Internal Ext groups

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1 Introduction

We detail our current understanding of Ext groups in homotopy type theory and their semantic counterparts in a Grothendieck ∞-topos. Homotopy type theory can be interpreted into any Grothendieck ∞-topos [Shu19], and abelian groups in the former are interpreted as abelian groups internal to the topos of 0-truncated objects in the latter — i.e. sheaves of abelian groups on the specified site. We'll refer to such sheaves as $\mathbb{Z}$-modules going forth; and in particular, our Ext groups are interpreted as $\mathbb{Z}$-modules. Section 2 begins with a brief description of how to think about constructions in homotopy type theory in these models, and contains the type-theoretic developments. Section 3 on the other hand takes place in a sheaf topos, and requires only familiarity with those.

Whereas categories of sheaves of abelian groups have enough injectives, this is not known (nor expected) to the case for abelian groups in homotopy type theory (nor for abelian groups in an elementary topos). We therefore define our $\mathcal{E}xt$ as an *internal* version of the so-called Yoneda Ext [Mac63], which is a resolution-free approach to Ext groups available in any abelian category. Explicitly, in Section 2.1 we define a type $h\mathcal{E}xt^1(B, A)$ consisting of all short exact sequences

$$0 \to A \to E \to B \to 0$$

and define $\mathcal{E}xt^1(B, A)$ to be the set-truncation of $h\mathcal{E}xt^1(B, A)$. The section is concluded by the internal analog of the usual characterisation of the acyclic objects of Ext.

In a sheaf topos, there are two distinct notions of Ext for the $\mathbb{Z}$-modules: the *external* Ext groups, and the *internal* $\mathcal{E}xt$ sheaves [Har77, Section III-6]. While the classical Yoneda Ext groups recover the external Ext groups, our $\mathcal{E}xt$ groups are *internal* and recover sheaf $\mathcal{E}xt$ defined using injective resolutions in these models. The acyclic objects in the first parameter are determined to be the *internally* projective $\mathbb{Z}$-modules, examples of which are the finite locally free $\mathbb{Z}$-modules (see Remark 3.2.2).

It is well-known classically that there exist abelian categories for which the Yoneda Ext groups are not sets, but proper classes. Our definition of $\mathcal{E}xt$ in homotopy type theory inherits this issue, meaning $\mathcal{E}xt^1(B, A)$ is a large type (see Remark 2.2.1 for related discussion). Nevertheless, in Section 2.2 we construct equivalences

$$h\mathcal{E}xt^1(B, A) \;\simeq\; \big(\mathrm{K}(B, n) \to_* \mathrm{BAut}(\mathrm{K}(A, n))\big), \qquad \text{for } n \geq 2$$

where the right-hand side is small. While all the parts of this equivalence are either classical or known, some parts have not previously been written out in type theory, and the application to proving smallness of $h\mathcal{E}xt^1$, and consequently $\mathcal{E}xt^1$, is new.

Despite the fact that injective resolutions always exist for $\mathbb{Z}$-modules, we have chosen to work with projectives, since these appear easier to work with in type theory. Section 2 is wrapped up by showing that our $\mathcal{E}xt$ groups can be computed in terms of projective resolutions, whenever such resolutions exist. The proof is classical in nature, but uses only tools available to us in homotopy type theory.

In Section 3 we explore the notion of an *internally* projective $\mathbb{Z}$-module and contrast it with the usual *external* notion of projectivity. Both external and internal injectivity has previously been studied for $\mathbb{Z}$-modules, and a little-known fact is that external and internal injectives coincide in this setting [Har83, Proposition 2.1]. In contrast, internal and external projectives differ in general and we demonstrate this through two examples. Only the example of an external projective that fails to be internally projective requires work, and the example presented is a novel modification of an argument due to Todd Trimble.

Finally, we conclude in Section 3.2 by computing a nontrivial $\mathcal{E}xt^2$ sheaf via an internal projective resolution. It is well-known [Har77, Proposition III-6.5] that sheaf $\mathcal{E}xt$ may be computed using locally free resolutions, however our resolution is not locally free. The computation generalises to show that the Sierpinski topos has internal homological dimension 2.

Everything presented here has been worked out in collaboration with my advisor, Dan Christensen.

2 Ext groups in homotopy type theory

Our setting is Martin-Löf type theory with a hierarchy of univalent universes, as in the HoTT Book [Uni13]. At any given moment of time, we'll use notation for at most two universes: $\mathcal{U} : \mathcal{U}'$, where the former consists of the *small* types and the latter of the *large* ones. When we say a large type is small, we mean it is equivalent to a small type.

We think of a **type** as an object in an ∞-topos $\mathscr{E}$, e.g. a space or a sheaf of spaces. Important operations on types that we need to understand are: forming **Σ-types**, **Π-types**, and **truncations**.

Given a type A, a **family of types over** A may be thought of in two ways. Firstly, as a map $p : B \to A$ where we see B as all the fibres $p^{-1}(a)$ of p glued together as $a : A$ varies. Secondly, as a map $P : A \to \mathcal{U}$ into the universe of types. The second viewpoint is convenient in type theory; and to go there from first viewpoint, one simply considers the association $a \mapsto p^{-1}(a) : A \to \mathcal{U}$, the fibre $p^{-1}(a)$ being itself a type.

Conversely, given a type family $P : A \to \mathcal{U}$, the **Σ-type** $\Sigma_{a:A}P(a)$ is the **total space** of a map into A whose fibre over $a : A$ is $P(a)$. For example, the type B above is the total space of $p^{-1}(-) : A \to \mathcal{U}$, i.e. B is equivalent to $\Sigma_{a:A}\ p^{-1}(a)$. An element of $\Sigma_{a:A}P(a)$ is a "dependent pair" (a, p) with $a : A$ and $p : P(a)$, and there's a projection

$$(a, p) \mapsto a : \Sigma_{a:A}P(a) \to A$$

making $\Sigma_{a:A}P(a)$ into "an object over A".

The type $\Pi_{a:A}P(a)$ is **the object of sections** of the bundle $\Sigma_{a:A}P(a) \to A$. An element of this type is a "dependent function" f such that $f(a) : P(a)$ for all $a : A$.

Finally, we require two notions of truncation: **set-truncation** $\|-\|_0$ and **propositional truncation** $\|-\|$. The former corresponds to forming the set of connected components $\|A\|_0$ of a type A, whereas the latter corresponds to the image of A under the unique map to the terminal object: $A \to \|A\| \to 1$. A type A is a **set** if the map $A \to \|A\|_0$ is an equivalence; a **proposition** (denoted $\mathrm{IsProp}(A)$) if the map $A \to \|A\|$ is one; and A is **contractible** (denoted $\mathrm{Contr}(A)$) if the map $A \to 1$ is an equivalence.

2.1 The type $h\mathcal{E}xt^1$ of abelian group extensions

Fix two abelian groups B and A in $\mathcal{U}$ throughout this section. Below, we define the type $h\mathcal{E}xt^1(B, A)$ of abelian group extensions, i.e. short exact sequences

$$0 \to A \xrightarrow{i} E \xrightarrow{p} B \to 0$$

for some abelian group $E : \mathrm{Ab}$. The type $h\mathcal{E}xt^1(B, A)$ is a 1-type whose set-truncation is defined to be $\mathcal{E}xt^1(B, A)$. By characterising paths in $h\mathcal{E}xt^1(B, A)$ we will identify the trivial extensions as the *merely split* extensions, and this produces the relevant notion of projectivity in homotopy type theory.

Definition 2.1.1. Given a homomorphism $f : \mathrm{Ab}(A, B)$, we say that f is an **epimorphism**, respectively **monomorphism**, if the following hold, respectively:

$$\mathrm{IsEpi}(f) := \Pi_{b:B}\,\mathrm{Contr}(\|f^{-1}(b)\|), \qquad \mathrm{IsMono}(f) := \mathrm{IsProp}(\ker(f))$$

Definition 2.1.2. Let $A \xrightarrow{i} E \xrightarrow{p} B$ be two composable group homomorphisms. Whenever the composite $p \circ i$ is trivial, there is an induced map $i' : A \to \ker(p)$. We say that i and p are **exact** if i' is an equivalence:

$$\mathrm{IsExact}(i, p) :\equiv \sum_{h:\prod_{a:A} p(i(a))=0} \mathrm{IsEquiv}(i')$$

Having defined monomorphisms, epimorphisms, and exactness, we begin constructing a Yoneda-style Ext in homotopy type theory:

Definition 2.1.3. The **type of extensions of** B **by** A is:

$$h\mathcal{E}xt^1(B, A) :\equiv \sum_{E:\mathrm{Ab}} \sum_{i:\mathrm{Ab}(A,E)} \sum_{p:\mathrm{Ab}(E,B)} \mathrm{IsMono}(i) \wedge \mathrm{IsEpi}(p) \wedge \mathrm{IsExact}(i, p) : \mathcal{U}'$$

An element of $h\mathcal{E}xt^1(B, A)$ is written as (E, i, p).

As defined, $h\mathcal{E}xt^1$ quantifies over Ab and is therefore a large type. It is moreover a 1-type, since Ab is and the fibre of the outermost sigma is a set. This mirrors the fact that the category of group extensions of B by A, whose morphisms are homomorphisms $E \to E'$ making the relevant triangles commute, is a groupoid. The following proposition makes that clear:

Proposition 2.1.4. *Let (E, i, p) and (F, j, q) be extensions of B by A. Paths between the extensions correspond to isomorphisms $E \cong F$ respecting the inclusions and projections:*

$$\big((E, i, p) =_{h\mathcal{E}xt^1(B,A)} (F, j, q)\big) \ \simeq \ \sum_{\varphi:E\cong F'} (\varphi \circ i = j) \wedge (p = q \circ \varphi)$$

Proof. Follows by characterisation of paths in Σ-types, and that transport in function types happens via pre- and post-composition. $\square$

In [Mac63], Mac Lane produces the set $\mathrm{Ext}^1(B, A)$ by applying π_0 to the groupoid of extensions. We now do the corresponding thing:

Definition 2.1.5. The **set of extensions of** B **by** A is $\mathcal{E}xt^1(B, A) :\equiv \|h\mathcal{E}xt^1(B, A)\|_0$.

Using this definition, we have previously verified that the arguments in [Mac63], with only minor modifications, make $\mathcal{E}xt^1(B, A)$ into an abelian group (via the Baer sum) and show functoriality in both variables. For this reason we may speak of $\mathcal{E}xt^1(C, A)$ as an abelian group going forth.

Perhaps surprisingly, our truncation kills more than Mac Lane's π_0.

Proposition 2.1.6. *Let (E, i, p) be an extension of B by A. Then E is trivial in $\mathcal{E}xt^1(B, A)$ if and only if p* merely splits*, i.e. the following proposition holds:*

$$\|\sum_{s:\mathrm{Ab}(B,E)} p \circ s = \mathrm{id}_B\|$$

Proof. First of all, by the characterisation of paths in truncations [Uni13, Theorem 7.3.12]

$$\big(|(E, i, p)|_0 =_{\mathcal{E}xt^1(B,A)} 0\big) \ \simeq \ \|(E, i, p) =_{h\mathcal{E}xt^1(B,A)} (A \oplus B, i_A, p_B)\|$$

Forgetting about truncations, the right-hand side holds if and only if p splits, by the usual argument. This in turn implies the statement on the truncations. $\square$

Remark 2.1.7. Given a short exact sequence of sheaves of abelian groups,

$$0 \to F \to G \to H \to 0$$

one may consider this sequence as an element of the *external* Yoneda Ext group, or as an element of the *internal* $\mathcal{E}xt^1(H, F)$ sheaf. The previous proposition indicates that the sequence can represent a trivial element in the latter, but a nontrivial sequence in the former. An example of this is given in Remark 3.1.3. It is however not the case that external Ext being zero implies the same for internal $\mathcal{E}xt$, since the latter may compensate by having partial elements.

Classically, an abelian group P is projective if and only if any group epi $p : E \to P$ splits and these are exactly the acyclic objects for $\mathcal{E}xt^1$ in the contravariant parameter. The previous proposition provides evidence that the corresponding notion of projectivity in our setting is that of p *merely* splitting. In section 3 the relationship between *internal* and *external* projectivity is explored; and in a general model they are incomparable.

We conclude this section by characterising projective abelian groups in homotopy type theory.

Proposition 2.1.8. *Let P be an abelian group. The following are equivalent:*

(i) Any epi $p : \mathrm{Ab}(A, P)$ merely splits, i.e. the following holds:

$$\|\{\ s : \mathrm{Ab}(P, A) \mid p \circ s = \mathrm{id}_P\ \}\|$$

(ii) For any epi $p : \mathrm{Ab}(A, B)$ and any $f : \mathrm{Ab}(P, B)$, there merely exists a lift:

$$\|\{\ g : \mathrm{Ab}(P, A) \mid p \circ g = f\ \}\|$$

(iii) $\mathcal{E}xt^1(P, A) = 0$ for all abelian groups A.

If any of the equivalent conditions above hold, we say that P is **projective***.*

Proof. The equivalence of (i) and (iii) follows from Proposition 2.1.6, and the equivalence between (i) and (ii) mirrors the classical argument by picking points. □

A dual argument characterizes the **injective** abelian groups Q as those such that $\mathcal{E}xt^1(B, Q) = 0$ for all abelian groups B.

2.2 BAut and smallness of $h\mathcal{E}xt^1$

A well-known issue with Yoneda Ext in a general abelian category is that it produces (a priori) large abelian groups, which are not equivalent to a set. The type $h\mathcal{E}xt^1(B, A)$ inherits this issue by living in a larger universe, but we show that it is nevertheless equivalent to a small type. Specifically, we construct equivalences

$$h\mathcal{E}xt^1(B, A) \ \simeq\ (\mathrm{K}(B, n) \to_* \mathrm{BAut}(\mathrm{K}(A, n))), \quad \text{for all } n \geq 2$$

where the right-hand side is small. Consequently, the set-truncation $\mathcal{E}xt^1(B, A)$ is small as well. Of course, these equivalences are also of independent interest.

Remark 2.2.1. Any Grothendieck ∞-topos contains a hierarchy of universes, and restricting to the sets (i.e. 0-truncated objects) produces a hierarchy of sheaves of sets. The meaning of $\mathcal{E}xt^1$ being large is that it ascends one step in this hierarchy. However, our $\mathcal{E}xt^1$ recovers the $\mathcal{E}xt^1$ sheaf in these models, which of course is small. Thus the fact that $\mathcal{E}xt^1$ is small is only of interest in models lacking resolutions, examples of which are not yet well understood.

Recall that any group G has a *classifying space* BG, which is a pointed and connected 1-type such that $\Omega BG \simeq_* G$. More generally, for a loop space G we write BG for its *delooping*, i.e. a pointed and connected type such that $\Omega BG \simeq_* G$. Important examples are the Eilenberg–MacLane spaces $K(G, n)$ for a group G. In homotopy type theory, these are constructed via higher inductive types, suspensions, and truncations and therefore inhabit the same universe as G.

Viewing a type $X : \mathcal{U}$ as a point in the universe, we naturally obtain loop spaces $(X =_{\mathcal{U}} X)$ which by univalence are equivalent to $\mathrm{Aut}(X)$, the type of auto-equivalences of a type X. This means that the path component of X in $\mathcal{U}$ merits the name $\mathrm{BAut}(X)$:

Definition 2.2.2. Given a type $X : \mathcal{U}$, denote $\mathrm{BAut}(X) :\equiv \sum_{Y:\mathcal{U}} \|X = Y\|_{-1}$ for the path component of X in the universe $\mathcal{U}$, pointed at $(X, |\mathrm{id}_X|_{-1})$.

Almost by definition, $\mathrm{BAut}(X)$ is a pointed and connected type. Characterisation of paths in Σ-types immediately tells us that the loop space is $(X =_{\mathcal{U}} X)$, justifying the notation. While elegant, this definition raises universe level by quantifying over $\mathcal{U}$. If we know X is a small set, then $\mathrm{Aut}(X)$ is a group and $\mathrm{BAut}(X)$ is equivalent to the classifying space $\mathrm{K}(\mathrm{Aut}(X), 1)$ described above. As such, when X is a set $\mathrm{BAut}(X)$ is small. However, when X isn't a set this is a known corollary of a theorem by Rijke.

Proposition 2.2.3. *For any small type X, the type* $\mathrm{BAut}(X)$ *is also small.*

Proof. This is a corollary of Theorem 4.6 in [Rij17], since we assume a hierarchy of univalent universes (which implies Rijke's "global" function extensionality), as well as closure under HITs (in particular graph quotients). The fact that $\mathrm{BAut}(X)$ is connected means the point inclusion $1 \to_* \mathrm{BAut}(X)$ is surjective, and the theorem implies the image (which is all of $\mathrm{BAut}(X)$) is equivalent to a small type $\mathrm{im}'(1)$. □

The equivalence $\mathcal{E}xt^1(B, A) \simeq (\mathrm{K}(B, n) \to_* \mathrm{BAut}(\mathrm{K}(A, n)))$ will be built up from several intermediate equivalences, and we start from the right.

Notation 2.2.4. For a type Y, write $\mathcal{U}/Y :\equiv \sum_{X:\mathcal{U}} X \to Y$ for the type of maps into Y. If Y is pointed, write $\mathcal{U}_*/Y :\equiv \sum_{X:\mathcal{U}_*} X \to_* Y$ for the type of pointed maps into Y.

Lemma 2.2.5. *Let F and Y be types and $y_0 : Y$. We have an equivalence:*

$$\Big(\sum_{p:\mathcal{U}/Y} F \simeq p^{-1}(y_0)\Big) \;\simeq\; (Y \to_* (\mathcal{U}, F))$$

where $(\mathcal{U}, F)$ denotes the universe pointed by F.

Proof. The type of pointed maps $Y \to_* (\mathcal{U}, F)$ is a sigma-type, and we have an equivalence between the two bases $\mathcal{U}/Y \simeq (Y \to \mathcal{U})$ since $\mathcal{U}$ is the object classifier [Uni13, Theorem 4.8.3]. Via this equivalence, a bundle $p : X \to Y$ is sent to its family of fibres $p^{-1}(-) : Y \to \mathcal{U}$, and so $p^{-1}(y_0) \simeq_* F$ asserts via univalence exactly that the latter map is pointed when considered into (U, F). □

Note that when Y is connected, the right-hand side of the equivalence above can be replaced by $Y \to_* \mathrm{BAut}(F)$ since Y maps into the connected component of F in $\mathcal{U}$.

When F is pointed, we wish to promote the left-hand side to a type of fiber sequences, which involve pointed maps. There is a natural map which forgets the point

$$U : \sum_{p:\mathcal{U}_*/Y} F \simeq_* p^{-1}(y_0) \longrightarrow \sum_{p:\mathcal{U}/Y} F \simeq p^{-1}(y_0)$$

where $p^{-1}(y_0)$ has a natural basepoint whenever p is a pointed map. In fact, it's possible to go the other way:

Definition 2.2.6. Let (Y, y_0) and (F, f_0) be pointed types. The "promotion" map

$$P : \sum_{p:\mathcal{U}/Y} F \simeq p^{-1}(y_0) \longrightarrow \sum_{p:\mathcal{U}_*/Y} F \simeq_* p^{-1}(y_0)$$

sends $p : X \to Y$ and $e : F \simeq p^{-1}(y_0)$ to the pointed map

$$(p, \mathrm{pr}_2\, e(f_0))) : (X, \mathrm{pr}_1(f_0)) \to Y$$

and definitionally points the equivalence, $(e, 1_{e(f_0)}) : F \simeq_* p^{-1}(y_0)$.

Promoting and forgetting points form an equivalence, and showing this requires knowledge of paths in $\mathcal{U}_*/Y$ and $\sum_{p:\mathcal{U}_*/Y} F \simeq_* p^{-1}(y_0)$.

Lemma 2.2.7. *Let (Y, y_0) be a pointed type, and let $p : X \to_* Y$ and $q : Z \to_* Y$ be two elements of $\mathcal{U}_*/Y$. Paths $p = q$ in $\mathcal{U}_*/Y$ correspond to pointed equivalences between the total spaces over Y, i.e.*

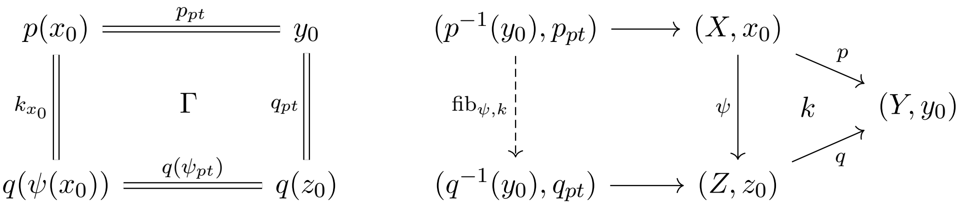
$$(p =_{\mathcal{U}_*/Y} q) \;\simeq \sum_{\psi:X \simeq_* Z} p =_* q \circ \psi$$

where $=_*$ denotes equality of pointed maps.

Proof. By the characterisation of paths in sigma-types, and transport in families of pointed maps. □

Notation 2.2.8. Let (X, x_0) and (Y, y_0) be pointed types, and $p : X \to_* Y$ a pointed map. We denote the path $p(x_0) = y_0$ by p_{pt}.

A pointed equivalence between total spaces over Y induces an equivalence of the fibres whose precise decription we now give. Consider a triangle over Y as on the right below, where $k : p =_* q \circ \psi$ is a path of pointed maps containing a 2-cell Γ on the left below asserting that p and $q \circ \psi$ are pointed by the same path.



On some $(x, \rho) : p^{-1}(y_0)$, the induced map $\mathrm{fib}_{\psi,k}$ is defined by

$$\mathrm{fib}_{\psi,k}(x, \rho) :\equiv (\psi(x), \overline{k_x} \cdot \rho)$$

where $\cdot$ is path composition and $\overline{(-)}$ is path inversion.

In order to show that $\mathrm{fib}_{\psi,k}$ is a pointed map, we need to produce a path

$$(\psi(x_0), \overline{k_{x_0}} \cdot p_{pt}) =_{q^{-1}(y_0)} (z_0, q_{pt})$$

where the left-hand side is $\mathrm{fib}_{\psi,k}(x_0, p_{pt})$. In the base, we have $\psi_{pt} : \psi(x_0) =_Z z_0$ which by transporting in the family $(z : Z \mapsto q(z) =_Y y_0)$ produces the goal

$$\overline{q(\psi_{pt})} \cdot \overline{k_{x_0}} \cdot p_{pt} = q_{pt}$$

and this holds since it's on the boundary of Γ.

Lemma 2.2.9. *Let (Y, y_0) and (F, f_0) be pointed types, let $p : X \to_* Y$ and $q : Z \to_* Y$ be elements of $\mathcal{U}_*/Y$, and let $e : F \simeq_* p^{-1}(y_0)$ and $e' : F \simeq_* q^{-1}(y_0)$ be pointed equivalences. Paths $(p, e) = (q, e')$ in $\sum_{p:\mathcal{U}_*/Y} F \simeq_* p^{-1}(y_0)$ correspond to equivalences $\psi : X \simeq_* Z$ over Y which induce equivalences on the fibres $\mathrm{fib}_\psi : p^{-1}(y_0) \simeq_* q^{-1}(y_0)$ respecting F:*

$$(p, e) = (q, e') \;\simeq\; \sum_{\psi:X\simeq_* Z} \; \sum_{k:p=_* q\circ\psi} \mathrm{fib}_{\psi,k} \circ e =_* e'$$

Proof. By Lemma 2.2.7, the characterisation of paths in Σ-types and transporting in

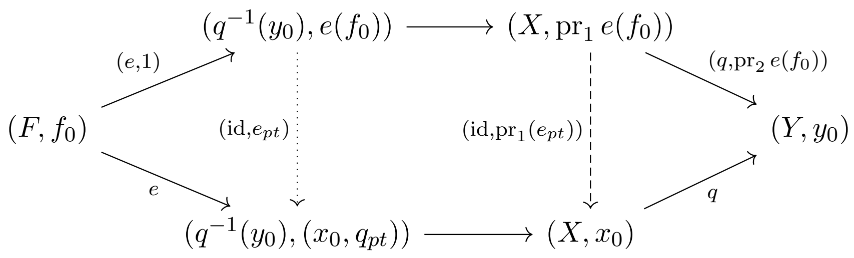
$$s \mapsto F \simeq_* s^{-1}(y_0) : \mathcal{U}_*/Y \to \mathcal{U}$$

□

Proposition 2.2.10. *Let (Y, y_0) and (F, f_0) be pointed types. We have an equivalence:*

$$P : \Big(\sum_{q:\mathcal{U}/Y} F \simeq q^{-1}(y_0) \Big) \;\simeq\; \Big(\sum_{q:\mathcal{U}_*/Y} F \simeq_* (q^{-1}(y_0), q_{pt}) \Big) : U$$

Proof. The direction $U \circ P = \mathrm{id}$ is definitional, so we concentrate on $P \circ U = \mathrm{id}$. Let $q : (X, x_0) \to_* Y$ be a pointed map and $e : F \simeq_* q^{-1}(y_0)$ a pointed equivalence, depicted (with explicit base points) as the bottom row of arrows below:



The top row is $P \circ U$ applied to the bottom row, and we now construct the required data for a path $UP(q, e) = (q, e)$. The rightmost triangle commutes definitionally (i.e. $k_x = 1$ for all $x : X$), so we need only argue that the pointings agree. To that end, we split e_{pt} into two components:

$$(e^1_{pt}, e^2_{pt}) : \sum_{\rho : \mathrm{pr}_1\, e(f_0) = x_0} q(\rho) \cdot q_{pt} = \mathrm{pr}_2\, e(f_0)$$

This means e^2_{pt} is a 2-cell of the form

$$\begin{array}{ccc} \mathrm{pr}_1\, q(e(f_0)) & \overset{\mathrm{pr}_2\, e(f_0)}{=\!=\!=} & y_0 \\ {\scriptstyle 1}\big\| & & \big\|{\scriptstyle q_{pt}} \\ \mathrm{pr}_1\, q(e(f_0)) & \overset{q(e^1_{pt})}{=\!=\!=} & q(x_0) \end{array}$$

which implies that the pointings in the rightmost triangle agree.

Since $k_x = 1_{p(x)}$, the induced map on the fibres (dotted line) is indeed the identity. This means the resulting pointing is e_{pt}, and the leftmost triangle clearly commutes. ☐

Theorem 2.2.11. *Let B and A be abelian groups. For any $n \geq 2$, we have an equivalence*

$$h\mathcal{E}xt^1(B, A) \;\simeq\; \big(\mathrm{K}(B, n) \to_* \mathrm{BAut}(\mathrm{K}(A, n))\big)$$

In particular, $h\mathcal{E}xt^1(B, A)$ and $\mathcal{E}xt^1(B, A)$ are both equivalent to a small type.

Proof. We may replace the right-hand side by $\mathrm{K}(B, n) \to_* (\mathcal{U}, \mathrm{K}(A, n))$, since $\mathrm{K}(B, n)$ is connected. Lemma 2.2.5 and Proposition 2.2.10 then combine to give an equivalence:

$$\sum_{p : \mathcal{U}_* / \mathrm{K}(B,n)} \mathrm{K}(A, n) \simeq_* p^{-1}(y_0) \;\simeq\; \big(\mathrm{K}(B, n) \to_* (\mathcal{U}, \mathrm{K}(A, n))\big)$$

Theorem 4 in [BDR18] produces the following equivalences of categories for $n \geq 2$:

$$\mathrm{K}(-, n) : \mathrm{Ab} \;\simeq\; \{\text{ pointed, } (n{-}1)\text{-connected } n\text{-types }\} : \pi_n$$

under which kernels and fibres correspond. This implies an equivalence

$$\Big(\sum_{E:\mathrm{Ab}}\ \sum_{p:\mathrm{Ab}(E,B)} A \cong \ker(p)\Big) \ \simeq\ \Big(\sum_{p:\mathcal{U}_*/\,\mathrm{K}(B,n)} \mathrm{K}(A,n) \simeq_* p^{-1}(y_0)\Big)$$

where the left-hand side is equivalent to $h\mathcal{E}xt^1(B,A)$. □

We wish to emphasize that the only new observation in this section is that $h\mathcal{E}xt^1$ is equivalent to a small type. Similar results about pointed bundles have been discussed in [Sco20], [Mye], and [NSS14].

2.3 $\mathcal{E}xt^n$ via projective resolutions

We have previously defined the higher Ext groups $\mathcal{E}xt^n$ in homotopy type theory and verified in detail that short exact sequences induce long exact sequences in the contravariant variable. (The covariant version is expected to hold as well, but we do not use it here.) In the interest of space we refrain from reproducing the definition of $\mathcal{E}xt^n$ here, as we only need its familiar abstract properties.

Having both projective abelian groups and higher Ext groups at hand, it is natural to ask whether projective resolutions can be used to compute $\mathcal{E}xt^n$. Below we show that this is indeed the case, when such resolutions exist. The argument is standard homological algebra, and the content is that it holds with the results available to us in homotopy type theory.

Let B be an abelian group, and assume a projective resolution $P_\bullet$ of B. Write $p_0 : P_0 \to B$ for the obvious map, and note that by definition $P_0/\operatorname{im}(P_1) = B$ and so P_1 surjects onto $B_1 :\equiv \ker(p_0)$. Continuing inductively, we may factor the projective resolution as follows:

$$\begin{array}{ccccccc} \cdots\to & P_2 & \longrightarrow & P_1 & \longrightarrow & P_0 \\ \nearrow & \downarrow{\scriptstyle p_2} & \overset{i_2}{\nearrow} & \downarrow{\scriptstyle p_1} & \overset{i_1}{\nearrow} & \downarrow{\scriptstyle p_0} \\ & B_2 & & B_1 & & B_0 \end{array}$$

where $B_{i+1} :\equiv \ker(p_i)$ and $B_0 :\equiv B$. Let $P_{-1} :\equiv 0$ and $i_0 :\equiv 0$ in the following.

Proposition 2.3.1. *Let B and A be abelian groups, and assume $P_\bullet$ is a projective resolution of B. Then $\mathcal{E}xt^n(B,A)$ is the n^{th} cohomology of the cochain complex*

$$\mathrm{Ab}(P_\bullet, A) :\equiv (\cdots \to \mathrm{Ab}(P_{n-1}, A) \to \mathrm{Ab}(P_n, A) \to \mathrm{Ab}(P_{n+1}, A) \to \cdots)$$

Proof. The short exact sequence $(B_{k+1} \xrightarrow{i_{k+1}} P_k \xrightarrow{p_k} B_k)$ induces a long exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Ab}(B_k,A) & \xrightarrow{p_k^*} & \mathrm{Ab}(P_k,A) & \xrightarrow{i_{k+1}^*} & \mathrm{Ab}(B_{k+1},A) \\ & & \mathcal{E}xt^1(B_k,A) & \longrightarrow & 0 & \longrightarrow & \mathcal{E}xt^1(B_{k+1},A) \\ & & \mathcal{E}xt^2(B_k,A) & \longrightarrow & 0 & \longrightarrow & \mathcal{E}xt^2(B_{k+1},A) \ \cdots\to \end{array}$$

where the zeros down the middle column appear due to P_k being projective, by Proposition 2.1.8. It follows from the first diagonal arrow that $\mathcal{E}xt^1(B_k, A) = \mathrm{cok}(i^*_{k+1})$, and moreover from the subsequent diagonal arrows that $\mathcal{E}xt^{n+1}(B_k, A) = \mathcal{E}xt^n(B_{k+1}, A)$ for $n \geq 1$. Applying the second equality recursively produces $\mathcal{E}xt^{n+1}(B, A) = \mathcal{E}xt^1(B_n, A)$, and so we may conclude by showing $\mathcal{E}xt^1(B_n, A) = H^n(\mathrm{Ab}(P_\bullet, A))$ for all $n \geq 0$.

The epis $p_{n+1} : P_{n+1} \to B_{n+1}$ become monos $p^*_{n+1} : \mathrm{Ab}(B_{n+1}, A) \to \mathrm{Ab}(P_{n+1}, A)$, and because of this we get that $\ker(\mathrm{Ab}(P_n, A) \to \mathrm{Ab}(P_{n+1}, A)) = \ker(i^*_{n+1})$ by the commuting triangle on the right:

$$\begin{array}{ccccc} \mathrm{Ab}(P_{n-1},A) & \longrightarrow & \mathrm{Ab}(P_n,A) & \longrightarrow & \mathrm{Ab}(P_{n+1},A) \\ & \overset{i_n^*}{\searrow} & \uparrow p_n^* & \overset{i_{n+1}^*}{\searrow} & \uparrow p_{n+1}^* \\ & & \mathrm{Ab}(B_n,A) & & \mathrm{Ab}(B_{n+1},A) \end{array}$$

The long exact sequence we started out with says moreover that this kernel is $\mathrm{Ab}(B_n, A)$, with p^*_n being the kernel inclusion. Consequently,

$$H^n(\mathrm{Ab}(P_\bullet, A)) = \ker(i^*_{n+1}) / \mathrm{im}(\mathrm{Ab}(P_{n-1}, A)) = \mathrm{Ab}(B_n, A) / \mathrm{im}(i^*_n) = \mathrm{cok}(i^*_n)$$

If $n = 0$, then this is $\mathrm{Ab}(B_0, A) \equiv \mathcal{E}xt^0(B, A)$, since $i_0 \equiv 0$, and for $n > 0$ we have

$$\mathrm{cok}(i^*_n) = \mathcal{E}xt^1(B_{n-1}, A) = \mathcal{E}xt^n(B, A)$$

by the equations above. □

3 Internal Ext groups in a topos

In this section we explore some of the statements in the previous section when interpreted into a Grothendieck ∞-topos. Statements about abelian groups in homotopy type theory become statements about sheaves of abelian groups, i.e. $\mathbb{Z}$-modules. The $\mathcal{E}xt$ groups defined Section 2 recover the sheaf $\mathcal{E}xt$ [Har77, Section III-6] defined via injective resolutions. The notions of projectivity and injectivity in homotopy type theory become *internal* projectivity and injectivity in a model, and we study the former notion in the next section. Thus Proposition 2.1.8, and its dual, identify the acyclic modules for sheaf $\mathcal{E}xt$ in both variables, and therefore justify computing sheaf $\mathcal{E}xt$ via internal projective and injective resolutions (since they are acyclic resolutions). The latter is nothing new, because internal and external injectives coincide [Har83, Proposition 2.1]; but we wrap things up with a detailed example of the former, which we haven't seen explicitly mentioned in the literature.

3.1 External and internal projectivity

We start by defining internally projective abelian groups in a topos, and then we demonstrate that this notion is incomparable to the usual one of (external) projectivity. Let $\mathscr{E}$ denote a topos and $\mathrm{Ab}_{\mathscr{E}}$ the category of abelian groups in $\mathscr{E}$. We'll use exponential notation F^G for the internal Hom in $\mathrm{Ab}_{\mathscr{E}}$.

Definition 3.1.1. An abelian group $P \in \mathrm{Ab}_{\mathscr{E}}$ is **internally projective** if the exponential functor $-^P : \mathrm{Ab}_{\mathscr{E}} \to \mathrm{Ab}_{\mathscr{E}}$ preserves epimorphisms.

Contrast this with the notion of P being *externally* projective, i.e. that the external Hom $\mathrm{Ab}_{\mathscr{E}}(P, -) : \mathrm{Ab}_{\mathscr{E}} \to \mathrm{Ab}$ preserves epimorphisms.

We demonstrate that the two notions of projectivity are incomparable through two examples.

Proposition 3.1.2. *Let G be a nontrivial group, and let $\mathscr{E}$ be the topos of G-sets. The constant G-set at $\mathbb{Z}$ is internally projective but not externally projective in* $\mathrm{Ab}_{\mathscr{E}}$.

Proof. It is immediate that $\mathbb{Z}$ is internally projective since it represents the identity functor on $\mathrm{Ab}_{\mathscr{E}}$. Let $\mathbb{Z}G$ denote the free abelian group on G acting on itself on the right (i.e. contravariantly). The natural epimorphism $\mathbb{Z}G \to \mathbb{Z}$ fails to have a section because G is nontrivial, so $\mathbb{Z}$ cannot be externally projective. □

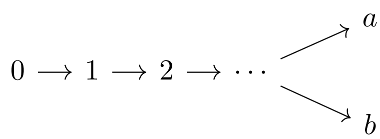
Remark 3.1.3. The previous proposition implies that $\mathcal{E}xt^i_{\mathbb{Z}G}(\mathbb{Z}, -) = 0$ for $i > 1$, since $\mathbb{Z}$ is internally projective. On the other hand, it is well known that the corresponding external Ext groups may be nontrivial. If for example $G = \mathbb{Z}/2$, then we have

$$\mathrm{Ext}^*_{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z}/2) = H^*(\mathbb{Z}/2, \mathbb{Z}/2) = (\mathbb{Z}/2)[u_1]$$

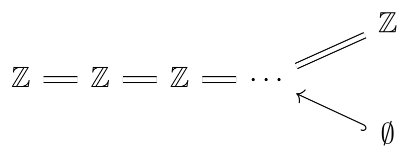
where u_1 has degree 1, so the above is a graded ring which is nontrivial in all degrees.

The example of an externally projective abelian group which fails to be internally projective is more involved. The following is a version in $\mathrm{Ab}_{\mathscr{E}}$ of an example in $\mathscr{E}$ due to Todd Trimble[1].

Consider the poset $\mathbf{C} := \mathbb{N} * \{a, b\}$ where a and b are greater than all $n \in \mathbb{N}$:



and let $\mathscr{E}$ be the category of presheaves on $\mathbf{C}$. Write $y : \mathbf{C} \to \mathscr{E}$ for the Yoneda embedding, and write $\mathbb{Z}(-) : \mathscr{E} \to \mathrm{Ab}_{\mathscr{E}}$ for the functor which pointwise constructs the free abelian group on a set. In particular, we may depict $\mathbb{Z}y(a)$ as follows:



Proposition 3.1.4. $\mathbb{Z}y(a)$ *is externally projective but not internally projective in* $\mathrm{Ab}_{\mathscr{E}}$.

[1] Written out at the nLab's presentation axiom (rev. 46).

Proof. It is immediate that $\mathbb{Z}y(a)$ is externally projective, since it represents evaluation at a. To show that $\mathbb{Z}y(a)$ isn't internally projective, we construct an epimorphism $\sigma : F \to G$ that isn't preserved by $(-)^{\mathbb{Z}y(a)}$.

Let $F : \mathbf{C}^{\mathrm{op}} \to \mathrm{Ab}$ be the following $\mathbb{Z}$-module, with index-wise inclusions:

$$\begin{array}{ccccccc} & & & & & & 0 \\ & & & & & \swarrow & \\ \bigoplus_{n\in\mathbb{N}} \mathbb{Z} & \hookleftarrow & \bigoplus_{n\geq 1} \mathbb{Z} & \hookleftarrow & \bigoplus_{n\geq 2} \mathbb{Z} \hookleftarrow \cdots & & \\ & & & & & \nwarrow & \\ & & & & & & 0 \end{array}$$

Define $G(n) := \mathbb{Z}$ for $n \in \mathbb{N}$ and $G(a) := 0 =: G(b)$, with all maps on $\mathbb{N}$ being identities. Then we have an epimorphism $\sigma : F \to G$ given by addition at $n \in \mathbb{N}$ and identities at a and b. However $\mathrm{Ab}_{\mathscr{E}}(G, F) = 0$ since any such map must factor through $\lim_n F(n) = 0$ on the $\mathbb{N}$-part of $\mathbf{C}$ (and is necessarily 0 on a and b).

Using that $\mathbb{Z} : \mathscr{E} \to \mathrm{Ab}_{\mathscr{E}}$ is strong monoidal, we see that

$$G = \mathbb{Z}(y(a) \times y(b)) = \mathbb{Z}y(a) \otimes \mathbb{Z}y(b)$$

and consequently that

$$F^{\mathbb{Z}y(a)}(b) = \mathrm{Ab}_{\mathscr{E}}(\mathbb{Z}y(a) \otimes \mathbb{Z}y(b), F) = \mathrm{Ab}_{\mathscr{E}}(G, F) = 0$$

On the other hand, $G^{\mathbb{Z}y(a)}(b) = \mathrm{Ab}_{\mathscr{E}}(\mathbb{Z}y(a) \otimes \mathbb{Z}y(b), G) = \mathrm{Ab}_{\mathscr{E}}(G, G)$ contains at least two elements: 0 and id_G. This means that $\sigma^{\mathbb{Z}y(a)}$ cannot be an epimorphism, since it isn't one at b. □

3.2 Computing sheaf $\mathcal{E}xt$ via internal projectives

We conclude by computing a nontrivial $\mathcal{E}xt^2$ sheaf using an internal projective resolution. Let $\mathscr{E}$ be the Sierpinski topos, i.e. the category of presheaves on a single arrow $(0 \to 1)$, or equivalently, sheaves on the two-point space S with a single closed point.

Lemma 3.2.1. *The functor* $-^{\mathbb{Z}y(0)} : \mathrm{Ab}_{\mathscr{E}} \to \mathrm{Ab}_{\mathscr{E}}$ *sends a presheaf* $F := (F(0) \leftarrow F(1))$ *to* $F^{y(0)} = (F(0) \xleftarrow{\mathrm{id}} F(0))$*, the constant presheaf at the zero component. In particular,* $\mathbb{Z}y(0)$ *is internally projective.*

The proof is straightforward.

Remark 3.2.2. A $\mathbb{Z}$-module F over a space X is **finite locally free** if there exists an open cover $(U_i)_{i\in I}$ of X such that the restriction of F to each U_i is free of finite rank, i.e. $F|_{U_i} \simeq \mathbb{Z}^{n_i}$, where the right-hand side is a constant sheaf on U_i. It is well-known [Har77, Proposition III-6.5] that sheaf $\mathcal{E}xt$ can be computed via (finite) locally free resolutions. However, the internal projective $\mathbb{Z}y(0)$ is not locally free since any open cover of the Sierpinski space S must contain S itself, and $\mathbb{Z}y(0)$ is not constant on S.

We now construct an internal projective resolution $P_\bullet$ of $B := (0 \leftarrow \mathbb{Z}/2) \in \mathrm{Ab}_{\mathscr{E}}$, aiming to compute a certain nontrivial $\mathcal{E}xt^2$. For reasons pointed out in the remark above, this resolution will not be locally free.

Diagrams in $\mathscr{E}$ are written with 1 at the top and 0 at the bottom, with the arrow pointing downwards due to contravariance. We have a short exact sequence in $\mathrm{Ab}_{\mathscr{E}}$,

$$\begin{array}{ccccc} \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & \longrightarrow & \mathbb{Z}/2 \\ \downarrow{\scriptstyle 2} & & \| & & \downarrow \\ \mathbb{Z} & = & \mathbb{Z} & \longrightarrow & 0 \end{array}$$

which we'll denote $K \to \mathbb{Z}y(1) \to B$.

One can check that K is not internally projective, and so it makes sense to choose the internal projective cover $p : \mathbb{Z}y(0) \oplus \mathbb{Z}y(1) \to K$ drawn as the right square below:

$$\begin{array}{ccccc} 0 & \dashrightarrow & \mathbb{Z} & \xrightarrow{1} & \mathbb{Z} \\ \downarrow & & \downarrow{\scriptstyle (0,1)} & & \downarrow{\scriptstyle 2} \\ \mathbb{Z} & \overset{(2,-1)}{\dashrightarrow} & \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{1+2} & \mathbb{Z} \end{array}$$

The left square is the kernel of p, thus we have a short exact sequence:

$$\mathbb{Z}y(0) \xrightarrow{f} \mathbb{Z}y(0) \oplus \mathbb{Z}y(1) \xrightarrow{p} K$$

where the kernel $\mathbb{Z}y(0)$ is internally projective. This means we have found a complete internal projective resolution $P_\bullet$ of B:

$$0 \to \mathbb{Z}y(0) \xrightarrow{f} \mathbb{Z}y(0) \oplus \mathbb{Z}y(1) \to \mathbb{Z}y(1) \to B \to 0$$

Now we compute $H^2(P_\bullet, \mathbb{Z}y(0))$, which is $\mathcal{E}xt^2(B, \mathbb{Z}y(0))$ by Proposition 2.3.1.

Proposition 3.2.3. $H^2(P_\bullet, \mathbb{Z}y(0)) = (0 \leftarrow \mathbb{Z}/2)$.

Proof. Since the resolution has length two, $H^2(P_\bullet, \mathbb{Z}y(0)) = \mathbb{Z}y(0)^{\mathbb{Z}y(0)}/\operatorname{im}(f^*)$ is the cokernel of f^*. Note that the codomain of f^* is $\mathbb{Z}y(0)^{\mathbb{Z}y(0)} \simeq \mathbb{Z}y(1)$, by our description of exponentiating with $\mathbb{Z}y(0)$ in Lemma 3.2.1. As for the domain, we further have:

$$\mathbb{Z}y(0)^{\mathbb{Z}y(0) \oplus \mathbb{Z}y(1)} \;\simeq\; \mathbb{Z}y(0)^{\mathbb{Z}y(0)} \oplus \mathbb{Z}y(0)^{\mathbb{Z}y(1)} \;\simeq\; \mathbb{Z}y(1) \oplus \mathbb{Z}y(0)$$

Now we determine f^* on the left below, from f on the right:

$$\begin{array}{ccc} \mathbb{Z} & \xleftarrow[f_1^*]{} & \mathbb{Z} \\ \| & & \downarrow{\scriptstyle (1,0)} \\ \mathbb{Z} & \xleftarrow[f_0^*]{} & \mathbb{Z} \oplus \mathbb{Z} \end{array} \qquad\qquad \begin{array}{ccc} 0 & \longrightarrow & \mathbb{Z} \\ \downarrow & & \downarrow{\scriptstyle (0,1)} \\ \mathbb{Z} & \xrightarrow{(2,-1)} & \mathbb{Z} \oplus \mathbb{Z} \end{array}$$

It suffices to think about f_0^* since this determines f_1^*. By definition,

$$f_0^* : \mathrm{Ab}_{\mathscr{E}}((\mathbb{Z}y(0) \oplus \mathbb{Z}y(1)) \otimes \mathbb{Z}y(0), \mathbb{Z}y(0)) \to \mathrm{Ab}_{\mathscr{E}}(\mathbb{Z}y(0) \otimes \mathbb{Z}y(0), \mathbb{Z}y(0))$$

is pre-composition with $f \otimes \mathbb{Z}y(0)$. Since tensoring is pointwise, everything happens in degree 0. This reduces the problem to computing the image of

$$(2, -1)^* : (\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}) \to (\mathbb{Z} \to \mathbb{Z})$$

which sends the two projections to multiplication by 2 and -1, respectively. Therefore,

$$f_0^*(a, b) = 2a - b : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$$

by the identifications made at the start. This determines $f_1^* : \mathbb{Z} \to \mathbb{Z}$ as multiplication by 2, and consequently the cokernel of f^* is $(0 \leftarrow \mathbb{Z}/2)$, as desired:

$$\begin{array}{ccccc} \mathbb{Z}/2 & \longleftarrow & \mathbb{Z} & \xleftarrow{\;2\;} & \mathbb{Z} \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle (1,0)} \\ 0 & \longleftarrow & \mathbb{Z} & \xleftarrow{\;2-1\;} & \mathbb{Z} \oplus \mathbb{Z} \end{array}$$

□

Theorem 3.2.4. $\mathrm{Ab}_{\mathscr{E}}$ *has internal homological dimension* 2*.*

In the proof we use the notation $\mathbb{Z}[-]$ for the free abelian group on a set; avoid confusing this with $\mathbb{Z}(-)$ which denotes the free abelian group on a presheaf. Let U denote the functor sending an abelian group to its underlying set.

Proof. Consider a general $\mathbb{Z}$-module $F := (F(0) \leftarrow F(1))$. We wish to construct an internal projective resolution $P_\bullet$ of F that ends at P_2. First we construct an internal projective cover $P_0 \to F$ on the right,

$$\begin{array}{ccccc} G(1) & \dashrightarrow & \mathbb{Z}[UF(1)] & \longrightarrow & F(1) \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle (0,1)} & & \downarrow \\ G(0) & \dashrightarrow & \mathbb{Z}[UF(0)] \oplus \mathbb{Z}[UF(1)] & \longrightarrow & F(0) \end{array}$$

whose kernel G is on the left. Since subgroups of free abelian groups are free, $G(0)$ and $G(1)$ are both free and we may choose sets of generators g_0 and g_1 so that $G(0) = \mathbb{Z}[g_0]$ and $G(1) = \mathbb{Z}[g_1]$ as abelian groups. This gives a natural candidate for an internal projective cover P_1 of G,

$$\begin{array}{ccccc} 0 & \dashrightarrow & \mathbb{Z}[g_1] & \xrightarrow{\;\mathrm{id}\;} & G(1) \\ \downarrow & & \downarrow{\scriptstyle (0,1)} & & \downarrow{\scriptstyle g} \\ K(0) & \dashrightarrow & \mathbb{Z}[g_0] \oplus \mathbb{Z}[g_1] & \xrightarrow{\;\mathrm{id}+g\;} & G(0) \end{array}$$

whose kernel K is on the left. Now, $K(0)$ must be a free abelian group and we may choose a generating set k_0 so that $K \simeq \mathbb{Z}(y(0) \times k_0)$. Hence K is internally projective, and gives the last step P_2 of an internal projective resolution of F. □

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