A First Look at Geometric Group Theory

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Abstract

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1 Introduction

We provide a short introduction to some basic ideas and themes of geometric group theory. Considering that this remains an active, wide-ranging research area, it goes without saying that any attempt to fence it in is doomed to exclude a large amount of interesting mathematics. However, in order to get anywhere, we need to settle on some (probably narrow) view of the topic. To this end, we declare *Geometric Group Theory* to be that part of mathematics which deals with the question "what can be learned about a finitely-generated group by studying the large-scale geometry of its Cayley graph?" It is only to this perspective that we attempt an introduction. The following quotation, taken from [8], describes this viewpoint in rather poetic terms.

> "A group G with a given system of generators carries a unique maximal left invariant distance function for which the distance from each generator and its inverse to the identity is 1. This distance function, called the word metric associated to the given system of generators, makes G as subject to a geometric scrutiny as any other metric space.
>
> This space may appear boring and uneventful to a geometer's eye since it is discrete and the traditional (e.g. topological and infinitesimal) machinery does not run in G. To regain the geometric perspective one has to change one's position and move the observation point far away from G. Then the metric in G seen from the distance d becomes the original distance divided by d and as d tends to infinity the points in G coalesce into a connected continuous solid unity which occupies the visual horizon without any gaps or holes and fills our geometer's heart with joy..."
>
> "...one may start to feel uncomfortable by realizing how much structure has been lost as one passed from G to the quasi-isometry class of G with its word metric. Indeed, one barters here the rigid crystalline beauty of a

group for a soft and flabby chunk of geometry where all measurements have built-in errors. But something amazing and unexpected happens here as was discovered by Mostow in 1968: the quasi-isometric (or large-scale) geometry turns out to be far more rich and powerful than appears at first sight. In fact one believes nowadays that most essential elements of an infinite group are quasi-isometry invariant."

– Mikhail Gromov

2 History

An early intrusion of geometrical ideas into group theory occurred in the work of Max Dehn in the early 20th century. In [3], motivated by problems in knot theory, Dehn was the first to pose some of the basic algorithmic questions concerning group presentations. In particular:

> "*The identity problem*[1]: An element of the group is given as a product of generators. One is required to give a method whereby it may be decided in a finite number of steps whether this element is the identity or not."
>
> -Max Dehn (translation by John Stillwell)

It is a sad fact of life that the word problem of a finitely-generated group is not generally solvable, i.e. there may not exist an algorithm which can be used to determine when two words represent the same group element. Rather strikingly, concrete presentations are known which exhibit this pathological behaviour. In [1], one finds an example of the form

$$G = \langle \text{ 10 generators } \mid \text{ 27 explicit relations } \rangle$$

whose word problem is not solvable.

What is it that causes a group to have a hard word problem? The basic issue is that, for a given word w representing 1, it may be necessary to insert a large number of relations, meanwhile vastly increasing the length of the w, before one succeeds in reducing w to 1. Here, we find some suggestion that the large-scale geometry of the Cayley graph might play a role since, in cases where $w = 1$ has a long proof, a large amount of the Cayley graph is used. Therefore, it is not unreasonable to suspect that, if we are able to exercise some kind of control over the geometry of the Cayley graph, the word problem may become more tractable.

Indeed, it is by looking at the Cayley graph that Dehn was able to prove in [3] that the fundamental group of the genus g surface S_g is solvable (in linear time) by what is now known as *Dehn's algorithm.* This group has a standard presentation with $2g$ generators and one relation

$$\pi_1(S_g) = \langle\ a_1, \ldots, a_n; b_1, \ldots, b_n \mid a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_n b_n a_n^{-1} b_n^{-1}\ \rangle.$$

arising from the picture (Figure 2) of S_g as an identification space of a $(4g)$-gon. The single relation reflects the contractibility of the boundary cycle.

[1] Nowadays, usually called the *word problem.*

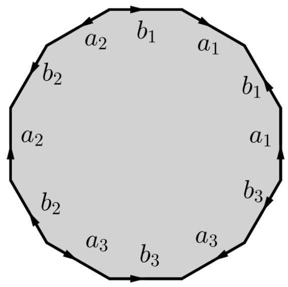

Figure 1: Planar diagram for surface of genus 3.

Although, ultimately, the validity of Dehn's algorithm can be established purely through combinatorial means (see [4]) the original proof made use of a choice of constant negative curvature metric on S_g. The universal cover of $\widetilde{S}_g$ is then a hyperbolic plane in which the Cayley graph of $\pi_1(S_g)$ can be inscribed. It was through consideration of this geometric situation that Dehn first discovered his eponymous algorithm.

Another development which set the stage for geometric group theory was gradual realization during the mid-1900s that, under certain natural hypotheses to be fleshed out in a later section, a discrete group G acting by isometries on a metric space X will actually resemble the space X at large scales. This result is often referred to as the *Schwarz-Milnor Lemma* or, sometimes, the *Fundamental Observation of Geometric Group Theory*, in both cases following the terminology of the influential book [2]. As is noted in that account, this result is actually rather difficult to attribute because various authors, Albert Schwarz and John Milnor among them, have published results of this nature in slightly different contexts.

Another precursor to geometric group theory was the discovery of rigidity phenomena in negatively curved Riemannian geometry, especially the landmark theorem of Mostow[2]. For a nice survey, see [13]. To appreciate Mostow's remarkable result, one needs to understand how spaces of constant curvature can be described as quotients of the model geometries. We briefly summarize this. By a *constant curvature manifold*, let us understand a connected, complete Riemann manifold with constant (sectional) curvature equal to $1, 0$ or -1. It turns out that every simply-connected constant curvature manifold X is isometric to one of S^n, $\mathbb{R}^n$ or $\mathbb{H}^n$, following the curvature. Now, given any constant curvature manifold M, the universal cover of M is isometric to one of these spaces X, and so M can be identified with X/Γ, where $\Gamma \subseteq \mathrm{Isom}(X)$ is the fundamental group of M. All constant curvature manifolds arise in this way. Furthermore, by lifting isometries to the universal cover, one has that $M_1 = X/\Gamma_1$ and $M_2 = X/\Gamma_2$ are isometric Riemann manifolds if and only if Γ_1 and Γ_2 are conjugate in $\mathrm{Isom}(X)$.

Generally, one expects there should exist plenty of pairs of constant curvature manifolds which are the same as topological manifolds, but different as Riemann manifolds. For example, any pair of linearly independent vectors in $\mathbb{R}^2$ determine commuting translation operators, hence an embedding $\iota : \mathbb{Z}^2 \hookrightarrow \mathbb{R}^2 \subseteq \mathrm{Isom}(\mathbb{R}^2)$. The quotient

[2] Sadly, George Mostow (1923-2017) passed away while I was researching this topic.

$\mathbb{R}^2/\iota(\mathbb{Z}^2)$ is always a topological torus, but two such subgroups $\iota(\mathbb{Z}^2), \iota'(\mathbb{Z}^2) \subseteq \mathbb{R}^2$ are only conjugate in $\mathrm{Isom}(\mathbb{R}^2)$ when they are rigid rotations of one another. So, at the isometry level, there are many, many distinct tori. Because of this expectation, it was a big surprise in 1968 when Mostow proved

Theorem 1 (Mostow Rigidity). *In dimension* 3 *and up, it holds that two closed Riemann manifolds of constant curvature* -1 *with the same fundamental group must be isometric.*

In particular, such manifolds are identical metrically if and only if they are identical topologically. The Mostow rigidity theorem was highly influential and attracted the attention of many outstanding mathematicians to the area. Indeed, Mostow's theorem played a role in the work for which each of Margulis, Perelman and Thurston received their Fields medals.

One can also not ignore the influence which the groundbreaking work of Thurston in low-dimensional topology[3] in the 1980s has had on geometric group theory in general, and Gromov in particular. The two had a collaboration in the 1980s which resulted in the two papers [11] and [10]. By Thurston's own account [14], he and Gromov were considering a number of similar questions at the time, in particular the problem of recovering group-theoretic information from information about the growth rate:

> "I was thinking about the problem of deducing structure of a group from the growth rate in the '70's, when Gromov was still in Russia. When we first met, soon after he arrived, we had much in common that we'd been thinking about, but he had not yet proven the theorem concerning groups of polynomial growth. I was stuck on trying to analyze groups of quadratic growth."
>
> -Bill Thurston

This brings us, after all, to the work of Gromov and the emergence of geometric group theory as an industry unto itself. The papers [6] and [8] and especially the monograph [7] have proven extremely influential and could be regarded as the birth of the subject. In [6], he proves his famous theorem on groups of polynomial growth. In one [7], he defines what is a hyperbolic group and initiates their study. In [8], one finds his vision for the study of groups up to quasi-isometry which was quoted in the introduction.

3 The Word Metric and the Cayley Graph

Geometric group theory begins with the definition of the word metric. Let S be a finite generating set for a group G. The generating set S induces a *length function* $|\cdot|_S : G \to \{0, 1, 2, \dots\}$ by setting $|g|_S = n$ where $g = s_1 \dots s_n$ is the shortest presentation of g as a word in $S \cup S^{-1}$. By convention, the product of the empty word equals 1 so that $|1|_S = 0$. Given the length function, there is a unique way to define a left-invariant metric d_S on G such that $d_S(1, g) = |g|_S$. One simply defines $d_S(g, h) = |g^{-1}h|_S$.

[3] Demonstrating, among other things, the special importance of hyperbolic geometry.

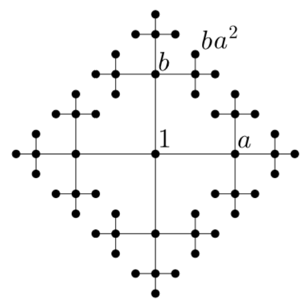

Figure 2: Standard Cayley graph for $F_2 = \langle a, b \rangle$.

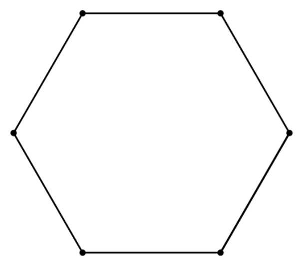

Figure 3: Possible Cayley graph for $\mathbb{Z}/6\mathbb{Z}$ and S_3.

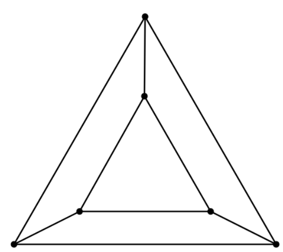

Figure 4: Another possible Cayley graph for $\mathbb{Z}/6\mathbb{Z}$ and S_3.

The distance function d_S is easily visualized as the path metric of the *Cayley graph* of G which, for us, is the undirected graph with vertex set consisting of the elements of G and an edge joining $g, h \in G$ whenever $g = sh$ for some $s \in S \cup S^{-1}$. A simple, but important, example is the case where $G = F_n$ is free of rank n and $S = \{a_1, \ldots, a_n\}$ is a free generating set. In this case, the Cayley graph is an infinite tree in which every vertex has degree $2n$, as shown in Figure 3.

For finite groups, the Cayley graph does precious little to recover the structure of the group. For example, the presentations $\mathbb{Z}/6\mathbb{Z} = \langle 1 \rangle$ and $S_3 = \langle$ 2 reflections $\rangle$ both have a 6-cycle as the Cayley graph. On the other hand, the presentations $\mathbb{Z}/6\mathbb{Z} = \langle 2, 3 \rangle$ and $S_3 = \langle$ 1 rotation, 1 flip $\rangle$ both have a triangular prism as the Cayley graph. The situation becomes much more interesting when we go to infinite, but still finitely-generated, groups. Even though the Cayley graph of a finitely-generated group is not well-defined, because different generating sets can give different Cayley graphs, it turns out there is a precise sense in which the large-scale structure of the Cayley graph is independent of the chosen generating set. The basic jumping off point is the following observation:

Fact 2. *If S and S' are two finite generating sets for a group G, then there exist constants $A, B > 0$ such that the path metrics d_S and $d_{S'}$ satisfy $d_S \leq A \cdot d_{S'}$ and $d_{S'} \leq B \cdot d_S$.*

To see this, one simply uses the fact that each element of S can be expressed as a word in S', and vice versa. In this way, any word w in S can be converted into a word in S' by substituting each S-generator for its expression in terms of S'-generators. Since only finitely many generators are involved here, the length of the word w can only increase by some bounded constant factor. The above fact tells us that the Cayley graph of a finitely-generated group is a well-defined metric space up to the notion of a quasi-isometry, which we now define.

Definition 3. A not-necessarily-continuous map of metric spaces $\phi : X \to Y$ is called a *quasi-isometry* if there exist constants constants $A \geq 1, B \geq 0, R \geq 0$ such that the estimate $\frac{1}{A} \cdot d(x, y) - B \leq d(\phi(x), \phi(y)) \leq A \cdot d(x, y) + B$ holds for all $x, y \in X$ and such that, for any $y \in Y$, there exists an $x \in X$ with $d(\phi(x), y)) \leq R$.

In a more casual language, a quasi-isometry is a map which does not distort any distance by more than some fixed affine function, and which is surjective, up to bounded error. One may check that "X is quasi-isometric to Y" $\Leftrightarrow$ "there exists a quasi-isometry $X \to Y$" defines an equivalence relation on metric spaces. Intuitively, metric spaces are quasi-isometric if they look the same at large scales. For example, when we look at $\mathbb{Z}$ from very far away, its points coalesce and it becomes hard to tell it apart from its continuous counterpart $\mathbb{R}$.

Example 4. The inclusion map $\mathbb{Z}^n \to \mathbb{R}^n$ is a quasi-isometry. The "round to the nearest integer" map $\mathbb{R}^n \to \mathbb{Z}^n$ is a quasi-isometry. If T_1 and T_2 are infinite trees such that each vertex in T_i has the same finite degree $\delta_i \geq 3$, then T_1 and T_2 are quasi-isometric (this requires some work to see).

4 The Schwarz-Milnor Lemma

When a group G arises "in the wild", it may not be obvious whether G is finitely-generated. Moreover, even if a fairly canonical finite generating set S *is* known, we may still have very little clue what the corresponding Cayley graph looks like. Thus, if the program of analyzing groups by analyzing the large-scale geometry of their Cayley graphs is to have any chance of success, it is crucially important that we have tools for (1) showing that a group G is finitely generated, and (2) determining the metric structure of the Cayley graph up to quasi-isometry. The Schwarz-Milnor lemma, which we shall now state, is exactly such a device. Provided we have, or can construct, a sufficiently nice action of $G \curvearrowright X$ of G by isometries on a metric space X, the Schwarz-Milnor Lemma tells us (1) that G is finitely generated, and (2) that the Cayley graph is quasi-isometric to X. Conventionally, X is taken to be a geodesically complete space, but, in an attempt to be slightly novel and in order to allow some actions on discrete spaces (which are never geodesically complete), we use a nonstandard definition.

Definition 5. We say that a metric space X is *1-walkable* if it has the following property. Fix any two points $x, y \in X$ and write $d(x,y) = n+r$ where n is a nonnegative integer and $0 \leq r < 1$. Then, there must exist $x = x_0, x_1, x_2, \ldots, x_n, x_{n+1} = y \in X$ such that $d(x_{i-1}, x_i) = 1$ for $i = 1, \ldots, n$ and $d(x_n, y) = r$.

Note that $d(x,y) = \sum_{i=1}^{n+1} d(x_{i-1}, x_i)$ above, the idea being we can walk from x to y in steps of size 1, possibly taking one extra step of size < 1. This definition is designed to cover the following two examples: (1) metric spaces in which every distance is realized as the length of a connecting geodesic, and (2) the vertex set of a graph in its path metric.

Theorem 6 (Schwarz-Milnor Lemma)**.** *Let G act by isometries on a 1-walkable metric space X and suppose the action is*

1. Cobounded: *There exists $x_0 \in X$ and $R \geq 0$ such that $X = \bigcup_{g \in G} \{x : d(gx_0, x) \leq R\}$.*
2. Metrically proper: *For all $x \in X$ and $M \geq 0$, the set $\{g \in G : d(gx, x) \leq M\}$ is finite.*

Then, the finite set
$$S = \{g \in G : d(gx_0, x_0) \leq 2R + 1\}$$
generates G and $g \mapsto gx_0$ defines a quasi-isometry from G in the word metric d_S to X. More precisely, we have the estimate
$$d_S(g,h) - 1 \leq d(gx_0, hx_0) \leq (2R+1) d_S(g,h)$$
for all $g, h \in G$.

One feels that no mathematical document, even an expository one, should fail to prove at least one statement, so we give a proof of this.

Proof. The coboundedness immediately gives that $g \mapsto gx_0$ is surjective up to bounded error so, to show this map is a quasi-isometry, it only remains to establish the affine estimate. Strictly speaking, we also should show that S generates G, but this is equivalent to finiteness of d_S, so will also follow from the bound. Also note that, by left translation invariance of the metrics, we just need to show that, for all $g \in G$,
$$|g|_S - 1 \leq d(gx_0, x_0) \leq (2R+1)|g|_S.$$
First we tackle the inequality on the left. The strategy is illustrated in Figure 4. As in Definition 5 above, we write $d(gx_0, x_0) = n + r$ and find corresponding $x = x_0, x_1, x_2, \ldots, x_{n+1} = gx_0$. By coboundedness, we can find $g_0, g_1, g_2, \ldots, g_{n+1} \in G$ such that $d(g_i x_0, x_i) \leq R$ for $i = 1, \ldots, n+1$. Indeed, for $i = 0, n+1$, we take $g_0 = 1$ and $g_{n+1} = g$ so as to have $g_i x_0 = x_i$ in those cases. Then, for each $i = 1, \ldots, n+1$, we have $d(g_{i-1}^{-1} g_i x_0, x_0) = d(g_i x_0, g_{i-1} x_0) \leq d(g_i x_0, x_i) + d(x_i, x_{i-1}) + d(x_{i-1}, g_{i-1} x_0) \leq 2R + 1$

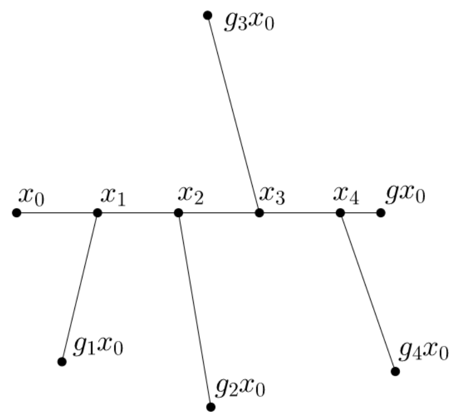

Figure 5: Schwarz-Milnor lemma proof

so that $g_{i-1}^{-1}g_i = s_i \in S_i$. Thus, we have $g = g_{n+1} = s_1s_2 \dots s_{n+1}$, and so $|g|_S - 1 \leq n \leq n + r = d(x, gx_0)$, as desired.

For the inequality on the left, i.e. $d(gx_0, x_0) \leq (2R+1)|g|_S$, suppose $|g|_S = n$ and write $g = s_1s_2 \dots s_n$ where $s_i \in S$. Define $x_i = s_1s_2 \dots s_ix_0$ for $i = 1, \dots, n$. In particular, $x_n = gx_0$. Observe $d(x_i, x_{i-1}) = d(s_ix_0, x_0) \leq 2R + 1$ by the translation invariance, and so, by the triangle inequality, $d(x_0, gx_0) \leq (2R+1)n = (2R+1)|g|_S$, as desired. □

We end this section by giving two examples which we hope indicate the wide applicability of this result.

Example 7. Let M be a closed Riemann manifold and $G = \pi_1(M, p)$ its fundamental group with respect to some basepoint. Let $X = \widetilde{M}$, the universal cover of M. Then, X is a geodesically complete metric space on which G acts isometrically (by deck transformations). It is easy to see this action is cobounded and proper, so the Schwarz-Milnor lemma tells us that $\pi_1(M, p)$ is finitely-generated, and quasi-isometric to the universal cover $\widetilde{M}$. In particular, if M has constant (sectional) curvature, its universal cover must be one of the model geometries $S^n, \mathbb{R}^n, \mathbb{H}^n$. Thus, $\pi_1(M, p)$ is quasi-isometric to either (1) a sphere, (2) Euclidean space, or (3) hyperbolic space. In the first case, the fundamental group is quasi-isometric to a bounded space, hence finite. In the second case, it turns out the fundamental group must have $\mathbb{Z}^n$ as a finite-index subgroup. In the third case, the fundamental group must be a so-called hyperbolic group.

Example 8. Let G be any finitely-generated group with word metric d, and let $H \subseteq G$ be a subgroup of finite index. Then, H acts by isometries on G, simply by left translation. It is easy to see this action is cobounded and proper, so the Schwarz-Milnor lemma tells us that (1) H is also finitely-generated, and (2) H is quasi-isometric to G. This example brings out a basic feature of geometric group theory: in this subject, one cannot tell the difference between a group and any of its finite index subgroups. In particular, all finite groups should be thought of as equivalent to the trivial group.

We hasten to point out that one is certainly able to prove that finite index subgroups of finitely-generated groups are finitely-generated without appealing to the Schwarz-Milnor lemma; this is just for purposes of illustration.

5 Examples of Quasi-isometry Invariant Properties

Since the Cayley graph of a finitely-generated group is a well-defined metric space, modulo quasi-isometry, any metric property of the Cayley graph which is quasi-isometry invariant is also a property of the group. Thus, it is of interest to look for interesting quasi-isometry invariant properties, and to see what these properties can tell us about the original group. In this final section, we rapidly survey three such properties: growth rate, asymptotic dimension and hyperbolicity. The goal is to demonstrate that one can indeed recover group theoretic information from the geometry of the Cayley graph.

Growth Rate

Once a finite-generating set S has been fixed for G, one can consider the growth function

$$f_S(x) = \#\{g \in G : |g|_S \leq x\}$$

which counts the number of elements in the ball of radius x. Of course, if the generating set S changes, so does the function f_S. However, up to some standard notion of equivalence, the *growth rate* of the function f_S is well-defined.

Example 9. If $G = \mathbb{Z}^n$, the growth is like $x \mapsto x^n$. If $G = F_n$, the free group on $n \geq 2$ generators, the growth is exponential (growth rate fails to detect the rank of a free group).

A question which has received considerable attention is whether or not there can exist groups of various intermediate growth rates. Can the growth rate be between two powers x^n and x^{n+1}? Can the growth be super-polynomial and also sub-exponential? The latter is a famous question posed by Milnor in [12] and eventually settled in the affirmative by Grigorchuk in [5].

A highly celebrated theorem concerning growth rates is the 1981 theorem of Gromov on polynomial growth. This a paradigmatic result in geometric group theory; a purely geometrical property is shown to be equivalent to a purely algebraic one.

Theorem 10 ([6])**.** *If a finitely generated group Γ has polynomial growth then Γ contains a nilpotent subgroup of finite index.*

It was known already from [15] that a nilpotent group has polynomial growth. Moreover, in this whole area, one cannot distinguish between the properties of a group and one of its finite index subgroups (see Example 8), so the above theorem of Gromov achieves an exact characterization of groups having polynomial growth.

Asymptotic Dimension

The asymptotic dimension of a metric space X was defined by Gromov in [8] as a large-scale analog of Lebesgue covering dimension.

Definition 11. We write $\operatorname{asdim}(X) \leq n$ if, for every $R > 0$, there is a covering $\mathscr{B}$ of X by bounded sets such that (1) there is a uniform bound on the diameters of the sets in $\mathscr{B}$ and (2) each ball of radius R in X intersects at most $n+1$ of the sets in $\mathscr{B}$.

This definition is very much dual to covering dimension. Considering that, in order to show that $\dim(\mathbb{R}) \leq 1$, one covers $\mathbb{R}$ with extremely small intervals which overlap in pairs. Similarly, in order to show that $\operatorname{asdim}(\mathbb{R}) \leq 1$, one covers $\mathbb{R}$ with extremely long intervals. Some standard examples include:

Example 12. $\operatorname{asdim}(\mathbb{R}^n) = \operatorname{asdim}(\mathbb{Z}^n) = \operatorname{asdim}(\mathbb{H}^n) = n$. The asymptotic dimension of an (infinite) regular tree is 1.

A famous application of this concept is the the following theorem of Guoliang Yu.

Theorem 13 ([16])**.** *Let Γ be a finitely generated group whose classifying space $B\Gamma$ has the homotopy type of a finite CW-complex. If Γ has finite asymptotic dimension as a metric space with a word length-metric, then the Novikov conjecture holds for Γ.*

This was proved by using a refinement of operator K-theory known as *controlled K-theory*[4] to establish the coarse Baum-Connes conjecture for groups of finite asymptotic dimension, and then applying a known descent procedure to get the Novikov conjecture.

Hyperbolicity

The definition of a *hyperbolic group* was first given by Gromov in [6]. Gromov's definition applies not only to groups, but actually to arbitrary metric spaces, thus opening up the study of "negatively curved geometry" in contexts very far from manifold theory.

There are several ways to define what is a hyperbolic metric space. A geometrically appealing approach, which works well when there are enough geodesics (or at least near-geodesics), is to first define what it means for a (geodesic) triangle to be D-thin. This means simply that there is some point p simultaneously within distance D of all three of the triangle's sides. If, for some fixed D, every triangle in the space is D-thin, one says the space is D-hyperbolic. When working up to quasi-isometry, distances can be multiplied, so it no longer makes sense to speak of a D-hyperbolic metric space. However, it turns out that the notion of a *hyperbolic metric space*, i.e. a metric space which is D-hyperbolic for some D, still makes sense up to quasi-isometry.

A great many things are known about hyperbolic groups. For instance:

- Every hyperbolic group can be finitely presented (i.e. presented with finitely many generators and relations).
- Every hyperbolic group has a word problem which is solvable in linear time.

[4] Invented by Yu for this purpose.

- Except in trivial cases[5], every hyperbolic group G contains the free group on 2 generators and so, in particular, has exponential growth.

Let us finish by mentioning that there is also a sense, which we will not make precise here, in which hyperbolicity is a "generic property" for groups. Roughly speaking, if we randomly prescribe the relations in a group with n-generators, the probability of getting a hyperbolic group approaches 1 as $n \to \infty$. This, too, is a theorem of Gromov [9].

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