :\equiv \mu(x, y)$. Any H-space structure yields a non-coherent H-space structure by forgetting the coherence. Suppose A has a (non)coherent H-space structure μ.

(4) If $\mu(a, -) : A \to A$ is an equivalence for all $a : A$, then μ is **left-invertible**, and we write $x \backslash y :\equiv \mu(x, -)^{-1}(y)$. **Right-invertible** is defined dually, and we write $x/y :\equiv \mu(-, y)^{-1}(x)$.
(5) The **twist** μ^T of μ is the natural (non)coherent H-space structure with operation

$$\mu^T(a_0, a_1) :\equiv \mu(a_1, a_0).$$

When we say "H-space" we mean the coherent notion—we will only say "coherent" for emphasis. The notion of H-space structure considered in [Uni13, Def. 8.5.4] corresponds to our *non-coherent* H-space structures. While many constructions can be carried out for non-coherent H-spaces (such as the Hopf construction), the coherent case is more natural for our purposes. Moreover, any non-coherent H-space can be made coherent by simply changing one of the unit laws:

Proposition 2.2. *Any non-coherent H-space structure on a pointed type A gives rise to a coherent H-space structure with the same underlying binary operation.*

Proof. Let (A, μ, μ_l, μ_r) be a non-coherent H-space. We define a new homotopy $\mu_r' : \mu(-, \mathsf{pt}) = \mathsf{id}$ as the concatenation of paths

$$\mu(x, \mathsf{pt}) \xlongequal{\mathsf{ap}_{\mu(x)}(\mu_r(\mathsf{pt}))^{-1}} \mu(x, \mu(\mathsf{pt}, \mathsf{pt})) \xlongequal{\mathsf{ap}_{\mu(x)}(\mu_l(\mathsf{pt}))} \mu(x, \mathsf{pt}) \xlongequal{\mu_r(x)} x.$$

We claim that $\mu_l(\mathsf{pt}) = \mu_r'(\mathsf{pt})$. To see this, it suffices to show that the square

$$\begin{tikzcd}
\mu(\mathsf{pt}, \mu(\mathsf{pt}, \mathsf{pt})) \arrow[r, equal, "\mathsf{ap}_{\mu(\mathsf{pt})}(\mu_l(\mathsf{pt}))"] \arrow[d, equal, "\mathsf{ap}_{\mu(\mathsf{pt})}(\mu_r(\mathsf{pt}))"'] & \mu(\mathsf{pt}, \mathsf{pt}) \arrow[d, equal, "\mu_r(\mathsf{pt})"] \\
\mu(\mathsf{pt}, \mathsf{pt}) \arrow[r, equal, "\mu_l(\mathsf{pt})"'] & \mathsf{pt}
\end{tikzcd}$$

commutes. We will show that the top path is equal to $\mu_l(\mu(\mathsf{pt}, \mathsf{pt}))$, and this turns the square into a naturality square for the homotopy μ_l, which always commutes. To see that

$$\mathsf{ap}_{\mu(\mathsf{pt})}(\mu_l(\mathsf{pt})) = \mu_l(\mu(\mathsf{pt}, \mathsf{pt})),$$

observe that μ_l is a homotopy $\mu(\mathsf{pt}) = \mathsf{id}$, and for any homotopy $H : f = \mathsf{id}$ we have $\mathsf{ap}_f\, H_x = H_{f(x)}$ for all x. □

The proposition implies that the types of non-coherent and coherent H-space structures on a pointed type are logically equivalent. However, they are not generally equivalent as types (see Remark 3.4).

We'll be interested in abelian and associative H-spaces later on.

Definition 2.3. Let A be an H-space with multiplication μ.

(1) If there is a homotopy $h : \Pi_{a,b}\mu(a, b) = \mu(b, a)$ then μ is **abelian**.
(2) If $\mu = \mu^T$ in HSpace(A) then μ is **coherently abelian**.
(3) If there is a homotopy $\alpha : \Pi_{a,b,c:A}\mu(\mu(a, b), c) = \mu(a, \mu(b, c))$ then μ is **associative**.

The following lemma gives a convenient way of constructing abelian H-space structures, and will be used in Theorem 4.19.

Lemma 2.4. *Let A be a pointed type with a binary operation μ, a symmetry $\sigma_{a,b} : \mu(a, b) = \mu(b, a)$ for every $a, b : A$ such that $\sigma_{\mathsf{pt},\mathsf{pt}} = \mathsf{refl}$, and a left unit law $\mu_l : \mu(\mathsf{pt}, -) = \mathsf{id}_A$. Then A becomes an abelian H-space with the right unit law induced by symmetry.*

Proof. For $b : A$, the right unit law is given by the path $\sigma_{b,\mathsf{pt}} \cdot \mu_l(b)$ of type $\mu(b, \mathsf{pt}) = b$. For coherence we need to show that the following triangle commutes:

$$\begin{tikzcd} \mu(\mathsf{pt},\mathsf{pt}) \arrow[rr, "\sigma_{\mathsf{pt},\mathsf{pt}}"] \arrow[dr, "\mu_l"'] & & \mu(\mathsf{pt},\mathsf{pt}) \arrow[dl, "\mu_l"] \\ & \mathsf{pt}\,. & \end{tikzcd}$$

By our assumption that $\sigma_{\mathsf{pt},\mathsf{pt}} = \mathsf{refl}$, the triangle is filled refl_{μ_l}. □

For any right-invertible H-space A, for $b : A$ one can define the two operations $(-)/b$ and $(-)\cdot(\mathsf{pt}/b)$ of type $A \to A$. If A is associative, then these coincide:

Lemma 2.5. *Let A be an associative H-space. For any $a, b : A$, we have that $a/b = a \cdot (\mathsf{pt}/b)$.*

Proof. For all $a, b : A$ we have $(a \cdot (\mathsf{pt}/b)) \cdot b = a \cdot ((\mathsf{pt}/b) \cdot b) = a \cdot \mathsf{pt} = a$. Thus by dividing by b on the right, we deduce $a \cdot (\mathsf{pt}/b) = a/b$. □

We collect a few basic facts about H-spaces. The following lemma generalizes a result of Evan Cavallo, who formalized the fact that unpointed homotopies between pointed maps into a *homogeneous type* A can be upgraded to pointed homotopies. Being a homogeneous type is logically equivalent to being a left-invertible H-space [Cav21]. Here we do not need to assume left-invertibility, and we factor this observation through a further generalization.

Lemma 2.6. *Let A be a pointed type, and consider the following three conditions:*

(1) A is an H-space.
(2) The evaluation map $(\mathsf{id}_A = \mathsf{id}_A) \to (\mathsf{pt} = \mathsf{pt})$ has a section.
(3) For every pointed type B and pointed maps $f, g : B \to_* A$, there is a map $(f = g) \to (f =_* g)$ which upgrades unpointed homotopies to pointed homotopies.

Then (1) implies (2) and (2) implies (3).

Proof. To show that (1) implies (2), suppose that A is an H-space, and let $p : \mathsf{pt} = \mathsf{pt}$. For any $x : A$ we define the path $p_x : x = x$ to be the concatenation

$$x \overset{\mu_r^{-1}}{=\!=} x \cdot \mathsf{pt} \overset{\mathsf{ap}_{\mu(x)}(p)}{=\!=\!=} x \cdot \mathsf{pt} \overset{\mu_r}{=\!=} x\,.$$

This defines a map $s : (\mathsf{pt} = \mathsf{pt}) \to (\mathsf{id}_A = \mathsf{id}_A)$. To see that this map is a section of the evaluation map, it suffices to show that the square

$$\begin{tikzcd} \mathsf{pt}\cdot\mathsf{pt} \arrow[r, equal, "\mathsf{ap}_{\mu(\mathsf{pt})}(p)"] \arrow[d, equal, "\mu_r"'] & \mathsf{pt}\cdot\mathsf{pt} \arrow[d, equal, "\mu_r"] \\ \mathsf{pt} \arrow[r, equal, "p"'] & \mathsf{pt} \end{tikzcd}$$

commutes. To see this, note that $\mu_r = \mu_l$. If we replace μ_r by μ_l in the above square, we obtain a naturality square of homotopies, which always commutes.

We next show that (2) implies (3). Let $f, g : B \to_* A$ be pointed maps and let $H : f = g$ be an unpointed homotopy. By path induction on H, we can assume we have a single function $f : B \to A$ with two pointings, f_{pt} and $f'_{\mathsf{pt}} : f(\mathsf{pt}) = \mathsf{pt}$. Our goal is to define a homotopy $K : f = f$ such that $K_{\mathsf{pt}} = r$, where $r :\equiv f_{\mathsf{pt}} \cdot \overline{f'_{\mathsf{pt}}} : f(\mathsf{pt}) = f(\mathsf{pt})$. By path induction on f_{pt}, we can assume that the basepoint of A is $f(\mathsf{pt})$. By (2), we have $s : (f(\mathsf{pt}) = f(\mathsf{pt})) \to (\mathsf{id}_A = \mathsf{id}_A)$ such that $s(p, f(\mathsf{pt})) = p$ for all $p : f(\mathsf{pt}) = f(\mathsf{pt})$. For $b : B$, define K_b to be $s(r, f(b))$. Then $K_{\mathsf{pt}} = r$, as required. □

The following result is straightforward and has been formalized, so we do not include a proof.

Proposition 2.7. *Suppose A is a (left-invertible) H-space. For any pointed type B, the mapping type $B \to_* A$ based at the constant map is naturally a (left-invertible) H-space under pointwise multiplication. Similarly, for any type B, the mapping type $B \to A$ based at the constant map is a (left-invertible) H-space under pointwise multiplication.* □

In particular, if A is left-invertible then for any $f : B \to_* A$ there is a self-equivalence of $B \to_* A$ which sends the constant map to f—namely, the pointwise multiplication by f on the left.

Our next goal is to rule out H-space structures on even spheres using Brunerie's computation of Whitehead products. (See [Bru16, Section 3.3] for their definition.) To do so, we prove some results about Whitehead products from [Whi46] which relate to H-spaces.

Definition 2.8. Let $\alpha : B \to_* A$ and $\beta : C \to_* A$ be pointed maps. An **(α, β)-extension** is a pointed map $f : B \times C \to_* A$ equipped with a pointed homotopy filling the following diagram:

$$\begin{tikzcd} B \vee C \arrow[rr, "\alpha\vee\beta"] \arrow[dr] & & A \\ & B \times C . \arrow[ur, "f"'] & \end{tikzcd}$$

Remark 2.9. It is equivalent to consider the type of *unpointed* (α, β)-extensions consisting of unpointed maps $f : B \times C \to A$ and unpointed fillers. The additional data in a pointed extension is a path $f_{\mathsf{pt}} : f(\mathsf{pt}, \mathsf{pt}) = \mathsf{pt}$ and a 2-path that determines f_{pt} in terms of the other data. These form a contractible pair.

When α and β are maps between spheres, Whitehead instead says that f is "of type (α, β)" but we prefer to stress that we work with a structure and not a property, as the following lemma illustrates:

Lemma 2.10. *H-space structures on a pointed type A correspond to $(\mathsf{id}_A, \mathsf{id}_A)$-extensions.* □

The proof consists of straightforward reshuffling of data.

Lemma 2.11. *If A is an H-space, then there is an (α, β)-extension for every pair $\alpha : B \to_* A$ and $\beta : C \to_* A$ of pointed maps.*

Proof. Using naturality of the left and right unit laws and coherence, one can show that the map $(b, c) \mapsto \alpha(b) \cdot \beta(c) : B \times C \to A$ is an (α, β)-extension. Alternatively, observe that the (α, β)-extension problem factors through the $(\mathsf{id}_A, \mathsf{id}_A)$-extension problem via the map $\alpha \times \beta : B \times C \to A \times A$. □

The lemmas explain the relation between H-space structures and (α, β)-extensions, which are in turn related to Whitehead products via the next two results.

Proposition 2.12 ([Whi46, Corollary 3.5])**.** *Let $m, n > 0$ be natural numbers and consider two pointed maps $\alpha : \mathbb{S}^m \to_* A$ and $\beta : \mathbb{S}^n \to_* A$. The type of (α, β)-extensions is equivalent to the type of witnesses that the map $[\alpha, \beta] : \mathbb{S}^{m+n-1} \to_* A$ is constant (as a pointed map).*

Proof. Consider the diagram of pointed maps below, where the composite of the top two maps is $[\alpha, \beta]$ and the left diamond is a pushout of pointed types:

$$\begin{tikzcd} & \mathbb{S}^m \vee \mathbb{S}^n \arrow[dr] \arrow[drr, bend left, "\alpha\vee\beta"] & & \\ \mathbb{S}^{m+n-1} \arrow[ur] \arrow[dr] & & \mathbb{S}^m \times \mathbb{S}^n \arrow[r, dashed, "f"] & A . \\ & 1 \arrow[ur] \arrow[urr, bend right] & & \end{tikzcd}$$

An (α, β)-extension is the same as a pointed map f along with a pointed homotopy filling the top-right triangle. Since the bottom-right triangle is filled by a unique pointed homotopy, an (α, β)-extension thus corresponds exactly to the data of a filler in the outer diagram, i.e., a homotopy witnessing that $[\alpha, \beta]$ is constant as a pointed map. □

With the notation of the previous proposition, we have the following:

Corollary 2.13 ([Whi46, Corollary 3.6])**.** *Suppose A is an H-space. Then $[\alpha, \beta]$ is constant.*

Proof. The follows from Lemma 2.11 and Proposition 2.12. □

Using the above results, we can rule out H-space structures on even spheres in positive dimensions.

Proposition 2.14. *The n-sphere merely admits an H-space structure if and only if $[\iota_n, \iota_n] = 0$. In particular, there are no H-space structures on the n-sphere when $n > 0$ is even.*

Proof. The implication $(\rightarrow)$ is immediate by Corollary 2.13. Conversely, Proposition 2.12 implies that $[\iota_n, \iota_n] = 0$ if and only if an $(\mathsf{id}_{\mathbb{S}^n}, \mathsf{id}_{\mathbb{S}^n})$-extension merely exists, which by Lemma 2.10 happens if and only if $\mathbb{S}^n$ merely admits an H-space structure.

Finally, Brunerie showed that $[\iota_n, \iota_n] = 2$ in $\pi_{2n-1}(\mathbb{S}^n)$ for even $n > 0$ [Bru16, Proposition 5.4.4], which by the above implies that $\mathbb{S}^n$ cannot admit an H-space structure. □

We also record the following result and a corollary.

Proposition 2.15. *Let A be a left-invertible H-space. The unit $\eta : A \to_* \Omega\Sigma A$ has a pointed retraction, given by the connecting map $\delta : \Omega\Sigma A \to_* A$ associated to the Hopf fibration of A.*

Proof. Let $\delta : \Omega\Sigma A \to_* A$ be the connecting map associated to the Hopf fibration of A. Recall that for a loop $p : N = N$, we have $\delta(p) :\equiv p_*(\mathsf{pt})$ where $p_* : A \to A$ denotes transport and A is the fibre above N. By definition of the Hopf fibration, a path $\mathsf{merid}(a) : N =_{\Sigma A} S$ sends an element x of the fibre A to $a \cdot x$. Now define a homotopy $\delta \circ \eta = \mathsf{id}$ by

$$\delta(\eta(a)) \equiv \delta(\mathsf{merid}(a) \cdot \mathsf{merid}(\mathsf{pt})^{-1}) = \mathsf{merid}(\mathsf{pt})_*^{-1}(\mathsf{merid}(a)_*(\mathsf{pt})) \equiv \mathsf{pt} \backslash (a \cdot \mathsf{pt}) = a.$$

Finally, we promote this to a pointed homotopy using Lemma 2.6. □

It follows that for any n-connected H-space A, the Freudenthal map $\pi_{2n+1}(A) \to \pi_{2n+2}(\Sigma A)$ is an isomorphism, not just a surjection. In particular, we have:

Corollary 2.16. *The natural map $\pi_5(\mathbb{S}^3) \to \pi_6(\mathbb{S}^4)$ is an isomorphism.* □

The fact that the unit $\eta : A \to_* \Omega\Sigma A$ has a retraction when A is a left-invertible H-space also follows from James' reduced product construction, as shown in [Jam55]. Using [Bru16], one can see that this goes through in homotopy type theory. However, the above argument is much more elementary. We don't know if this argument had been observed before.

2.2. **Evaluation fibrations.** We now begin our study of *evaluation fibrations* and their relation to H-space structures and (α, β)-extensions from the previous section. Given a pointed map $f : B \to_* A$, we will simply write $\mathbf{ev} : (B \to A, f) \to_* A$ for the map which evaluates at $\mathsf{pt} : B$. This map is pointed since f is. If no map f is specified, then we mean that $f \equiv \mathsf{id}$.

In a moment we will define evaluation fibrations to be the restriction of $\mathbf{ev}$ to a component, but first we make a useful observation.

Definition 2.17. Let $e : X \to_* A$ and $g : B \to_* A$ be pointed maps. A **pointed lift of g through e** consists of a pointed map $s : B \to_* X$ along with a pointed homotopy $e \circ s =_* g$. If $g \equiv \mathsf{id}$, then s is more specifically a **pointed section of e**.

Proposition 2.18. *Let $f : B \to_* A$ and $g : C \to_* A$ be pointed maps. The type of (f, g)-extensions is equivalent to the type of pointed lifts of g through $\mathbf{ev} : (B \to A, f) \to_* A$.* □

We stress that the domain of $\mathbf{ev}$ is the type of *unpointed maps* $B \to A$, pointed by (the underlying map of) f. The proof of the statement is a straightforward reshuffling of data. Diagrammatically, it gives a correspondence between the dashed arrows below, with pointed homotopies filling the triangles:

$$\begin{tikzcd} B \vee C \arrow[r, "f \vee g"] \arrow[d] & A \\ B \times C \arrow[ur, dashed] & \end{tikzcd} \qquad \begin{tikzcd} & (B \to A, f) \arrow[d, "\mathbf{ev}"] \\ C \arrow[r, "g"] \arrow[ur, dashed] & A \end{tikzcd}$$

Combining Lemma 2.10 with the previous proposition, we deduce:

Corollary 2.19. *Let A be a pointed type. The type of H-space structures on A is equivalent to the type of pointed sections of* $\mathbf{ev} : (A \to A, \mathsf{id}) \to_* A$. □

Remark 2.20. Phrased another way, an H-space structure on a pointed type A is equivalent to a family

$$\mu : \Pi_{(a:A)}(A, \mathsf{pt}) \to_* (A, a).$$

If A is a higher inductive type with a point pt, one can define $\mu(\mathsf{pt}) :\equiv \mathsf{id}$ to simplify the task.

Definition 2.21. Let A be a type and $a : \|A\|_0$. The **path component of $\boldsymbol{a}$ in $\boldsymbol{A}$** is

$$A_{(a)} :\equiv \Sigma_{a':A}(|a'|_0 = a).$$

If $a : A$ then we abuse notation and write $A_{(a)}$ for $A_{(|a|_0)}$, and in this case $A_{(a)}$ is pointed at (a, refl).

Definition 2.22. For any pointed map $\alpha : B \to_* A$, the **evaluation fibration (at $\boldsymbol{\alpha}$)** is the pointed map $\mathbf{ev}_\alpha : (B \to A)_{(\alpha)} \to_* A$ induced by evaluating at the base point of B.

Observe that the component $(A \to A)_{(\mathsf{id})}$ is equivalent to $(A \simeq A)_{(\mathsf{id})}$, since being an equivalence is a property of a map. We permit ourselves to pass freely between the two.

Since pointed maps out of connected types land in the component of the base point of the codomain, we have the following consequence of Corollary 2.19.

Corollary 2.23. *Let A be a pointed, connected type. The type of H-space structures on A is equivalent to the type of pointed sections of* $\mathbf{ev}_{\mathsf{id}} : (A \simeq A)_{(\mathsf{id})} \to_* A$. □

For certain H-spaces, various evaluation fibrations become trivial:

Proposition 2.24. *Suppose A is a left-invertible H-space. We have a pointed equivalence over A*

$$\begin{tikzcd} (A \to A) \arrow[rr, "\sim"] \arrow[dr, "\mathbf{ev}"'] & & (A \to_* A) \times A \arrow[dl, "\mathsf{pr}_2"] \\ & A, & \end{tikzcd}$$

where the mapping spaces are both pointed at their identity maps. This pointed equivalence restricts to pointed equivalences $(A \simeq A) \simeq_* (A \simeq_* A) \times A$ over A, and $(A \to A)_{(\mathsf{id})} \simeq_* (A \to_* A)_{(\mathsf{id})} \times A_{(\mathsf{pt})}$ over $A_{(\mathsf{pt})}$.

Proof. Define $e : (A \to A) \to (A \to_* A) \times A$ by $e(f) :\equiv (a \mapsto f(\mathsf{pt})\backslash f(a), f(\mathsf{pt}))$ where the first component is a pointed map in the obvious way. Clearly e is a map over A, and moreover e is pointed. It is straightforward to check that the triangle above is filled by a pointed homotopy. (One could also apply Lemma 2.6, but a direct inspection suffices in this case.)

Finally, it's straightforward to check that e has an inverse given by

$$(g, a) \mapsto (x \mapsto a \cdot g(x)).$$

Hence e is an equivalence, as desired. The restrictions to equivalences and path components follow by functoriality. □

The hypotheses of the proposition are satisfied, for example, by connected H-spaces.

Example 2.25. We obtain three pointed equivalences for any abelian group A and $n \geq 1$:

$$\begin{aligned} \big(\mathrm{K}(A,n) \to \mathrm{K}(A,n)\big) &\simeq_* \mathrm{Ab}(A,A) \times \mathrm{K}(A,n), \\ \big(\mathrm{K}(A,n) \simeq \mathrm{K}(A,n)\big) &\simeq_* \mathsf{Aut}_{\mathrm{Ab}}(A) \times \mathrm{K}(A,n), \text{ and} \\ \big(\mathrm{K}(A,n) \to \mathrm{K}(A,n)\big)_{(\mathsf{id})} &\simeq_* \mathrm{K}(A,n). \end{aligned}$$

Example 2.26. Taking $A :\equiv \mathbb{S}^3$ in the previous proposition, by virtue of the H-space structure on the 3-sphere constructed in [BR18], we get three pointed equivalences:

$$(\mathbb{S}^3 \to \mathbb{S}^3) \simeq_* \Omega^3\mathbb{S}^3 \times \mathbb{S}^3, \quad (\mathbb{S}^3 \simeq \mathbb{S}^3) \simeq_* \Omega^3_{\pm 1}\mathbb{S}^3 \times \mathbb{S}^3, \quad \text{and} \quad (\mathbb{S}^3 \simeq \mathbb{S}^3)_{(\mathsf{id})} \simeq_* (\mathbb{S}^3 \simeq_* \mathbb{S}^3)_{(\mathsf{id})} \times \mathbb{S}^3,$$

where $\Omega^3_{\pm 1}\mathbb{S}^3 :\equiv (\Omega^3\mathbb{S}^3)_{(1)} \sqcup (\Omega^3\mathbb{S}^3)_{(-1)}$ and 1 and -1 refer to the corresponding elements of $\pi_3(\mathbb{S}^3) = \mathbb{Z}$.

By combining our results thus far, we obtain the following equivalence which generalizes a classical formula of [Cop59, Theorem 5.5A], independently shown by [AC63], for counting *homotopy classes* of H-space structures on certain spaces.

Theorem 2.27. *Let A be a left-invertible H-space. The type* $\mathrm{HSpace}(A)$ *of H-space structures on A is equivalent to $A \wedge A \to_* A$.*

Proof. By Corollary 2.19, the type of H-space structures on A is equivalent to the type of pointed sections of $\mathbf{ev} : (A \to A) \to A$. By Proposition 2.24, this type is equivalent to the type of pointed sections of $\mathrm{pr}_2 : (A \to_* A) \times A \to A$, which are simply pointed maps $A \to_* (A \to_* A, \mathsf{id})$, where the codomain is pointed at the identity. The latter type is equivalent to $A \to_* (A \to_* A)$, where the codomain is pointed at the constant map, by Proposition 2.7. Finally, this type is equivalent to $A \wedge A \to_* A$ by the smash–hom adjunction for pointed types [vDoo18, Theorem 4.3.28]. □

Example 2.28. It follows from the proposition that $\mathrm{HSpace}(\mathbb{S}^1) \simeq \mathbf{1}$ and $\mathrm{HSpace}(\mathbb{S}^3) \simeq \Omega^6\mathbb{S}^3$.

We record the following result which relates Whitehead products and evaluation fibrations.

Proposition 2.29 ([Han74, Lemma 2.2])**.** *Let $n, m \geq 2$ and let $\alpha : \pi_m(\mathbb{S}^n)$. The evaluation fibration $\mathbf{ev}_\alpha : (\mathbb{S}^m \to \mathbb{S}^n)_{(\alpha)} \to \mathbb{S}^n$ merely has a section if and only if the Whitehead product $[\alpha, \iota_n] : \pi_{n+m-1}(\mathbb{S}^n)$ vanishes.*

Proof. As we are proving a proposition, we may pick a representative $\alpha : \mathbb{S}^m \to_* \mathbb{S}^n$ throughout. Using Proposition 2.18 and that $\mathbb{S}^n$ is connected, we see that $[\alpha, \iota_n]$ vanishes if and only if there merely exists a pointed section of $\mathbf{ev}_\alpha$. The fibre of the forgetful map from pointed sections of $\mathbf{ev}_\alpha$ to unpointed sections of $\mathbf{ev}_\alpha$ over some section (s, h) is equivalent to

$$\sum_{k : s(\mathsf{pt}, -) = \alpha} h(\mathsf{pt}) =_{s(\mathsf{pt},\mathsf{pt}) = \mathsf{pt}} k(\mathsf{pt}) \cdot \alpha_{\mathsf{pt}},$$

where $\alpha_{\mathsf{pt}} : \alpha(\mathsf{pt}) = \mathsf{pt}$ is the pointing of α. This fibre is (-1)-connected since s lands in the component of α and the inner part of the Σ-type is a double path space of $\mathbb{S}^n$ with $n \geq 2$. In other words, this forgetful map is an epimorphism. A pointed section of $\mathbf{ev}_\alpha$ therefore merely exists if and only if an unpointed section merely exists, completing the proof. □

2.3. **Unique H-space structures.** We collect results about H-space structures which are unique, in the sense that the type of H-space structures is contractible. In particular, we give elementary proofs that such H-space structures are automatically coherently abelian and associative. Moreover, pointed self-maps of such are automatically H-space self-maps.

Lemma 2.30. *Let A be a pointed type and suppose* $\mathrm{HSpace}(A)$ *is contractible. Then the unique H-space structure μ on A is coherently abelian.*

Proof. Since $\mathrm{HSpace}(A)$ is contractible, there is an identification $\mu = \mu^T$ of H-space structures. (Here, μ^T is the twist, defined in Definition 2.1.) □

For the next result, we use the definition of the smash product from [vDoo18, Definition 4.3.6] (see also [CS20, Definition 2.29]) which avoids higher paths. For pointed types (X, x_0) and (Y, y_0), the **smash product** $X \wedge Y$ is the higher inductive type with point constructors $\mathsf{sm} : X \times Y \to X \wedge Y$ and $\mathsf{auxl}, \mathsf{auxr} : X \wedge Y$, and path constructors $\mathsf{gluel} : \prod_{y:Y} \mathsf{sm}(x_0, y) = \mathsf{auxl}$ and $\mathsf{gluer} : \prod_{x:X} \mathsf{sm}(x, y_0) = \mathsf{auxr}$. It is pointed by auxl. The smash product was shown to be associative in [vDoo18, Definition 4.3.33].

Proposition 2.31. *Suppose A is a pointed type with a unique H-space structure, and suppose moreover that this H-space structure is left-invertible. Then any pointed map $f : A \to_* A$ is an H-space map, i.e., we have $f(a \cdot b) = f(a) \cdot f(b)$ for all $a, b : A$.*

Proof. Let $f : A \to_* A$ be a pointed map. We will define an associated map $\nu : A \wedge A \to_* A$, which records how f deviates from being an H-space map. We define $\nu(\mathsf{sm}(a,b)) :\equiv \big(f(a \cdot b)/f(b)\big)/f(a)$, $\nu(\mathsf{auxl}) :\equiv \mathsf{pt}$, and $\nu(\mathsf{auxr}) :\equiv \mathsf{pt}$. For $b : A$, we have a path $\nu(\mathsf{sm}(\mathsf{pt}, b)) \equiv \big(f(\mathsf{pt} \cdot b)/f(b)\big)/f(\mathsf{pt}) = \big(f(b)/f(b)\big)/\mathsf{pt} = \mathsf{pt}/\mathsf{pt} = \mathsf{pt}$, and similarly for the other path constructor. Since A admits a unique H-space structure, the type $A \wedge A \to_* A$ is contractible by Theorem 2.27. Consequently, ν is constant, whence for all $a, b : A$ we have $\big(f(a \cdot b)/f(b)\big)/f(a) = \mathsf{pt}$, and therefore

$$f(a \cdot b) = f(a) \cdot f(b).$$

□

Remark 2.32. Note that when A and B are two pointed types, each with unique H-space structures, it is not necessarily the case that every pointed map $f : A \to_* B$ is an H-space map. For example, the squaring operation gives a natural transformation $H^2(X; Z) \to H^4(X; Z)$ which is represented by a map $\mathrm{K}(\mathbb{Z}, 2) \to_* \mathrm{K}(\mathbb{Z}, 4)$. But since squaring isn't a homomorphism, this map isn't an H-space map.

Proposition 2.33. *Suppose A is a pointed type with a unique H-space structure which is left-invertible. Then the H-space structure is necessarily associative.*

Proof. Let $a : A$. Define a map $\nu : A \wedge A \to_* A$ as follows. We let $\nu(\mathsf{sm}(b,c)) :\equiv ((a \cdot b) \cdot c)/(a \cdot (b \cdot c))$, $\nu(\mathsf{auxl}) :\equiv \mathsf{pt}$, and $\nu(\mathsf{auxr}) :\equiv \mathsf{pt}$. For $c : A$, we have a path $\nu(\mathsf{sm}(\mathsf{pt}, c)) \equiv ((a \cdot \mathsf{pt}) \cdot c)/(a \cdot (\mathsf{pt} \cdot c)) = (a \cdot c)/(a \cdot c) = \mathsf{pt}$, and similarly for the other path constructor. Since A admits a unique H-space structure, the type $A \wedge A \to_* A$ is contractible by Theorem 2.27. Consequently, for each a, ν is constant. It follows that for all $a, b, c : A$ we have $((a \cdot b) \cdot c)/(a \cdot (b \cdot c)) = \mathsf{pt}$, and therefore

$$(a \cdot b) \cdot c = a \cdot (b \cdot c).$$

□

Note that if $A \wedge A \to_* A$ is contractible, then it follows from the smash-hom adjunction that $A^{\wedge n} \to_* A$ is contractible for each $n \geq 2$, where $A^{\wedge n}$ denotes the smash power.

3. Central types

In this and the next section we focus on pointed types which we call *central*. Centrality is an elementary property with remarkable consequences. For example, in the next section we will see that every central type is an infinite loop space (Corollary 4.20). To show this, we require a certain amount of theory about central types. We first show that every central type has a unique H-space structure. When A is already known to be an H-space, we give several conditions which are equivalent to A being central. From this, it follows that every Eilenberg–Mac Lane space $\mathrm{K}(G, n)$, with G abelian and $n \geq 1$, is central. We also prove several other results which we will need in the next section.

Definition 3.1. Let A be a pointed type. The **center of A** is the type $ZA :\equiv (A \to A)_{(\mathsf{id})}$, which comes with a natural map $\mathbf{ev}_{\mathsf{id}} : ZA \to_* A$ (see Definition 2.22). If the map $\mathbf{ev}_{\mathsf{id}}$ is an equivalence, then A is **central**.

Remark 3.2. The terminology "central" comes from higher group theory. Suppose $A :\equiv BG$ is the delooping of an ∞-group G. The center of G is the ∞-group $ZG :\equiv \Pi_{x:G}(x = x)$ with delooping $BZG :\equiv (BG \simeq BG)_{(\mathsf{id})}$, which is our ZA.

Central types and H-spaces are connected through evaluation fibrations:

Proposition 3.3. *Suppose that A is central. Then A admits a unique H-space structure. In addition, A is connected, so this H-space structure is both left- and right-invertible.*

Proof. Since $\mathbf{ev}_{\mathsf{id}}$ is an equivalence, it has a unique section. By Corollary 2.23, we deduce that A has a unique H-space structure μ. It follows from Lemma 2.30 that it is coherently abelian. Finally, the equivalence $\mathbf{ev}_{\mathsf{id}} : (A \to A)_{(\mathsf{id})} \simeq A$ implies that A is connected. Then, since $\mu(\mathsf{pt}, -)$ and $\mu(-, \mathsf{pt})$ are both equal to the identity, it follows that μ is left- and right-invertible. □

It follows from Proposition 2.33 and Lemma 2.30 that the unique H-space structure on a central type is associative and coherently abelian.

Remark 3.4. In contrast, the type of non-coherent H-space structures on a central type A is rarely contractible. We'll show here that it is equivalent to the loop space ΩA. First consider the type of binary operations $\mu : A \to (A \to A)$ which *merely* satisfy the left unit law. This is equivalent to the type of maps $A \to (A \to A)_{(\mathsf{id})}$, since A is connected. Such a map μ satisfies the right unit law if and only if the composite $\mathbf{ev}_{\mathsf{id}} \circ \mu : A \to A$ is the identity map. In other words, μ must be a section of the equivalence $\mathbf{ev}_{\mathsf{id}}$, so there is a contractible type of such μ.

The left unit law says that μ sends pt to id. After post-composing with $\mathbf{ev}_{\mathsf{id}}$, it therefore says that it sends pt to $\mathsf{id}(\mathsf{pt})$, which equals pt. So the type of left unit laws is $\mathsf{pt} = \mathsf{pt}$, i.e., the loop space ΩA. Note that we imposed the left unit law both merely and purely, but that doesn't change the type. So it follows that the type of all non-coherent H-space structures on a central type A is ΩA.

We give conditions for an H-space to be central, in which case the H-space structure is the unique one coming from centrality. For the next two results, write

$$F :\equiv \Sigma_{f:A \to_* A} \| f = \mathsf{id} \|$$

for the fibre of $\mathbf{ev}_{\mathsf{id}} : (A \to A)_{(\mathsf{id})} \to_* A$ over $\mathsf{pt} : A$. Note that the equality $f = \mathsf{id}$ is in the type of *unpointed* maps $A \to A$.

Lemma 3.5. *Suppose that A is a connected H-space. Then $F \simeq (A \to_* A)_{(\mathsf{id})}$.*

Proof. By our assumptions, Proposition 2.24 gives a trivialization of $\mathbf{ev}_{\mathsf{id}}$ over A:

$$t : (A \to A)_{(\mathsf{id})} \simeq_* (A \to_* A)_{(\mathsf{id})} \times A.$$

Passing to the fibres of $\mathbf{ev}_{\mathsf{id}}$ and pr_2 over $\mathsf{pt} : A$ gives the desired equivalence. □

The lemma can also be shown using Lemma 2.6.

Proposition 3.6. *Let A be a pointed type. Then the following are logically equivalent:*

(1) A is central;
(2) A is a connected H-space and $A \to_* A$ is a set;
(3) A is a connected H-space and $A \simeq_* A$ is a set;
(4) A is a connected H-space and $A \to_* \Omega A$ is contractible;
(5) A is a connected H-space and $\Sigma A \to_* A$ is contractible.

Proof. (1) $\implies$ (2): Assume that A is central. Then Proposition 3.3 implies that A is a connected H-space. Since A is a left-invertible H-space, so is $A \to_* A$, by Proposition 2.7. Therefore all components of $A \to_* A$ are equivalent to $(A \to_* A)_{(\mathsf{id})}$, and thus to F by Lemma 3.5. Now, F is contractible since $\mathbf{ev}_{\mathsf{id}}$ is an equivalence, and consequently $A \to_* A$ is a set since all of its components are contractible.

(2) $\implies$ (3): This follows from the fact that $A \simeq_* A$ embeds into $A \to_* A$.

(3) $\implies$ (1): If $(A \simeq_* A)$ is a set, then its component $(A \to_* A)_{(\mathsf{id})}$ is contractible. Therefore F is contractible, by Lemma 3.5. It follows that $\mathbf{ev}_{\mathsf{id}}$ is an equivalence, since A is connected. Hence A is central.

(3) $\iff$ (4): Since A is a left-invertible H-space, so is $A \to_* A$. The latter is therefore a set if and only if the component of the constant map is contractible, which is true if and only if the loop space $\Omega(A \to_* A)$ is contractible. Finally, the equivalence $\Omega(A \to_* A) \simeq (A \to_* \Omega A)$ shows that this is true if and only if $A \to_* \Omega A$ is contractible.

(4) $\iff$ (5): This follows from the equivalence $(A \to_* \Omega A) \simeq (\Sigma A \to_* A)$. □

Example 3.7. Consider the Eilenberg–Mac Lane space $\mathrm{K}(G, n)$ for $n \geq 1$ and G an abelian group. It is a pointed, connected type. Since $\mathrm{K}(G, n) \simeq \Omega\, \mathrm{K}(G, n+1)$, it is an H-space. By [BvDR18, Theorem 5.1], $\mathrm{K}(G, n) \simeq_* \mathrm{K}(G, n)$ is equivalent to the set of automorphisms of G. It therefore follows from Proposition 3.6 that $\mathrm{K}(G, n)$ is central. We will see in Proposition 5.9 a more self-contained proof of this result.

Example 3.8. Brunerie showed that $\pi_4(\mathbb{S}^3) \simeq \mathbb{Z}/2$ [Bru16]. Therefore, $\mathbb{S}^4 \to_* \mathbb{S}^3$ is not contractible, and so $\mathbb{S}^3$ is not central, by Proposition 3.6(5). Since this is in the stable range, it follows that $\mathbb{S}^n$ is not central for $n \geq 3$.

Remark 3.9. For a pointed type A, we have seen that A being central is logically equivalent to A being a connected H-space such that $A \simeq_* A$ is a set. It is natural to ask whether the reverse implication holds without the assumption that A is an H-space. However, this is not the case. Consider, for example, the pointed, connected type $\mathrm{K}(G,1)$ for a non-abelian group G. Then $\mathrm{K}(G,1) \simeq_* \mathrm{K}(G,1)$ is equivalent to the set of group automorphisms of G. If $\mathrm{K}(G,1)$ were central, then G would be twice deloopable, which would contradict G being non-abelian.

By the previous proposition, the type $A \to_* A$ is a set whenever A is central. Presently we observe that it is in fact a ring.

Corollary 3.10. *For any central type A, the set $A \to_* A$ is a ring under pointwise multiplication and function composition.*

Proof. It follows from A being a commutative and associative H-space that the set $A \to_* A$ is an abelian group. The only nontrivial thing we need to show is that function composition is linear. Let $f, g, \phi : A \to_* A$, and consider $a : A$. By Proposition 2.31, ϕ is an H-space map. Consequently,

$$\big(\phi \circ (f \cdot g)\big)(a) \equiv \phi(f(a) \cdot g(a)) = \phi(f(a)) \cdot \phi(g(a)) \equiv \big((\phi \circ f) \cdot (\phi \circ g)\big)(a). \qquad \square$$

The following remark gives some insight into the nature of the ring $A \to_* A$.

Remark 3.11. If BG is an ∞-group and X is a pointed type, recall that a bundle over X is *G-principal* if it is classified by a map $X \to_* BG$ (see e.g. [Sco20, Def. 2.23] for a formal definition which easily generalizes to arbitrary ∞-groups). In particular, it is not hard to see that the Hopf fibration of G (as the loop space of BG) is a G-principal bundle, i.e., classified by a map $\Sigma G \to_* BG$.

In Proposition 4.4 we will see that any central type A has a delooping $\mathsf{BAut}_1(A)$. This means we have equivalences

$$(A \to_* A) \;\simeq\; \big(A \to_* (A \simeq A)_{(\mathsf{id})}\big) \;\simeq\; (\Sigma A \to_* \mathsf{BAut}_1(A)).$$

Thus we see that $A \to_* A$ is *the ring of principal A-bundles over* ΣA. The equivalence above maps the identity $\mathsf{id} : A \to_* A$ to the Hopf fibration of A (as a principal A-bundle), meaning the Hopf fibration is the multiplicative unit from this perspective.

In the remainder of this section we collect various results which are needed later on. The first result is that "all" of the evaluation fibrations of a central type A are equivalences:

Proposition 3.12. *Let A be a central type and let $f : A \to_* A$ be a pointed map. The evaluation fibration $\mathbf{ev}_f : (A \to A)_{(f)} \to_* A$ is an equivalence, with inverse given by $a \mapsto a \cdot f(-)$.*

Proof. The type $A \to A$ is a left-invertible H-space via pointwise multiplication, by Proposition 2.7. So there is an equivalence $(A \to A)_{(\mathsf{id})} \to (A \to A)_{(f)}$ sending g to $f \cdot g$. Since f is pointed, we have

$$\mathbf{ev}_f(f \cdot g) \equiv (f \cdot g)(\mathsf{pt}) \equiv f(\mathsf{pt}) \cdot g(\mathsf{pt}) = \mathsf{pt} \cdot g(\mathsf{pt}) = g(\mathsf{pt}) = \mathbf{ev}_{\mathsf{id}}(g).$$

In other words, $\mathbf{ev}_f \circ (f \cdot -) = \mathbf{ev}_{\mathsf{id}}$, which shows that $\mathbf{ev}_f$ is an equivalence. Since f is pointed, the stated map is a section of $\mathbf{ev}_f$, hence is an inverse. $\square$

Corollary 3.13. *Let A be a central type, let $f : A \to_* A$, and let $g : (A \to A)_{(f)}$. Then for all $a : A$, we have $g(a) = g(\mathsf{pt}) \cdot f(a)$.* $\square$

Any central type has an inversion map, which plays a key role in the next section.

Definition 3.14. Suppose that A is central. The **inversion map** $\mathsf{id}^* : A \to A$ sends a to $a^* :\equiv \mathsf{pt}/a$.

The defining property of a^* is that $a^* \cdot a = \mathsf{pt}$. Since A is abelian, we also have $a \cdot a^* = \mathsf{pt}$, so it would have been equivalent to define the inversion to be $a \mapsto a \backslash \mathsf{pt}$. From associativity of a central H-space it follows that $\mathsf{pt}^* = \mathsf{pt}$ and $a^{**} = a$ for all a, so the inversion map id^* is a pointed self-equivalence of A and an involution.

A curious property is that on the component of id^*, inversion of equivalences is homotopic to the identity. This comes up in the next section.

Proposition 3.15. *The map* $\phi \mapsto \phi^{-1} : (A \simeq A)_{(\mathsf{id}^*)} \to (A \simeq A)_{(\mathsf{id}^*)}$ *is homotopic to the identity.*

Proof. Let $\phi : (A \simeq A)_{(\mathsf{id}^*)}$. We need to show that $\phi = \phi^{-1}$, or equivalently that $\phi(\phi(\mathsf{pt})) = \mathsf{pt}$, since $\mathbf{ev}_{\mathsf{id}}$ is an equivalence. (Note that $\phi \circ \phi : (A \simeq A)_{(\mathsf{id})}$.) Using Corollary 3.13, we have that

$$\phi(\phi(\mathsf{pt})) = \phi(\mathsf{pt}) \cdot \phi(\mathsf{pt})^* = \mathsf{pt}. \qquad \square$$

4. Bands and torsors

We begin in Section 4.1 by defining and studying types *banded* by a central type A, also called A-*bands*. We show that the type $\mathsf{BAut}_1(A)$ of banded types is a delooping of A, that A has a unique delooping, and that every pointed self-map $A \to_* A$ has a unique delooping.

In Section 4.2, we show that $\mathsf{BAut}_1(A)$ is itself an H-space under a tensoring operation, from which it follows that it is again a central type. Thus we may iteratively consider banded types to obtain an infinite loop space structure on A, which is unique. As a special case, taking A to be $\mathrm{K}(G, n)$ for some abelian group G produces a novel description of the infinite loop space structure on $\mathrm{K}(G, n)$, as described in Section 5.2.

In Section 4.3, we define the type of A-torsors, which we show is equivalent to the type of A-bands when A is central, thus providing an alternate description of the delooping of A. The type of A-torsors has been independently studied by David Wärn, who has shown that it is a delooping of A under the weaker assumption that A has a unique H-space structure.

4.1. Types banded by a central type.

We now study types *banded* by a central type A. On this type we will construct an H-space structure, which will be seen to be central.

Definition 4.1. For a type A, let $\mathsf{BAut}_1(A) :\equiv \Sigma_{X:\mathcal{U}} \|A = X\|_0$. The elements of $\mathsf{BAut}_1(A)$ are types which are **banded** by A or ***A*-bands**, for short. We denote A-bands by X_p, where $p : \|A = X\|_0$ is the **band**. The type $\mathsf{BAut}_1(A)$ is pointed by $A_{|\mathsf{refl}|_0}$.

Given a band $p : \|A = X\|_0$, we will write $\tilde{p} : \|X \simeq A\|_0$ for the associated equivalence.

Remark 4.2. It's not hard to see that $\mathsf{BAut}_1(A)$ is a connected, locally small type—hence essentially small, by the join construction [Rij17].

The characterization of paths in Σ-types tells us what paths between banded types are.

Lemma 4.3. *Consider two A-bands X_p and Y_q. A path $X_p = Y_q$ of A-bands corresponds to a path $e : X = Y$ between the underlying types making the following triangle of truncated paths commute:*

$$\begin{tikzcd} & A \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X \arrow[rr, "|e|_0"] & & Y . \end{tikzcd}$$

In other words, there is an equivalence $(X_p = Y_q) \simeq (X = Y)_{(\tilde{p} \cdot q)}$. $\square$

For the remainder of this section, let A be a central type. We begin by showing that the type of A-bands is a delooping of A.

Proposition 4.4. *We have that* $\Omega\, \mathsf{BAut}_1(A) \simeq A$.

Proof. We have $\Omega\, \mathsf{BAut}_1(A) \simeq (A = A)_{(\mathsf{refl})} \simeq (A \simeq A)_{(\mathsf{id})} \simeq A$, where the first equivalence uses Lemma 4.3 and the last equivalence is by centrality. $\square$

Corollary 4.5. *The unique H-space structure on A is deloopable.* □

Note that this gives an independent proof that it is associative (cf. Proposition 2.33).

Theorem 4.6. *The type A has a unique delooping.*

Proof. We must show that the type $\Sigma_{B:\mathcal{U}_*}(\Omega B \simeq_* A)$ is contractible. We will use $\mathsf{BAut}_1(A)$, with the equivalence ψ from Proposition 4.4, as the center of contraction. Let $B : \mathcal{U}_*$ be a pointed type with a pointed equivalence $\phi : \Omega B \simeq_* A$. Given $x : B$, consider $\mathsf{pt} =_B x$. Since A is connected, B is simply connected. Therefore, to give a banding on $\mathsf{pt} =_B x$, it suffices to do so when x is pt, in which case we use ϕ. So we have defined a map $f : B \to \mathsf{BAut}_1(A)$, and it is easy to see that it is pointed.

We claim that the following triangle commutes:

$$\begin{tikzcd}
\Omega B \arrow[rr, "\Omega f"] \arrow[dr, "\sim", "\phi"'] & & \Omega\,\mathsf{BAut}_1(A) \arrow[dl, "\sim"', "\psi"] \\
& A\,. &
\end{tikzcd}$$

Let $q : \mathsf{pt} =_B \mathsf{pt}$. Then $(\Omega f)(q)$ is the path associated to the equivalence

$$A \simeq (\mathsf{pt} =_B \mathsf{pt}) \simeq (\mathsf{pt} =_B \mathsf{pt}) \simeq A.$$

The first equivalence is ϕ^{-1} and the last is ϕ, as these give the pointedness of f. The middle equivalence is the map sending p to $p \cdot q$. The map ψ comes from the evaluation fibration, so to compute $\psi((\Omega f)(q))$ we compute what happens to the base point of A. It gets sent to refl, then q, and then $\phi(q)$. This shows that the triangle commutes.

It follows that Ωf is an equivalence. Since B and $\mathsf{BAut}_1(A)$ are connected, f is an equivalence as well. So f and the commutativity of the triangle provide a path from (B, ϕ) to $(\mathsf{BAut}_1(A), \psi)$ in the type of deloopings. □

We conclude this section by showing how to deloop maps $A \to_* A$.

Definition 4.7. Given $f : A \to_* A$, define $Bf : \mathsf{BAut}_1(A) \to_* \mathsf{BAut}_1(A)$ by

$$Bf(X_p) :\equiv (X \to A)_{(f^* \circ \tilde{p}^{-1})},$$

where $f^* :\equiv f \circ \mathsf{id}^*$, and we have used that $f^* \circ \tilde{p}^{-1}$ is well-defined as an element of the set-truncation. To give a banding of $(X \to A)_{(f^* \circ \tilde{p}^{-1})}$ we may induct on p and use Proposition 3.12. The same argument shows that Bf is a pointed map.

Note that $f(a^*) = f(a)^*$ for any $a : A$, since f is an H-space map by Proposition 2.31, so there's no choice involved in this definition.

Let $g : \mathsf{BAut}_1(A) \to_* \mathsf{BAut}_1(A)$. Given a loop $q : \mathsf{pt} = \mathsf{pt}$, the loop $(\Omega g)(q)$ is the composite

$$\mathsf{pt} = g(\mathsf{pt}) = g(\mathsf{pt}) = \mathsf{pt},$$

which uses pointedness of g and $\mathsf{ap}_g(q)$. We will identify $(\mathsf{pt} = \mathsf{pt})$ with A and then write

$$\Omega' g : A \simeq_* (\mathsf{pt} = \mathsf{pt}) \xrightarrow{\Omega g}_* (\mathsf{pt} = \mathsf{pt}) \simeq_* A.$$

Proposition 4.8. *We have that $\Omega' Bf = f$ for any $f : A \to_* A$.*

Proof. The following diagram describes how Bf acts on a loop $p : \mathsf{pt} =_{\mathsf{BAut}_1(A)} \mathsf{pt}$:

$$\begin{tikzcd}
A_{\mathsf{refl}} \arrow[d, equal, "p"] & (A \to A)_{(f^*)} \arrow[d, equal, "g \mapsto g \circ \tilde{p}^{-1}"] & A \arrow[l, "\sim"'] \\
A_{\mathsf{refl}} & (A \to A)_{(f^*)} \arrow[r, "\sim"] & A
\end{tikzcd}$$

Since $\tilde{p}$ is in the component of the identity, we have $\tilde{p}(a) = x \cdot a$ for all $a : A$, where $x = \tilde{p}(\mathsf{pt})$. So $\tilde{p}^{-1}(a) = x \backslash a$. Then the composite $A \simeq A$ on the right is seen to be

$$a \mapsto \mathbf{ev}_{f^*}\left(\big(a \cdot f^*(-)\big) \circ \tilde{p}^{-1}\right) = \mathbf{ev}_{f^*}\left(a \cdot f^*\big(x\backslash(-)\big)\right) = a \cdot f(x^{**}) = a \cdot f(x).$$

The domain $A_{\mathsf{refl}} = A_{\mathsf{refl}}$ is identified with A by sending a path p to $\tilde{p}(\mathsf{pt})$, which in this case is the x above. The codomain $(A \simeq A)_{(\mathsf{id})}$ is identified with A using $\mathbf{ev}_{\mathsf{id}}$, which sends the displayed function to $\mathsf{pt} \cdot f(x)$, which equals $f(x)$. So we have that $\Omega Bf = f$. By Lemma 2.6, they are equal as pointed maps. □

Proposition 4.9. *We have that $B\Omega' g = g$ for any $g : \mathsf{BAut}_1(A) \to_* \mathsf{BAut}_1(A)$.*

Proof. Given an A-band X_p, we need to show that $g(X_p) = (X \to A)_{((\Omega' g)^* \circ \tilde{p}^{-1})}$. First we construct a map of the underlying types from left to right. For $y : g(X_p)$, define the map

$$G_y : X \xrightarrow{\sim} (\mathsf{pt} = X_p) \simeq (X_p = \mathsf{pt}) \xrightarrow{\mathsf{ap}_g} (g(X_p) = g(\mathsf{pt})) \simeq (\mathsf{pt} = \mathsf{pt}) \to A,$$

where the second map is path inversion, and the fourth map uses the trivialization of $g(X_p)$ associated to y and pointedness of g. The identification $\mathsf{pt} = g(\mathsf{pt})$ corresponds to a unique point $y_0 : g(\mathsf{pt})$. To check that G_y lies in the right component, we may induct on p and assume $y \equiv y_0$ since $g(\mathsf{pt})$ is connected. We then get the map

$$G_{y_0} : A \xrightarrow{\mathsf{id}^*} A \simeq (\mathsf{pt} = \mathsf{pt}) \xrightarrow{\Omega g} (\mathsf{pt} = \mathsf{pt}) \to A,$$

since path inversion on $(\mathsf{pt} = \mathsf{pt})$ corresponds to inversion on A, and y_0 corresponds to the pointing of g. This map is precisely the definition of $(\Omega' g)^*$, so G lands in the desired component.

To check that G defines an equivalence of bands we may again induct on p. Write $\widetilde{y_0} : \mathsf{pt} \simeq g(\mathsf{pt})$ for the equivalence associated to the point $y_0 : g(\mathsf{pt})$, which is a lift of the (equivalence associated to the) banding of $g(\mathsf{pt})$. It then suffices to check that the diagram

$$\begin{tikzcd} g(\mathsf{pt}) \arrow[rr, "G"] \arrow[dr, "\widetilde{y_0}^{-1}"'] & & (A \to A)_{((\Omega' g)^*)} \arrow[dl, "\mathbf{ev}_{(\Omega' g)^*}"] \\ & \mathsf{pt} & \end{tikzcd}$$

commutes. Let $y : g(\mathsf{pt})$, which we identify with a trivialization $y' : \mathsf{pt} = g(\mathsf{pt})$. Chasing through the definition of G and using that $\mathsf{ap}_g(\mathsf{refl}) = \mathsf{refl}$, we see that

$$G_y(\mathsf{pt}) = \mathbf{ev}(y' \bullet \overline{y_0}) = \widetilde{y_0}^{-1}(y'(\mathsf{pt})) \equiv \widetilde{y_0}^{-1}(y),$$

where $\mathbf{ev} : (\mathsf{pt} = \mathsf{pt}) \to A$ is the last map in the definition of G_y, which transports pt along a path. Thus we see that the triangle above commutes, whence G is an equivalence of bands, as required. □

Theorem 4.10. *We have inverse equivalences*

$$\Omega' : (\mathsf{BAut}_1(A) \to_* \mathsf{BAut}_1(A)) \simeq (A \to_* A) : B.$$

In particular, the type $\mathsf{BAut}_1(A) \to_* \mathsf{BAut}_1(A)$ is a set.

Proof. Combine Propositions 4.8 and 4.9. □

4.2. **Tensoring bands.** In this section, we will construct an H-space structure on $\mathsf{BAut}_1(A)$, where we continue to assume that A is a central type. This H-space structure is interesting in its own right, and also implies that $\mathsf{BAut}_1(A)$ is itself central. It that follows that A is an infinite loop space.

This elementary lemma will come up frequently.

Lemma 4.11. *Let $P : \mathsf{BAut}_1(A) \to \mathcal{U}$ be a set-valued type family. Then $\prod_{X_p} P(X_p)$ is equivalent to $P(\mathsf{pt})$.*

Proof. Since each $P(X_p)$ is a set, $\prod_{X_p} P(X_p)$ is equivalent to $\prod_{X:\mathcal{U}} \prod_{p:A=X} P(X_{|p|_0})$. By path induction, this is equivalent to $P(A_{|\mathsf{refl}|_0})$, i.e., $P(\mathsf{pt})$. □

A consequence of the following result is that any pointed A-band is trivial.

Proposition 4.12. *Let X_p be an A-band. Then there is an equivalence $(\mathsf{pt} =_{\mathsf{BAut}_1(A)} X_p) \to X$.*

Proof. By Lemma 4.3, there is an equivalence $(\mathsf{pt} =_{\mathsf{BAut}_1(A)} X_p) \simeq (A \simeq X)_{(\tilde{p})}$. We will show that $\mathbf{ev}_p : (A \simeq X)_{(\tilde{p})} \to X$ is an equivalence. By Lemma 4.11, it's enough to prove this when $X_p \equiv \mathsf{pt}$, and this holds because A is central. □

We now show that path types between A-bands are themselves banded. This underlies the main results of this section.

Proposition 4.13. *Let X_p and Y_q be A-bands. The type $X_p =_{\mathsf{BAut}_1(A)} Y_q$ is banded by A.*

Proof. We need to construct a band $\|A = (X_p = Y_q)\|_0$. Since the goal is a set, we may induct on p and q, thus reducing the goal to $\|A = (\mathsf{pt} =_{\mathsf{BAut}_1(A)} \mathsf{pt})\|_0$. Using that $(\mathsf{pt} =_{\mathsf{BAut}_1(A)} \mathsf{pt}) \simeq (A \simeq A)_{(\mathsf{id})}$ and that A is central, we may give the set truncation of the inverse of the evaluation fibration at id_A. □

The following is an immediate corollary of Proposition 4.12.

Corollary 4.14. *For any A-band X_p, the A-band $(X_p = X_p)$ is trivial.* □

We next define a tensor product of banded types, using the notion of duals of bands. Write $\mathsf{refl}^* : A = A$ for the self-identification of A associated to the inversion map id^* (Definition 3.14) via univalence.

Definition 4.15. Let X_p be an A-band. The band $p^* :\equiv |\mathsf{refl}^*| \cdot p$ is the **dual of p**, and the A-band $X_p^* :\equiv X_{p^*}$ is the **dual of X_p**.

Since id^* is an involution, it follows that taking duals defines an involution on $\mathsf{BAut}_1(A)$, meaning that $X_p^{**} = X_p$.

Lemma 4.16. *We have $\mathsf{pt} = \mathsf{pt}^*$ in $\mathsf{BAut}_1(A)$.*

Proof. The underlying type of pt^* is A, which has a base point, so this follows from Proposition 4.12. □

We now show how to tensor types banded by A.

Definition 4.17. For $X_p, Y_q : \mathsf{BAut}_1(A)$, define $X_p \otimes Y_q :\equiv (X_p^* = Y_q)$, with the banding from Proposition 4.13.

It follows from Lemma 4.3 that the type $X_p \otimes Y_q$ is equivalent to $(X = Y)_{(\overline{p^*} \cdot q)}$. Since taking duals is an involution, we also have equivalences $X_p \otimes Y_q \equiv (X_p^* = Y_q) \simeq (X_p = Y_q^*) \simeq (X = Y)_{(\overline{p} \cdot q^*)}$. Moreover, from Corollary 4.14, we see that $X_p^* \otimes X_p = \mathsf{pt}$.

Tensoring defines a binary operation on $\mathsf{BAut}_1(A)$, and we now show that this operation is symmetric.

Proposition 4.18. *For any $X_p, Y_q : \mathsf{BAut}_1(A)$, there is a path $\sigma_{(X_p, Y_q)} : X_p \otimes Y_q =_{\mathsf{BAut}_1(A)} Y_q \otimes X_p$ such that $\sigma_{\mathsf{pt},\mathsf{pt}} = 1$.*

Proof. By univalence and the characterization of paths between bands, we begin by giving an equivalence between the underlying types. The equivalence will be path-inversion, as a map

$$(X = Y)_{(\overline{p} \cdot q^*)} \longrightarrow (Y = X)_{(\overline{q} \cdot p^*)}.$$

To see that this is valid it suffices to show that the inversion of $\overline{p} \cdot q^*$ is $\overline{q} \cdot p^*$. We have:

$$\overline{\overline{p} \cdot q^*} \equiv \overline{\overline{p} \cdot \mathsf{refl}^* \cdot q} = \overline{\mathsf{refl}^* \cdot q} \cdot p = \overline{q} \cdot \overline{\mathsf{refl}^*} \cdot p = \overline{q} \cdot \mathsf{refl}^* \cdot p \equiv \overline{q} \cdot p^*,$$

where we have used associativity of path composition, and that $\overline{\mathsf{refl}^*} = \mathsf{refl}^*$ by Proposition 3.15.

To prove the transport condition, we may path induct on both p and q which then yields the following triangle:

$$\begin{tikzcd}
(A = A)_{(\mathsf{refl}^*)} \arrow[rr, "p \mapsto \overline{p}"] \arrow[dr, "\mathbf{ev}_{\mathsf{refl}^*}"'] & & (A = A)_{(\mathsf{refl}^*)} \arrow[dl, "\mathbf{ev}_{\mathsf{refl}^*}"] \\
& A\,. &
\end{tikzcd}$$

Here we are writing $\mathbf{ev}_{\mathsf{refl}^*}$ for the composite $(A = A)_{(\mathsf{refl}^*)} \simeq (A \simeq A)_{(\mathsf{id}^*)} \xrightarrow{\mathbf{ev}_{\mathsf{id}^*}} A$. The horizontal map is given by path-inversion, which is homotopic to the identity by Proposition 3.15, hence the triangle commutes.

Paths between paths between banded types correspond to homotopies between the underlying equivalences. Thus $\sigma_{\mathsf{pt},\mathsf{pt}} = 1$ since path-inversion on $(A = A)_{(\mathsf{refl}^*)}$ is homotopic to the identity. $\square$

We now use Lemma 2.4 to make $\mathsf{BAut}_1(A)$ into an H-space.

Theorem 4.19. *The binary operation $\otimes$ makes $\mathsf{BAut}_1(A)$ into an abelian H-space.*

Proof. We start by showing the left unit law. Since $\mathsf{pt}^* = \mathsf{pt}$, we instead consider the goal $(\mathsf{pt} = X_p) = X_p$. An equivalence between the underlying types is given by Proposition 4.12, which after inducting on p clearly respects the bands. Using Proposition 4.18 and Lemma 2.4, we obtain the desired H-space structure. $\square$

Corollary 4.20. *For a central type A, the type $\mathsf{BAut}_1(A)$ is again central. Therefore, A is an infinite loop space, in a unique way. Moreover, every pointed map $A \to_* A$ is infinitely deloopable, in a unique way.*

Proof. That $\mathsf{BAut}_1(A)$ is central follows from condition (2) of Proposition 3.6, using Theorems 4.10 and 4.19 as inputs. That A is a infinite loop space then follows from Proposition 4.4: writing $\mathsf{BAut}_1^0(A) :\equiv A$ and $\mathsf{BAut}_1^{n+1}(A) :\equiv \mathsf{BAut}_1(\mathsf{BAut}_1^n(A))$, we see that $\mathsf{BAut}_1^n(A)$ is an n-fold delooping of A. The uniqueness follows from Theorem 4.6. That every pointed self-map is infinitely deloopable in a unique way follows by iterating Theorem 4.10. $\square$

Note that $\mathsf{BAut}_1(A)$ is essentially small (Remark 4.2), so these deloopings can be taken to be in the same universe as A.

From Theorem 4.19 we deduce another characterization of central types:

Proposition 4.21. *A pointed, connected type A is central if and only if $\Sigma_{X:\mathsf{BAut}_1(A)} X$ is contractible.*

Proof. If A is central, then by the left unit law of Theorem 4.19, we have

$$\Sigma_{X:\mathsf{BAut}_1(A)} X \;\simeq\; \Sigma_{X:\mathsf{BAut}_1(A)} (\mathsf{pt}^* =_{\mathsf{BAut}_1(A)} X) \;\simeq\; 1.$$

Conversely, if $\Sigma_{X:\mathsf{BAut}_1(A)} X$ is contractible, then so is its loop space. But the loop space is equivalent to $\Sigma_{f:A \to_* A} \|f = \mathsf{id}\|$, i.e., the fibre of $\mathbf{ev}_{\mathsf{id}}$ over the base point. Thus $\mathbf{ev}_{\mathsf{id}}$ is an equivalence, since A is connected. $\square$

4.3. **Bands and torsors.** Let A be a central type. We define a notion of A-torsor which turns out to be equivalent to the notion of A-band from the previous section. Under our centrality assumption, it follows that the resulting type of A-torsors is a delooping of A. An equivalent type of A-torsors has been independently studied by David Wärn, who has also shown that it gives a delooping of A under the weaker hypothesis that A has a unique H-space structure.

Definition 4.22. An **action** of A on a type X is a map $\alpha : A \times X \to X$ such that $\alpha(\mathsf{pt}, x) = x$ for all $x : X$. If X has an A-action, we say that it is an ***A*-torsor** if it is merely inhabited and $\alpha(-, x)$ is an equivalence for every $x : X$. The type of ***A*-torsor structures** on a type X is

$$T_A(X) :\equiv \sum_{\alpha:A\times X\to X} (\alpha(\mathsf{pt}, -) = \mathsf{id}_X) \times \|X\|_{-1} \times \prod_{x:X} \mathsf{IsEquiv}\, \alpha(-, x),$$

and the type of A-torsors is $\sum_{X:\mathcal{U}} T_A(X)$.

Since A is connected, an A-action on X is the same as a pointed map $A \to_* (X \simeq X)_{(\mathsf{id})}$. Normally one would require at a minimum that this map sends multiplication in A to composition. We explain in Remark 4.28 why our definition suffices.

The condition that $\alpha(-,x)$ is an equivalence for all x is equivalent to requiring that for every x_0, $x_1 : X$, there exists a unique $a : A$ with $\alpha(a, x_0) = x_1$. It is also equivalent to saying that $(\alpha, \mathsf{pr}_2) : A \times X \to X \times X$ is an equivalence.

For any type X, write $\mathbf{ev}_{\simeq} : (A \simeq X) \to X$ for the evaluation fibration which sends an equivalence e to $e(\mathsf{pt})$. For a map f, write $\mathsf{Sect}(f)$ for the type of (unpointed) sections of f.

Lemma 4.23. *For any X, we have an equivalence*

$$T_A(X) \simeq \|X\|_{-1} \times \mathsf{Sect}(\mathbf{ev}_{\simeq}).$$

Proof. This is simply a reshuffling of the data. The map from left to right sends a torsor structure with action $\alpha : A \times X \to X$ to the map $X \to (A \to X)$ sending x to $\alpha(-,x)$. By assumption, this lands in the type of equivalences, and the condition $\alpha(\mathsf{pt}, -) = \mathsf{id}_X$ says that it is a section. We leave the remaining details to the reader. □

Lemma 4.24. *Let X be an A-torsor. Then X is connected.*

Proof. Since X is merely inhabited and our goal is a proposition, we may assume that we have $x_0 : X$. Then we have an equivalence $\alpha(-, x_0) : A \to X$. A is connected by Proposition 3.3, so it follows that X is. □

Proposition 4.25. *Let X be an A-torsor. Then X is banded by A.*

Proof. Associated to the torsor structure on X is a section $X \to (A \simeq X)$ of $\mathbf{ev}_{\simeq}$. Since X is 0-connected, it lands in a component of $A \simeq X$. By univalence, this determines a banding of X. □

Theorem 4.26. *Let X be a type. There is an equivalence $T_A(X) \simeq \|A = X\|_0$. Therefore, there is an equivalence between the type of A-torsors and $\mathsf{BAut}_1(A)$.*

Proof. Proposition 4.25 gives a map f. We check that the fibres are contractible. Let $p : \|A = X\|_0$ be a banding of X. An A-torsor structure t on X with $f(t) = p$ consists of a section s of $\mathbf{ev}_{\simeq}$ that lands in the component $(A \simeq X)_{(\tilde{p})}$, where $\tilde{p}$ denotes the equivalence associated to p. But by Proposition 4.12, the evaluation fibration $(A \simeq X)_{(\tilde{p})} \to X$ is an equivalence, so it has a unique section. □

Remark 4.27. It follows that $T_A(X)$ is a set. One can also show this using Corollary 4.14 and Proposition 3.6.

Remark 4.28. Let X be an A-torsor, or equivalently, an A-band. By Corollary 4.14, we have an equivalence $e : A \simeq (X \simeq X)_{(\mathsf{id})}$. Since A has a unique H-space structure, this equivalence is an equivalence of H-spaces, where the codomain has the H-space structure coming from composition. Since A is connected, the A-action on X gives a map $\alpha' : A \to_* (X \simeq X)_{(\mathsf{id})}$. (In fact, $\alpha' = e$, but we won't use this fact.) Using the equivalence e, it follows from Theorem 4.10 that any map with the same type as α' is deloopable in a unique way. That is, it has the structure of a group homomorphism in the sense of higher groups (see [BvDR18]). This explains why our naive definition of an A-action is correct in this situation.

5. Examples and non-examples

We show that the Eilenberg–Mac Lane spaces $\mathrm{K}(G, n)$ are central whenever G is abelian and $n > 0$. In addition, we produce examples of products of Eilenberg–Mac Lane spaces which are central and examples which are not central. At present, we do not know whether there exist central types which are not products of Eilenberg–Mac Lane spaces. Along the way, we use our results to give a self-contained, independent construction of Eilenberg–Mac Lane spaces. To this end, we begin by discussing the base case $\mathrm{K}(G, 1)$.

5.1. **The H-space of G-torsors.** Given a group G, we construct the type TG of G-torsors and show that it is a $\mathrm{K}(G,1)$. Specifically, a pointed type X **is a $\mathbf{K}(G,1)$** if it is connected and comes equipped with a pointed equivalence $\Omega X \simeq_* G$ which sends composition of loops to multiplication in G. (We always point ΩX at refl.)

When G is abelian, we can tensor G-torsors to obtain an H-space structure on TG which is analogous to the tensor product of bands of Theorem 4.19. These constructions are all classical and we therefore omit some details.

Definition 5.1. Let G be a group. A ***G*-set** is a set X with a group homomorphism $\alpha : G \to \mathsf{Aut}(X)$. If the set X is merely inhabited and the map $\alpha(-,x) : G \to X$ is an equivalence for every $x : X$, then (X,α) is a ***G*-torsor**. We write TG for the type of G-torsors. Given two G-sets X and Y, we write $X \to_G Y$ for the set of G-equivariant maps from X to Y, defined in the usual way.

We may write $g \cdot x$ instead of $\alpha(g,x)$ when no confusion can arise. The following is straightforward to check:

Lemma 5.2. *Let X and Y be G-torsors. There is a natural equivalence $(X =_{TG} Y) \simeq (X \to_G Y)$. In particular, a G-equivariant map between G-torsors is automatically an equivalence.* □

Any group G acts on itself by left translation, making G into a G-torsor which constitutes the base point pt of both TG and the type of G-sets. Since a G-equivariant map $\mathsf{pt} \to_G X$ is determined by where it sends $1 : G$, the map $(\mathsf{pt} \to_G X) \to X$ that evaluates at 1 is an equivalence. It is clear that the type TG is a 1-type, which implies that its loop space is a group.

Proposition 5.3. *We have a group isomorphism $\Omega TG \simeq G$.*

We only sketch a proof since this is a classical result.

Proof. Since paths between G-torsors correspond to G-equivariant maps, we have equivalences of sets

$$(\mathsf{pt} =_{TG} \mathsf{pt}) \simeq (\mathsf{pt} \to_G \mathsf{pt}) \simeq G,$$

where the second equivalence is given by evaluation at 1. The first equivalence sends path composition to composition of maps, which reverses the order—i.e., it's an anti-isomorphism. The second equivalence evaluates a map at $1 : G$. Thus, for $\phi, \psi : \mathsf{pt} \to_G \mathsf{pt}$ we have

$$\phi(\psi(1)) = \phi(\psi(1) \cdot 1) = \psi(1) \cdot \phi(1),$$

where $\cdot$ denotes the multiplication in G. In other words, evaluation at 1 is an anti-isomorphism, meaning the composite $(\mathsf{pt} =_{TG} \mathsf{pt}) \simeq G$ is an isomorphism of groups. □

The following proposition says that the G-torsors are precisely those G-sets which lie in the component of the base point.

Proposition 5.4. *A G-set (X,α) is a G-torsor if and only if there merely exists a G-equivariant equivalence from pt to X.*

Proof. Suppose X is a G-torsor. To produce a mere G-equivariant equivalence $\mathsf{pt} \simeq_G X$ we may assume we have some $x : X$, since X is merely inhabited. Then $(-) \cdot x : G \to X$ yields an equivalence which is clearly G-equivariant, as required.

Conversely, assume that there merely exists a G-equivariant equivalence from pt to X. Since being a G-torsor is a proposition, we may assume we have an actual G-equivariant equivalence. But then we are done since pt is a G-torsor. □

It follows that TG is connected. Thus by Proposition 5.3 we deduce:

Corollary 5.5. *The type TG is a* $\mathrm{K}(G,1)$.

For the remainder of this section, let G be an abelian group.

Proposition 5.6. *For any two G-torsors S and T, the path type $S =_{TG} T$ is again a G-torsor.*

Proof. First we make $S =_{TG} T$ into a G-set. This path type is equivalent to the type $S \to_G T$. Using that G is abelian, it's easy to check that the map

$$(g, \phi) \longmapsto \big(s \mapsto g \cdot \phi(s)\big) \ : \ G \times (S \to_G T) \longrightarrow (S \to_G T)$$

is well-defined and makes $S \to_G T$ into a G-set.

To check that the above yields a G-torsor, we may assume that $S \equiv \mathsf{pt} \equiv T$, by the previous lemma. One can check that Proposition 5.3 gives an equivalence of G-sets, where $\mathsf{pt} \to_G \mathsf{pt}$ is equipped with the G-action just described. Thus $\mathsf{pt} \to_G \mathsf{pt}$ is a G-torsor, as required. □

In order to describe the tensor product of G-torsors, we first need to define duals.

Definition 5.7. Let (X, α) be a G-torsor. The **dual $\boldsymbol{X^*}$** of X is the G-torsor X with action

$$\alpha^*(g, x) :\equiv \alpha(g^{-1}, x).$$

The tensor product of G-torsors is now defined as $X \otimes Y :\equiv (X^* =_{TG} Y)$.

Proposition 5.8. *The tensor product of G-torsors makes TG into an H-space.*

Proof. We verify the hypotheses of Lemma 2.4. Thus our first goal is to construct a symmetry

$$\sigma_{X,Y} : (X^* =_{TG} Y) \ =_{TG} \ (Y^* =_{TG} X).$$

After identifying paths of G-torsors with G-equivariant equivalences, we may consider the map which inverts such an equivalence. A short calculation shows that if $\phi : X^* \to_G Y$ is G-equivariant, then $\phi^{-1} : Y^* \to_G X$ is again G-equivariant. We need to check that the map sending ϕ to ϕ^{-1} is itself G-equivariant, so let $\phi : X^* \to_G Y$ and let $g : G$. Since the inverse of $g \cdot (-)$ is $g^{-1} \cdot (-)$, we have:

$$(g \cdot \phi)^{-1} = \phi^{-1}(g^{-1} \cdot (-)) = g \cdot \phi^{-1}(-),$$

using that $\phi^{-1} : Y^* \to_G X$ is G-equivariant. Thus inversion is G-equivariant, yielding the required symmetry σ.

Now we argue that $\sigma_{\mathsf{pt},\mathsf{pt}} = \mathsf{refl}$, or, equivalently, that maps $\mathsf{pt}^* \to_G \mathsf{pt}$ are their own inverses. Such a map is uniquely determined by where it sends $1 : G$, so it suffices to show that $\phi(\phi(1)) = 1$ for every $\phi : \mathsf{pt}^* \to_G \mathsf{pt}$. Fortunately, we have

$$\phi(\phi(1)) = \phi(\phi(1) \cdot 1) = \phi(1)^{-1} \cdot \phi(1) = 1.$$

Lastly, it is straightforward to check that the map $(\mathsf{pt}^* \to_G X) \to X$ which evaluates at $1 : G$ is G-equivariant, for any G-torsor X. This yields the left unit law for the tensor product $\otimes$. As such we have fulfilled the hypotheses of Lemma 2.4, giving us the desired H-space structure. □

Using Proposition 3.6, one can check that TG is a central H-space. (See Proposition 5.9.)

5.2. **Eilenberg–Mac Lane spaces.** We now use our results to give a new construction of Eilenberg–Mac Lane spaces. For an abelian group G, recall that a pointed type X **is a $\mathbf{K}(\boldsymbol{G}, \mathbf{1})$** if it is connected and there is a pointed equivalence $\Omega X \simeq_* G$ which sends composition of paths to multiplication in G. For $n > 1$, a pointed type X **is a $\mathbf{K}(\boldsymbol{G}, \boldsymbol{n}+\mathbf{1})$** if it is connected and ΩX is a $\mathrm{K}(G, n)$. It follows that such an X is an n-connected $(n+1)$-type with $\Omega^{n+1} X \simeq_* G$ as groups.

In the previous section we saw that the type TG of G-torsors is a $\mathrm{K}(G, 1)$ and is central whenever G is abelian. The following proposition may be seen as a higher analog of this fact.

Proposition 5.9. *Let G be an abelian group and let $n > 0$. If a type A is a $\mathrm{K}(G, n)$ and an H-space, then A is central and $\mathsf{BAut}_1(A)$ is a $\mathrm{K}(G, n+1)$ and an H-space.*

The fact that $\mathsf{BAut}_1(A)$ is a $\mathrm{K}(G, n+1)$ also follows from [Shu], using the fact that $\mathsf{BAut}_1(A)$ is the 1-connected cover of $\mathsf{BAut}(A)$.

Proof. Suppose that A is a $\mathrm{K}(G, n)$ and an H-space. Then $A \to_* \Omega A$ is contractible, since it is equivalent to $\|A\|_{n-1} \to_* \Omega A$, and $\|A\|_{n-1}$ is contractible. So Proposition 3.6 implies that A is central. By Proposition 4.4, $\Omega\, \mathsf{BAut}_1(A) \simeq A$, so $\mathsf{BAut}_1(A)$ is a $\mathrm{K}(G, n+1)$. By Theorem 4.19, $\mathsf{BAut}_1(A)$ is also an H-space. □

We can use the previous proposition to *define* $\mathrm{K}(G,n)$ for all $n>0$ by induction. For the base case $n\equiv 1$ we let $K(G,1):\equiv TG$, the type of G-torsors from the previous section. When G is abelian, we saw that TG is an H-space, which lets us apply the previous proposition. By induction, we obtain a $\mathrm{K}(G,n)$ for all n. Note that this construction produces a $\mathrm{K}(G,n)$ which lives $n-1$ universes above the given $\mathrm{K}(G,1)$, but that it is essentially small by the join construction [Rij17].

5.3. **Products of Eilenberg–Mac Lane spaces.** Here is our first example of a central type that is not an Eilenberg–Mac Lane space.

Example 5.10. Let $K=\mathrm{K}(\mathbb{Z}/2,1)=\mathbb{R}P^\infty$ and $L=\mathrm{K}(\mathbb{Z},2)=\mathbb{C}P^\infty$, and consider $A=K\times L$. This is a connected H-space, and

$$\begin{aligned}
\big(K\times L\to_* \Omega(K\times L)\big) &\simeq \big(K\to_*\Omega(K\times L)\big) && \text{since } K=\|K\times L\|_1\\
&\simeq \big(K\to_*\Omega L\big) && \text{since } K \text{ is connected}\\
&\simeq \big(\mathbb{Z}/2\to_{\mathbf{Ab}}\mathbb{Z}\big) && \text{by [BvDR18, Theorem 5.1]}\\
&\simeq 1.
\end{aligned}$$

So it follows from Proposition 3.6(4) that A is central.

On the other hand, not every product of Eilenberg–Mac Lane spaces is central.

Example 5.11. Let $K=\mathrm{K}(\mathbb{Z}/2,1)=\mathbb{R}P^\infty$ and $L'=\mathrm{K}(\mathbb{Z}/2,2)$. A calculation like the above shows that $K\times L'\to_*\Omega(K\times L')$ is not contractible, so $K\times L'$ is not central.

As another example, [Cur68, Proposition Ia] shows that $\mathrm{K}(\mathbb{Z},1)\times\mathrm{K}(\mathbb{Z},2))$ (i.e., $\mathbb{S}^1\times\mathbb{C}P^\infty$) has infinitely many distinct H-space structures classically. So it is not central, by Proposition 3.3.

Clearly both of these examples can be generalized to other groups and shifted to higher dimensions.

By Proposition 3.3, centrality of a type implies that it has a unique H-space structure. The converse fails, as we now demonstrate. We are grateful to David Wärn for bringing our attention to this example.

Example 5.12. The type $A:\equiv\mathrm{K}(\mathbb{Z},2)\times\mathrm{K}(\mathbb{Z},3)$ is not central, by a computation similar to the one in the previous example. However, we note that it admits a unique H-space structure. Since A is a loop space it admits an H-space structure, and the type of H-space structures is given by $A\wedge A\to_* A$ according to Theorem 2.27. Since A is 1-connected, by [CS20, Corollary 2.32] the smash product $A\wedge A$ is 3-connected. It follows that $A\wedge A\to_* A$ is contractible, since A is 3-truncated. In other words, the space of H-space structures on A is contractible.

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University of Nottingham, Nottingham, United Kingdom
Email address: ulrik.buchholtz@nottingham.ac.uk

University of Western Ontario, London, Ontario, Canada
Email address: jdc@uwo.ca

University of Western Ontario, London, Ontario, Canada
Email address: jtaxers@uwo.ca

University of Ljubljana, Ljubljana, Slovenia
Email address: e.m.rijke@gmail.com