Research Statement

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November 2023

Abstract

My primary research areas are operator theory and differential geometry. My PhD dissertation (completed in 2021, supported by an NSERC PGS-D scholarship) centered on groupoids and operator algebras associated to singular foliations, building on work of Androulidakis and Skandalis. My postdoctoral research has also branched out into cyclic homology, complex geometry and index theory.

Contents

1 Introduction to singular foliations and holonomy

To give the flavour of my research and introduce major objects and themes, I include an informal overview of foliations and holonomy, highlighting the differences between the singular context, where I have done work, and the regular context.

Definition 1 ([AS09], Definition 1.1)**.** A (possibly singular) *foliation* $\mathcal{F}$ of a smooth manifold M is a locally finitely-generated $C^\infty(M)$-module of compactly-supported, smooth vector fields on M that is closed under Lie bracket.

The set of points accessible from a given point using the flows of the vector fields in $\mathcal{F}$ is called a *leaf*. By work of Stefan and Sussmann ([Ste74], [Sus73]), the leaves of $\mathcal{F}$ constitute a partition of M into immersed submanifolds. If all the leaves have the same dimension, the foliation is said to be *regular*. Otherwise, the foliation is *singular*. Besides being a classical topic in geometry, foliations play an important role in classical mechanics and optimal control theory. In the regular setting, the module of vector fields can be recovered from the partition, but this fails in the singular setting. Indeed, considering different modules which determine the same partition is a prominent theme in my work.

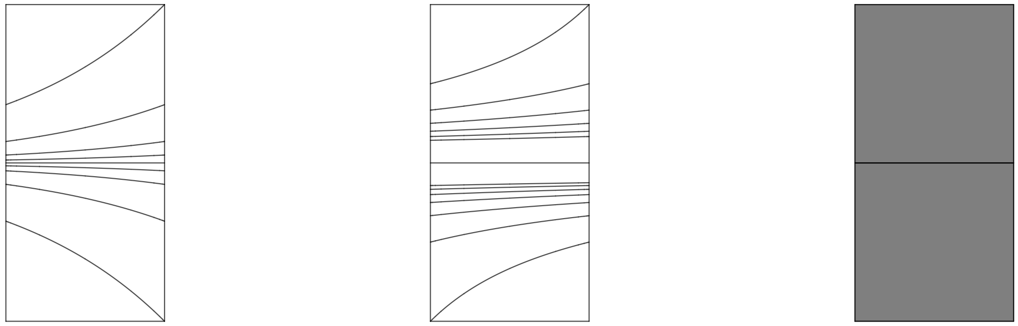
Recall that the solution operators of certain PDEs can be usefully represented by smooth integral kernels, e.g. in the case of the heat equation on a Riemannian manifold M. The value of the kernel at $(x, y) \in M \times M$ may be understood as a measure of how much the operator propagates from y to x. Whereas heat flow propagates in all directions, in other important situations one would like to consider operators which only propagate along the leaves of some foliation, e.g. the level sets of a quantity that is invariant for the time evolution. A kernel representing such an operator should then be a function on the equivalence relation whose classes are the leaves, and not the whole manifold $M \times M$. This motivates the following problem:

Problem 2. *What is a smooth function on the leaf equivalence relation of a given foliation?*

The problem is complicated by the fact the leaf equivalence relation need not be a submanifold of $M \times M$. The reason, and also the remedy, for this failure of smoothness is an interesting and important phenomenon known as *holonomy*. The *holonomy groupoid* $G(\mathcal{F})$ of a foliation $\mathcal{F}$ tries to desingularize the equivalence relation, somewhat in the spirit of blowups in algebraic geometry. When $G(\mathcal{F})$ is a Lie groupoid (the so-called *almost regular* case), the natural solution to Problem 2 is "a smooth function on $G(\mathcal{F})$". The holonomy groupoid was defined for regular foliations by Winkelnkemper [Win83] and extended to singular cases by various authors. A very general construction of $G(\mathcal{F})$ was given by Androulidakis and Skandalis in [AS09]. The use of $G(\mathcal{F})$ in operator theory was pioneered by Connes [Con82].

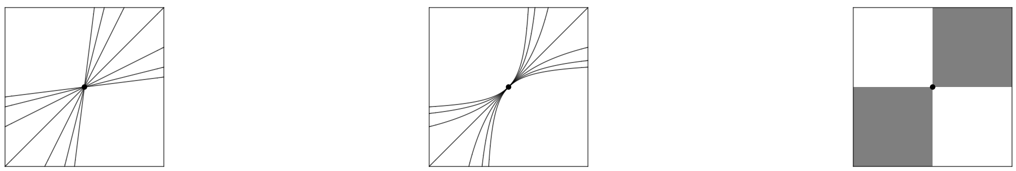
Figure 1 illustrates several foliations of the cylinder $S^1 \times \mathbb{R}$, regarded as having coordinates (x, y), where x is $\mathbb{Z}$-periodic. I have used the notation $\mathcal{F}\{X_1, \ldots, X_n\}$ to denote a foliation generated by a finite set of vector fields $X_1, \ldots, X_n$.

The first two foliations in Figure 1 are regular while the third is singular; its leaves are $S^1 \times (0, \infty)$, $S^1 \times \{0\}$ and $S^1 \times (-\infty, 0)$. All three foliations determine nonsmooth equivalence



$\mathcal{F}\{\frac{d}{dx} + y\frac{d}{dy}\}$ $\mathcal{F}\{\frac{d}{dx} + y^2\frac{d}{dy}\}$ $\mathcal{F}\{\frac{d}{dx}, y\frac{d}{dy}\}$.

Figure 1: Leaves of some foliations of $S^1 \times \mathbb{R}$.



$\mathcal{F}\{\frac{d}{dx} + y\frac{d}{dy}\}$ $\mathcal{F}\{\frac{d}{dx} + y^2\frac{d}{dy}\}$ $\mathcal{F}\{\frac{d}{dx}, y\frac{d}{dy}\}$.

Figure 2: Equivalence relations of some foliations of $S^1 \times \mathbb{R}$, restricted to $T = \{0\} \times \mathbb{R}$.

relations. The issue is visible when we restrict attention to the transversal $T = \{0\} \times \mathbb{R}$ passing through $p = (0, 0)$. The resulting subsets of $T \times T$ are depicted in Figure 2.

The holonomy groupoid, however, is smooth for all these foliations. In terms of Figure 2, what occurs is the problematic point (p, p) at the origin gets blown up and replaced with the *holonomy group* H_p. For a regular foliation, given a point p and a transversal T through the leaf of p, H_p may be viewed as the discrete group consisting of all germs of diffeomorphisms of T fixing p which can be obtained using flows of vector fields in $\mathcal{F}$. For both the regular foliations shown above, H_p is infinite cyclic, and it is easy to imagine how such a replacement can resolve the singularity.

A key difference between the regular and singular settings is that, whereas for regular foliations holonomy is purely a discrete phenomenon, for singular foliations one can also have *continuous holonomy*. For the singular foliation $\mathcal{F}\{\frac{d}{dx}, y\frac{d}{dy}\}$ shown above, H_p is isomorphic to the Lie group $\mathbb{R}$. However, this is just one many foliations of $S^1 \times \mathbb{R}$ whose leaves are $S^1 \times (0, \infty)$, $S^1 \times \{0\}$ and $S^1 \times (-\infty, 0)$. With the exception of some pathological examples, the holonomy group H_p for any such foliation is naturally realized, for some positive integer k, as a one-dimensional subgroup of the group $J^k(\mathbb{R})$ of k-jets of orientation-preserving diffeomorphisms of $\mathbb{R}$ which fix 0. Explicitly, $J^k(\mathbb{R}) = \{a_1 y + a_2 y^2 + \ldots a_\ell y^k : a_i \in \mathbb{R}, a_1 \neq 0\}$

under the operation "compose and truncate". Some examples are tabulated below:

$$\begin{aligned}
\mathcal{F}\{\tfrac{d}{dx}, y\tfrac{d}{dy}\} &\quad\rightsquigarrow\quad H_p \cong \{e^t y : t \in \mathbb{R}\} \subseteq J^1(\mathbb{R}) \\
\mathcal{F}\{\tfrac{d}{dx}, y^2\tfrac{d}{dy}\} &\quad\rightsquigarrow\quad H_p \cong \{y + ty^2 : y \in \mathbb{R}\} \subseteq J^2(\mathbb{R}) \\
\mathcal{F}\{\tfrac{d}{dx} + y\tfrac{d}{dy}, y^2\tfrac{d}{dy}\} &\quad\rightsquigarrow\quad H_p \cong \{e^n y + ty^2 : n \in \mathbb{Z}, t \in \mathbb{R}\} \subseteq J^2(\mathbb{R}) \\
\mathcal{F}\{\tfrac{d}{dx} + y^2\tfrac{d}{dy}, y^4\tfrac{d}{dy}\} &\quad\rightsquigarrow\quad H_p \cong \{y + ny^2 + n^2y^3 + ty^4 : n \in \mathbb{Z}, t \in \mathbb{R}\} \subseteq J^4(\mathbb{R})
\end{aligned}$$

Note that H_p is diffeomorphic to $\mathbb{R}$ in the first two cases and $\mathbb{R} \times \mathbb{Z}$ in the second two cases. The precise details of how (p, p) is blown up into a copy of H_p depend also on two natural orderings of group $J^k(\mathbb{R})$, associated to the postive and negative half lines. As Figure 3 shows, the topological possibilities for the blowup space are actually quite rich, especially given how simple the leaf space of these foliations is. The last two surfaces are not homeomorphic, as can be seen by counting the number of topological "ends".

$\mathcal{F}\{\frac{d}{dx}, y\frac{d}{dy}\}$ $\quad$ $\mathcal{F}\{\frac{d}{dx}, y^2\frac{d}{dy}\}$ $\quad$ $\mathcal{F}\{\frac{d}{dx} + y\frac{d}{dy}, y^2\frac{d}{dy}\}$ $\quad$ $\mathcal{F}\{\frac{d}{dx} + y^2\frac{d}{dy}, y^4\frac{d}{dy}\}$

Figure 3: Holonomy groupoids of some singular foliations of $S^1 \times \mathbb{R}$, restricted to $T = \{0\} \times \mathbb{R}$.

2 Postdoctoral research

2.1 A Dixmier-Malliavin theorem for Lie groupoids (published)

A famous theorem of Dixmier and Malliavin ([DM78], 3.1 Théorème) states that every smooth, compactly-supported function on a Lie group can be expressed as a finite sum in which each term is the convolution (with respect to Haar measure) of two such functions. This result has applications to the representation theory of real reductive groups.

To complete the proof of the main theorem in [Fra20], I needed to know that every element of the smooth convolution algebra of certain singular foliations could be expressed as a finite sum of convolution products. In other words, a Lie groupoid version of the Dixmier-Malliavin theorem was needed. In [Fra22], I took up the general form of this problem and extended Dixmier-Malliavin's result to the setting of arbitrary Lie groupoids.

Theorem 3 ([Fra22])**.** *Let G be a Lie groupoid with a smooth Haar system and form the smooth convolution algebra $C_c^\infty(G)$. Then, every $f \in C_c^\infty(G)$ can be expressed in the form*

$$f = g_1 * h_1 + \ldots + g_n * h_n$$

for some positive integer n and $g_1, h_1, \ldots, g_n, h_n \in C_c^\infty(G)$.

In the same article, I obtained results on the multiplication structure of certain ideals in $C_c^\infty(G)$ arising from functions vanishing to given order along a given invariant submanifold Z of the unit space. These results on ideals are only interesting after one has generalized to the groupoid setting. In the group case, the unit space consists of a single point and these ideals do not arise at all.

2.2 H-Unitality of Smooth Groupoid Algebras (preprint)

In [Fra23a], I was able to extend the techniques of [Fra22] in order to show that the convolution algebra $C_c^\infty(G)$ of smooth, compactly-supported functions on a Lie groupoid G is homologically unital in the sense of Wodzicki. Consequently, this algebra has the excision property for Hochschild and cyclic homology. I was also able to establish homological unitality for the infinite order vanishing ideals $J_Z^\infty \subseteq C_c^\infty(G)$ associated to invariant submanifolds of the unit space discussed above. This is a noncommutative generalization of a classical result of Wodzicki.

My work improves our understanding of localization around invariant subsets in calculations of the cyclic and Hochschild homology of convolution algebras of Lie groupoids. For example, my results yield the following corollary.

Corollary 4 ([Fra23a]). *For any Lie groupoid G, for any closed, invariant subset Z of the unit space of G, the exact sequence*

$$0 \longrightarrow J_Z^\infty \longrightarrow C_c^\infty(G) \longrightarrow C_c^\infty(G)/J_Z^\infty \longrightarrow 0$$

induces corresponding long exact sequences in Hochschild and cyclic homology.

In the same article, I furthermore establish analogous H-unitality and excision results for noncommutative algebras of Whitney functions. Such calculations fall squarely within Connes's noncommutative geometry program. One may see [PPT23] for recent progress in this area.

2.3 The Newlander-Nirenberg theorem for complex b-manifolds (preprint)

Melrose [Mel93] introduced b-geometry (also called log-geometry) as an organizational framework for studying partial differential operators on a smooth manifold M that suffer a first order degeneracy along a given hypersurface Z. The b-tangent bundle ${}^bTM$ is the vector bundle whose sections are smooth vector fields defined on all of M and tangent along Z. Many classical geometries admit "b-analogues" in which the b-tangent bundle fills the role of the usual tangent bundle (so one has symplectic b-geometry, Riemannian b-geometry, etc). Mendoza [Men14] defined a complex b-structure to be an involutive subbundle ${}^bT^{0,1}M$ of the complexified b-tangent bundle such that $\mathbb{C}{}^bTM = \overline{{}^bT^{0,1}M} \oplus {}^bT^{0,1}M$.

In joint work with Tatyana Barron [BF23b], I established that complex b-manifolds have a single local model depending only on dimension. This can be thought of as the Newlander-Nirenberg theorem for complex b-manifolds: there are no "local invariants" in complex b-geometry.

Theorem 5 ([BF23b]). *Around any point in the hypersurface Z of a complex b-manifold M, there are local coordinates $(x_0, y_0, \ldots, x_n, y_n)$, with x_0 vanishing on Z, such that*

$$\tfrac{1}{2}(x_0\partial_{x_0} + i\partial_{y_0}) \qquad \text{and} \qquad \tfrac{1}{2}(\partial_{x_j} + i\partial_{y_j}), j = 1, \ldots, n$$

constitute a local frame for ${}^bT^{0,1}M$.

The challenge in proving this stems from the fact that the analogue of the Dolbeault operator in b-geometry is nonelliptic.

2.4 Automorphisms of complex b^k-manifolds (accepted)

Scott [Sco16] generalized b-calculus by introducing b^k-manifolds, where k encodes the order of degeneracy along the hypersurface $Z \subseteq M$. From the point of view of Scott's theory, ordinary b-geometry is the case $k = 1$. In another joint article with Tatyan Barron [BF23a], I extend Mendoza's definition of complex b-manifold in the spirit of Scott's work by defining what is a complex b^k-manifold for $k > 1$. The case $k = 2$ is of particular interest, having connections to hyperbolic geometry. We then restrict attention to the (real) two-dimensional case and investigate the local and global automorphisms of complex b^k-manifolds. We also discuss candidates for function spaces one can attach to a complex b^k-manifold.

Looking to the future, I expect that complex b^k-manifolds also satisfy a Newlander-Nirenberg theorem analogous to that of [BF23b] and plan to address the problem of local normal forms for complex b^k-manifolds in future work. Some broader research objectives include studying the global structure of complex b^k-manifolds (it is known this can be rich thanks to a holonomy invariant constructed in my PhD thesis) and spaces of b^k-holomorphic functions. For example, considering analogues of classical Bergman spaces leads to a variety of interesting questions relating to spectral theory and dimension counting.

2.5 The fundamental class of certain singular folations (project)

Suppose that $(M, \mathcal{F})$ is a *regular*, codimension-1, transversely-oriented foliation. By a result of Connes ([Con94], Theorem 3.6.9), there is an associated nontorsion K-theory class $[V/\mathcal{F}] \in K_1(C^*(\mathcal{F}))$ called the *transverse orientation class* which may be defined by pushing forward the K-theory orientation class of any appropriate transversal. I have devised a related construction to define the tranverse fundamental class $[V/\mathcal{F}]$ of certain *singular* foliations having exactly one singular leaf of codimension-1. For reasons relating to the Connes-Thom isomorphism [Con81], this class $[V/\mathcal{F}]$ instead lives in $K_0(C^*(\mathcal{F}))$. Morally, this happens because the leaf space is zero-dimensional in this singular context.

Problem 6. *Show that this K-theory class $[V/\mathcal{F}] \in K_0(C^*(\mathcal{F}))$ is also nontorsion (and in particular nonzero).*

Having a nonzero canonically-defined orientation class is desirable because, for example, it provides a path to defining and calculating numerical invariants of the singular foliation by pairing the orientation class with elements of the dual theory. For instance, this opens up the possiblity of defining a Godbillon-Vey invariant in a singular context. Indeed, one natural approach to this problem is to construct a corresponding transverse fundamental class in the cyclic cohomology of $C_c^\infty(G)$, where G is the holonomy groupoid of $\mathcal{F}$, in such a way that the pairing with $[V/\mathcal{F}]$ is nonzero. This follows a similar thread to Connes. However, for reasons relating to the counterpart to the Connes-Thom isomorphism in cyclic theory [ENN88], the transverse fundamental class lives in $H^2(C_c^\infty(G))$ and is therefore represented by a 2-trace. This makes formulas more complicated than in Connes's case.

In this singular context, the holonomy group at any point on the singular leaf is a certain 1-dimensional Lie group $\Gamma_\mathbb{R}$ (an extension of a solvable discrete group Γ by $\mathbb{R}$). In another line of attack, I have shown that Problem 6 is equivalent to showing that a certain homomorphism $\mathrm{ind} : K_1(C^*(\Gamma_\mathbb{R})) \to \mathbb{Z} \oplus \mathbb{Z}$ is compatible with a naturally defined automorphism of $\Gamma_\mathbb{R}$ corresponding to reversing the orientation of the real line. This compatibility can be checked in a variety of examples.

2.6 Characterizing the smooth convolution algebra of singular foliations induced by a Lie groupoids (project)

Let $G \rightrightarrows M$ be a Lie groupoid and let $\mathcal{F}$ be the singular foliation of M induced by G. It is known that the smooth convolution algebra $\mathcal{A}(\mathcal{F})$ of Androulidakis and Skandalis fits into an exact sequence

$$0 \to I \to C_c^\infty(G) \to \mathcal{A}(\mathcal{F}) \to 0$$

(fixing a smooth Haar system on G to make sense of convolution). However, the description of the ideal I is very indirect and involves quantification over an infinite number of relators taking the form of "roof diagrams" (Section 4.3, [AS09]). I am interested in describing the ideal I in more explicit terms. A useful test case for this problem is the singular foliation of $\mathbb{R}^2$ associated to the action of $\mathrm{SL}(2,\mathbb{R})$. Consider the quadratic mapping $Z : \mathbb{R}^2 \to \mathfrak{sl}(2,\mathbb{R})$ defined by

$$Z(x,y) = \begin{bmatrix} xy & -x^2 \\ y^2 & -xy \end{bmatrix}.$$

One may show that Z defines a bi-invariant vector field on the transformation groupoid $\mathbb{R}^2 \rtimes \mathrm{SL}(2,\mathbb{R})$ that generates the singular foliation of $\mathbb{R}^2 \rtimes \mathrm{SL}(2,\mathbb{R})$ consisting of vector fields which are vertical with respect to both the source and target projections. Using this vector field, I was able to obtain the following.

Theorem 7. *Let $\mathcal{F}$ be the singular foliation of $\mathbb{R}^2$ generated by the natural action of $\mathrm{SL}(2,\mathbb{R})$. Then, an element $f \in C_c^\infty(\mathbb{R}^2 \rtimes \mathrm{SL}(2,\mathbb{R}))$ belongs to the kernel of the natural quotient map $C_c^\infty(\mathbb{R}^2 \rtimes \mathrm{SL}(2,\mathbb{R})) \to \mathcal{A}(\mathcal{F})$ if and only if $f = Zg$ for $g \in C_c^\infty(\mathbb{R}^2 \rtimes \mathrm{SL}(2,\mathbb{R}))$.*

The proof of the above result is related to the "Moser trick" of symplectic geometry. I am working on generalizing my arguments in order to obtain a concrete model for the smooth convolution algebra for singular foliations induced by other Lie groupoids.

Problem 8. *Let $G \rightrightarrows M$ be a Lie groupoid, satisfying some reasonable hypotheses, and let $\mathcal{F}$ the singular foliation induced on M. Describe the kernel of the quotient mapping $C_c^\infty(G) \to \mathcal{A}(\mathcal{F})$ in terms of the singular foliation of G consisting of vector fields which are vertical with respect to both the source and target projections.*

2.7 Grassmannian manifolds as singular foliations (project)

Unlike in the setting of regular foliations, there can exist interesting singular foliations having only a finite number of leaves. I am particularly interested in studying Grassmannian manifolds and their cell structures from the perspective of singular foliations. For example, consider the canonical cell structure of the real projective plane:

$$\mathbb{RP}^n = \mathbb{R}^0 \cup \mathbb{R}^1 \cup \mathbb{R}^2 \cup \ldots \cup \mathbb{R}^n.$$

To view this as a singular foliation, one must furthermore specify a module of vector fields inducing the partition. A simple choice is the module $\mathcal{F}$ determined by the action of the n-dimensional Lie group $G \subseteq GL(n+1, \mathbb{R})$ consisting of matrices of the form:

$$\begin{bmatrix} 1 & a_1 & a_2 & a_3 & \ldots & a_n \\ 0 & 1 & a_1 & a_2 & \ldots & a_{n-1} \\ 0 & 0 & 1 & a_1 & \ldots & a_{n-2} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 1 \end{bmatrix}.$$

The orbits of the natural action of G on $\mathbb{RP}^n$ are exactly the parition under consideration. The group G is commutative and, indeed, isomorphic to $\mathbb{R}^n$. The action of G on the top-dimensional leaf is free and transitive and this can be used to show that the holonomy groupoid of the singular foliation associated to this action is just the transformation groupoid $\mathbb{RP}^n \rtimes G$. Correspondingly, the foliation C*-algebra $C^*(\mathcal{F})$ is isomorphic to the crossed product C*-algebra $C(\mathbb{RP}^n) \rtimes G$. The K-theory of $C^*(\mathcal{F})$ can therefore be computed using the Connes-Thom isomorphism [Con81]. One obtains $K_i(C^*(\mathcal{F})) \cong K^{i+n}(\mathbb{RP}^n)$.

In general, Grassmanians and other spaces admitting a natural finite stratications promise to be a rich source of examples for further study.

3 Doctoral research

Note that, except for minor changes, my PhD dissertation [Fra21] reproduces the contents of the articles [Fra22], [Fra20] and Francis [Fra23b].

3.1 The smooth algebra of a one-dimensional singular foliation (preprint)

Given any singular foliation $\mathcal{F}$ of a smooth manifold M, it was shown in [AS09] how to construct a holonomy groupoid $G(\mathcal{F})$, a smooth convolution algebra $\mathcal{A}(\mathcal{F})$ and a C*-algebra[1] $C^*(\mathcal{F})$. In the article [Fra20], I consider a specific family of singular foliations of the real line and obtain a complete classification of their smooth convolution algebras and C*-algebras. The main findings may be summarized as follows.

Theorem 9 ([Fra20], Theorems 3,4,5)**.** *For each positive integer k, let $\mathcal{F}^k_{\mathbb{R}}$ denote the singular foliation of the real line singly-generated by $y^k \frac{d}{dy}$.*

1. *The smooth convolution algebras of the $\mathcal{F}^k_{\mathbb{R}}$ are pairwise nonisomorphic.*
2. *The C*-algebras of the $\mathcal{F}^k_{\mathbb{R}}$ are of two isomorphism types that are determined by the parity of k.*
3. *The C*-algebras of the $\mathcal{F}^k_{\mathbb{R}}$ are represented in a natural way on $L^2(\mathbb{R})$. The images of these representations are pairwise distinct.*

This demonstrates the principle that there can be information stored in the smooth algebra which is washed away when one passes to the C*-algebra.

3.2 On certain singular foliations with finitely many leaves (preprint)

In [Fra23b], I define and analyze a class of singular foliations which I call *transversely order-k foliations*. These are foliations which have exactly one singular leaf L of codimension one around which the transverse structure is modeled on the one-dimensional foliation of $\mathcal{F}^k_{\mathbb{R}}$ in the theorem above.

Unlike in the context of regular foliations, a loop in L does not determine a holonomy transformation in the usual sense of a diffeomorphism germ on a transversal. I show, however, that one does have a well-defined holonomy mapping at the level of $(k-1)$-jets. In this way, I assign an invariant to a transversely-order k foliation taking the form of a homomorphism (well-defined up to conjugation) $\pi_1(L) \to J^{k-1}$ where J^{k-1} denotes the group of $(k-1)$-jets of diffeomorphisms of $\mathbb{R}$ fixing the origin.

Theorem 10 ([Fra23b])**.** *The restriction of a transversely order k foliation to a small neighbourhood of its singular leaf L is uniquely determined by the above invariant. Moreover, the possible values of this invariant are exhausted by transversely order k foliations.*

I furthermore obtain a concrete description of the holonomy groupoid and C*-algebra of a transversely order k foliation in terms of its holonomy invariant.

[1] Actually, multiple C*-completions can be considered, including a reduced and a maximal version. For the examples considered here, all the standard completions agree.

3.3 Subgraph-avoiding minimum decycling sets and k-conversion sets in graphs (published)

My mathematical interests are quite varied and I enjoy interacting with researchers from other areas. A collaboration with Professors Kieka Mynhardt and Jane Wodlinger resulted in the article [FMW19] (published in the Australian Journal of Combinatorics). The main result of this article is stated below. A *minimum decycling set* in a graph G (finite, with no loops or multiple edges) is a set of vertices which breaks every cycle of G and has as few vertices as possible.

Theorem 11 ([FMW19]). *With the exception of the complete graph on $r+1$ vertices, every finite graph G with maximum degree r has a minimum decycling set S whose induced subgraph $G[S]$ does not contain any $(r-2)$-regular subgraph.*

This result has several corollaries including the classical Brooks' theorem, and the statement that (except for the complete graph on 4 vertices), every graph of maximum degree 3 admits a minimum decycling set which is also an independent set.

4 Master's research

My master's thesis [Fra14] provides a self-contained account of the following result of Connes:

Theorem 12 ([Con81]). *Suppose A is a C^*-algebra with $\mathbb{R}$-action α and α-invariant trace τ. Then, $\widehat{\tau}_*\phi^1_\alpha[u] = \frac{1}{2\pi i}\tau(\delta(u)u^{-1})$ holds, where $\widehat{\tau}$ is the dual trace, and $\phi^1_\alpha : K_1(A) \to K_0(A \rtimes_\alpha \mathbb{R})$ is the Connes-Thom isomorphism, $\delta = \frac{d}{dt}|_{t=0}$, and u is a suitable unitary.*

One novel aspect of my thesis is its avoidance of von Neumann algebraic methods. Another is a modern proof of following quantum mechanical theorem:

Theorem 13 (Bargmann-Wigner, c. 1960). *If α_t is a strongly continuous 1-parameter group of $*$-automorphisms of the compact operators on a separable Hilbert space, then there exists a strongly continuous 1-parameter unitary group U_t such that $\alpha_t = \mathrm{Ad}(U_t)$.*

Together with Stone's theorem on 1-parameter unitary groups, the above forms part of the chain of reasoning that justifies the practice of expressing the time-evolution of a quantum system by Schrodinger's equation for a given Hamiltonian $i\hbar\frac{\partial}{\partial t}\Psi = H\Psi$. The usual method of proof is to first implement α_t by a measurable family of unitaries, then correct that family to a 1-parameter group using a measurable, circle-valued cocycle and, finally, appeal to an automatic continuity result of von Neumann. I gave a new proof of the above theorem based on Connes' lemma that, for any projection e, one can explicitly define a continuous unitary cocycle u_t such that $\mathrm{Ad}(u_t) \circ \alpha_t$ leaves e invariant.

Also during my masters, I wrote an expository article [Fra12] on a 1910 theorem of Brouwer characterizing the Cantor set and 1920 theorem of Sierpinski characterizing the rationals. Due to its expository nature, I did not seek its publication, but it nonetheless attracted positive attention on mathoverflow [Kjo] and was cited in the general topology literature [EHW18].

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