On the Melnikov functions and limit cycles near a double homoclinic loop with a nilpotent saddle of order $\hat{m}$ ★

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Abstract

For a centrally symmetric near-Hamiltonian system, we develop a method for computing all the coefficients in the expansions of three Melnikov functions near a double homoclinic loop. Moreover, we give a new estimation on the lower bound of $H(2\hat{n}, 5)$ for $11 \leq \hat{n} \leq 23$, where $H(2\hat{n}, 5)$ is the maximal number of limit cycles for a kind of Liénard system, $\dot{x} = y$, $\dot{y} = -g(x) + \varepsilon f(x)y$, with $\deg g(x) = 5$ and $\deg f(x) = 2\hat{n}$.

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1. Introduction

In 1900, Hilbert posed 23 mathematical problems in the Second International Congress of Mathematicians, among which the 16th problem has not been completely solved. The second part of the Hilbert’s 16th problem is to find the maximal number of limit cycles, denoted by \( H(n) \), and their distribution for a polynomial system,

\[
\dot{x} = P_n(x, y), \quad \dot{y} = Q_n(x, y),
\]

where \( P_n(x, y) \) and \( Q_n(x, y) \) are polynomials in \( x, y \) with \( \max \{ \deg P_n, \deg Q_n \} = n \). Many results have been obtained on this problem for relevant works and introductions (see, for example, [1–4]). The Hilbert’s 16th problem is extremely difficult and is even not completely solved for the simplest case \( n = 2 \). For this problem, Smale [5] pointed out that “Except for the Riemann hypothesis, it seems to be the most elusive of Hilbert’s problems”. Thus, the weakened Hilbert’s 16th problem was proposed by Arnold [6] to reduce the difficulty of solving the problem, which is described below.

Consider the following near-Hamiltonian system,

\[
\dot{x} = H_y + \varepsilon p(x, y, \delta), \quad \dot{y} = -H_x + \varepsilon q(x, y, \delta), \tag{1.1}
\]

where \( \varepsilon \in \mathbb{R} \) is a bounded small parameter, \( H, p \) and \( q \) are analytic functions in \( x \) and \( y \), \( \delta \in \mathbb{R}^n (n \in \mathbb{N}^*) \) is a vector parameter. When \( \varepsilon = 0 \), system (1.1) becomes a Hamiltonian system,

\[
\dot{x} = H_y, \quad \dot{y} = -H_x. \tag{1.2}
\]

Suppose system (1.2) has at least a family of periodic orbits, defined by \( H(x, y) = h, h \in J \) with \( J \) an open interval. The first order Melnikov function of system (1.1) is

\[
M(h, \delta) = \oint_{H(x,y)=h} q \, dx - p \, dy.
\]

The weakened Hilbert’s 16th problem is to find the maximal number of isolated zeros of the above Melnikov function, which has a close relation with the number of limit cycles of system (1.1). One of the efficient methods of finding the number of isolated zeros of the Melnikov function is to study the expansion of \( M(h, \delta) \) near a center or a closed orbit. This method has been generalized to consider piecewise smooth systems, for example, see [7–9]. General theory on the expansions and the computation of the first few coefficients can be found in the survey article [10]. Many works about the expansion of the Melnikov function have been done (see, for example, [11–17]). In the following, we briefly summarize some results on the expansion of the Melnikov function near a homoclinic loop and a double homoclinic loop, respectively.

Suppose system (1.2) has a homoclinic loop \( L_0 \) passing through a hyperbolic saddle at the origin, and \( H(x, y) = h, 0 < -h \ll 1 \) defines a family of periodic orbits. Roussarie [13] proved that \( M(h, \delta) \) has the following expansion near the homoclinic loop,

\[
M(h, \delta) = \sum_{j \geq 0} (c_{2j} + c_{2j+1} h \ln |h|) h^j, \quad 0 < -h \ll 1.
\]
It is easy to see that \( \mathcal{C}_0 = \oint_{\mathcal{L}_0} q \, dx - p \, dy \). The formulas for the coefficients \( c_1 \) and \( c_2 \) were obtained by Han and Ye in [14], and \( c_3 \) by Han et al. in [15]. Then no progress was reported on the coefficients \( c_j, j \geq 4 \), for almost one decade. In 2017, Tian and Han [18] developed an approach to compute the coefficients \( c_j \) for \( j \geq 4 \) under some restrictions. The method is also valid for studying double homoclinic bifurcation.

In this paper, we generalize this method to a degenerate case. For this purpose, suppose system (1.2) has a nilpotent saddle of order \( \hat{m} (\hat{m} \in \mathbb{N}^*) \) at the origin. Without loss of generality, further assume the Hamiltonian function has the following form,

\[
H(x, y) = \frac{1}{2} y^2 + \sum_{i+j \geq 3} h_{ij} x^i y^j, \tag{1.3}
\]

for \((x, y)\) near the origin. There exists a unique analytic function \( \varphi(x) = \sum_{j \geq 2} e_j x^j \) such that \( H_y(x, \varphi(x)) = 0 \) for \(|x|\) small. By Definition 1 in Han et al. [19] we know that for \(|x|\) small \( H(x, \varphi(x)) \) can be expressed as

\[
H(x, \varphi(x)) = \sum_{j \geq k} h_j x^j, \quad h_k < 0, \quad k = 2\hat{m} + 2. \tag{1.4}
\]

Suppose the equation \( H(x, y) = 0 \) defines a double homoclinic loop \( L_0^* \) passing through the nilpotent saddle. By [17], we know that there are two possibilities for \( L_0^* \), i.e., cuspidal type and smooth type. See Fig. 1. Consequently, system (1.1) has three Melnikov functions,

\[
M(h, \delta) = \oint_{\mathcal{L}_h} q \, dx - p \, dy, \quad \tilde{M}(h, \delta) = \oint_{\tilde{\mathcal{L}}_h} q \, dx - p \, dy, \quad M^*(h, \delta) = \oint_{L^*_h} q \, dx - p \, dy. \tag{1.5}
\]

In 2012, Han et al. [17] gave the formal expansions of \( M, \tilde{M} \) and \( M^* \) for \( 0 < |h| \ll 1 \). In particular, for \( \hat{m} = 1 \), the authors obtained the formulas for the first seven coefficients in the expansions of \( M, \tilde{M} \), and the formulas for the first five coefficients in the expansion of \( M^* \). Later, for any \( \hat{m} \geq 1 \), Yang et al. [20] obtained the formulas for the first few coefficients in the
expansions of $M$, $\tilde{M}$ and $M^*$. It is hard to get higher-order coefficients, and there are no any progress reported so far.

In this paper, we develop a method for computing all the coefficients in the expansions of the three Melnikov functions $M$, $\tilde{M}$ and $M^*$ for $0 < |h| < 1$ for the case that system (1.1) is centrally symmetric and has a double homoclinic loop $L^*_0$. The idea on deriving the formulas for the coefficients of $h^j$, $j \geq 2$ is motivated by the work of Tian and Han [18]. We further give general conditions for generating limit cycles bifurcating from the double homoclinic loop $L^*_0$.

Let $H(n, m)$ denote the maximal number of limit cycles for a generalized Liénard system [21],

$$\dot{x} = y, \quad \dot{y} = -g(x) - \varepsilon f(x)y,$$

where $\varepsilon > 0$ is sufficiently small, $f(x)$ and $g(x)$ are polynomials in $x$ with $\deg g = m$ and $\deg f = n$.

There have been many studies on the lower bound of $H(n, m)$, see [22–26] and references therein. For $m = 5$, Kazemi et al. [27] proved that $H(4, 5) \geq 3$. Xu and Li [28,29] gave some lower bounds of $H(n, 5)$ for $n = 2, 4, 6, 8$ and 10. Xiong [30] studied the number of limit cycles near a compound loop with a cusp and a hyperbolic saddle and gave lower bounds of $H(n, 5)$ for $1 \leq n \leq 10$. Recently, Li and Yang [31] gave new estimations on the lower bounds of $H(n, 5)$ for $10 \leq n \leq 20$.

However, for $n \geq 21$, to the best of our knowledge, there are no any results for the lower bounds of $H(n, 5)$. As an application of our main results, we obtain the lower bounds of $H(n, 5)$ for $22 \leq n \leq 46$ with $n$ even.

In the next section, we present some lemmas which are needed for proving our main results. The main results and their proofs are given in Section 3. In Section 4, an example is particularly considered to demonstrate the applicability of our theoretical results.

2. Preliminaries

For the asymptotic property of the three Melnikov functions in (1.5) at $h = 0$, Han et al. [17] obtained the following results.

**Lemma 2.1.** ([17]) Let (1.3) and (1.4) hold, and $L^*_0$, defined by $H(x, y) = 0$, be a double homoclinic loop of cuspidal type. Then the functions $M(h, \delta)$, $\tilde{M}(h, \delta)$ and $M^*(h, \delta)$ in (1.5) have the following expansions.

(i) For $0 < -h \ll 1$,

$$M(h, \delta) = \varphi(h, \delta) - \frac{h \ln |h|}{2k} I^*_{1, \frac{k}{2} - 1}(h) + |h|^\frac{1}{2} + \frac{1}{2} \sum_{r=0}^{k-2} \sum_{r \neq \frac{k}{2} - 1} \tilde{A}_r I^*_{1, r}(h)|h|^\frac{r}{2},$$

(ii) For $0 < -h \ll 1$,

$$\tilde{M}(h, \delta) = \tilde{\varphi}(h, \delta) + (-1)^{k/2} \frac{h \ln |h|}{2k} I^*_{1, \frac{k}{2} - 1}(h) + |h|^\frac{1}{2} + \frac{1}{2} \sum_{r=0}^{k-2} \sum_{r \neq \frac{k}{2} - 1} (-1)^r \tilde{A}_r I^*_{1, r}(h)|h|^\frac{r}{2},$$

$$M^*(h, \delta) = \varphi(h, \delta) - \frac{h \ln |h|}{2k} I^*_{1, \frac{k}{2} - 1}(h) + |h|^\frac{1}{2} + \frac{1}{2} \sum_{r=0}^{k-2} \sum_{r \neq \frac{k}{2} - 1} \tilde{A}_r I^*_{1, r}(h)|h|^\frac{r}{2},$$

30
where \( \varphi(h, \delta) \in C^\infty \), \( \tilde{A}_r \) are constants satisfying Lemma 2.5 in [17] and for \( 0 \leq r \leq k - 1 \) \( I_{1,r}^* \) satisfy

\[
I_{1,r}^*(h) = \sum_{m,j \geq 0} \tilde{r}_{mk+r,j} \alpha_{mk+r,j}^* \beta_{mk+r}^* h^{j+m},
\]

(2.2)

with \( \tilde{r}_{mk+r,j} \) given by (18) in [17] and

\[
\alpha_{i,j}^* = \begin{cases} 
\frac{\frac{3}{2}k \cdot \frac{5}{2}k \cdots \frac{2j+1}{2}k}{(\frac{3}{2}k + i + 1) \cdots (\frac{2j+1}{2}k + i + 1)}, & i \geq 0, \ j \geq 1, \\
1, & i \geq 0, \ j = 0,
\end{cases}
\]

(2.3)

\[
\beta_{r_{mk+r}}^* = \begin{cases} 
(-1)^m (r + 1)(k + r + 1) \cdots ((m - 1)k + r + 1), & m \geq 1, \ 0 \leq r \leq k - 1, \\
(\frac{1}{2}k + r + 1)(\frac{1}{2}k + r + 1) \cdots (\frac{2m+1}{2}k + r + 1), & m = 0, \ 0 \leq r \leq k - 1.
\end{cases}
\]

(ii) For \( 0 < h \ll 1 \),

\[
M^*(h, \delta) = \begin{cases} 
\varphi^*(h, \delta) + 2h^{\frac{1}{2}+\frac{1}{4}} \sum_{r=0}^{\frac{k}{2}-1} \tilde{A}_r h^{\frac{3r}{2}} J_{1,r}^*(h), & \text{if } r_1 \text{ is not an integer}, \\
\varphi^*(h, \delta) - \frac{h \ln h}{k} J_{1,r_1}^*(h) + 2h^{\frac{1}{2}+\frac{1}{4}} \sum_{r=0}^{\frac{k}{2}-1} \tilde{A}_r h^{\frac{3r}{2}} J_{1,r}^*(h), & \text{if } r_1 \text{ is an integer},
\end{cases}
\]

(2.4)

where \( 2r_1 = \frac{k}{2} - 1 \), \( \varphi^*(h, \delta) \in C^\infty \), \( \tilde{A}_r \) are constants satisfying Lemma 2.5 in [17], and for \( 0 \leq r \leq \frac{k}{2} - 1 \), \( J_{1,r}^* \) satisfy

\[
J_{1,r}^*(h) = \sum_{m=0, j \geq 1 \text{ odd}} \sum_{i=\frac{mk}{r} + r}^{(1)} \tilde{r}_{i,j} \alpha_{i,j} \beta_i h^{m + \lfloor \frac{j}{2} \rfloor},
\]

(2.5)

with \( r_{i,j}^{(1)} \) given by (20) in [17] and

\[
\alpha_{i,j} = \begin{cases} 
1, & i \geq 0, \ j = 1, \\
1 - \frac{\frac{3}{2}k \cdot \frac{5}{2}k \cdots \frac{3}{2}k}{(\frac{3}{2}k + 2i + 1)(\frac{5}{2}k + 2i + 1) \cdots (\frac{3}{2}k + 2i + 1)}, & i \geq 0, \ j \geq 3 \text{ odd},
\end{cases}
\]

(2.6)

\[
\beta_i = \begin{cases} 
1, & 0 \leq i \leq \frac{k}{2} - 1, \\
(-1)^m (2i + 1 - k)(2i + 1 - 2k) \cdots (2i + 1 - mk), & i = \frac{mk}{2} + r, \ m \geq 1, \\
(2i + 1 + \frac{1}{2}k)(2i + 1 - \frac{1}{2}k) \cdots (2i + 1 - \frac{2m-3}{2}k), & 0 \leq r \leq \frac{k}{2} - 1.
\end{cases}
\]
Further, Yang et al. [20] gave the formulas of the first few coefficients in the expansions of $M, \tilde{M}$ and $M'$. The expansions of $M(h, \delta)$ and $\tilde{M}(h, \delta)$ are given in the following lemma.

**Lemma 2.2.** ([20]) Let (1.3) and (1.4) hold with $h_k < 0$ ($k = 2\tilde{m} + 2, \tilde{m} \geq 1$) and $L_0^* = L_0 \cup \tilde{L}_0$ be a double homoclinic loop of cuspidal type. Then for the expansions of $M(h, \delta)$ and $\tilde{M}(h, \delta)$ in (2.1) we have

$$
M(h, \delta) = c_0 + \sum_{r=0}^{\frac{k}{2} - 2} c_{r+1} |h|^{\frac{r+1}{2}} + \frac{1}{2} c_r h \ln |h| + c_{\frac{k}{2} + 1} h + \sum_{r=\frac{k}{2}}^{k-2} c_{r+2} |h|^{\frac{r+1}{2}} + \frac{1}{2} c_r h \ln |h| + O(h^2), \quad 0 < -h \ll 1,
$$

$$
\tilde{M}(h, \delta) = \tilde{c}_0 + \sum_{r=0}^{\frac{k}{2} - 2} (-1)^r c_{r+1} |h|^{\frac{r+1}{2}} + \frac{1}{2} c_r h \ln |h| + \tilde{c}_{\frac{k}{2} + 1} h + \sum_{r=\frac{k}{2}}^{k-2} (-1)^r c_{k+1+r} |h|^{\frac{r+1}{2}} + \frac{1}{2} c_r h \ln |h| + O(h^2), \quad 0 < -h \ll 1,
$$

where

$$
c_0 = M(0, \delta) = \oint_{L_0} q \, dx - p \, dy, \quad c_{r+1} = \tilde{A}_r \tilde{r}_{r,0}, \quad r = 0, 1, \ldots, \frac{k}{2} - 2, \quad c_{\frac{k}{2}} = -\frac{1}{2k} \tilde{r}_{\frac{k}{2} - 1,0},
$$

$$
c_{\frac{k}{2} + 1} = \oint_{L_0} \left( p_x + q_y - \tilde{b}_{0,0} - \tilde{b}_{1,0} x - \tilde{b}_{2,0} x^2 - \cdots - \tilde{b}_{\frac{k}{2} - 1,0} x^{\frac{k}{2} - 1} \right) \, dt + \sum_{i=1}^{k} O_1(c_i),
$$

$$
c_{r+2} = \tilde{A}_r \tilde{r}_{r,0}, \quad r = \frac{k}{2}, \frac{k}{2} + 1, \ldots, k - 2,
$$

$$
c_{k+1+r} = \frac{3}{2} \tilde{A}_r \left( \frac{(r+1)\tilde{r}_{k+r,0} - \frac{3}{2} k \tilde{r}_{r,1}}{\tilde{f}_{\frac{k}{2} + 1}} \right), \quad r = 0, 1, \ldots, \frac{k}{2} - 2,
$$

$$
c_{\frac{k}{2} + \frac{k}{2}} = \frac{1}{8k} \left( \tilde{r}_{\frac{k}{2} - 1,0} - 3 \tilde{r}_{\frac{k}{2} - 1,1} \right),
$$

$$
\tilde{c}_0 = \tilde{M}(0, \delta) = \oint_{L_0} q \, dx - p \, dy,
$$

$$
\tilde{c}_{\frac{k}{2} + 1} = \oint_{L_0} \left( p_x + q_y - \tilde{b}_{0,0} - \tilde{b}_{1,0} x - \tilde{b}_{2,0} x^2 - \cdots - \tilde{b}_{\frac{k}{2} - 1,0} x^{\frac{k}{2} - 1} \right) \, dt + \sum_{i=1}^{k} O_1(c_i),
$$

(2.8)
in which \(O_1(c)\) denotes \(c\) multiplied by a constant, and based on the programs in [16], \(\tilde{r}_{i,0}, \tilde{r}_{i,1}\) and \(\tilde{b}_{i,0}\) can be obtained by the method given in [17]. For example, some \(\tilde{b}_{i,0}\) are given as follows:

\[
\begin{align*}
\tilde{b}_{0,0} &= a_{10} + b_{01}, & \tilde{b}_{1,0} &= 2a_{20} + b_{11}, & \tilde{b}_{2,0} &= 3a_{30} + b_{21} - h_{21}(a_{11} + 2b_{02}), \\
\tilde{b}_{3,0} &= 4a_{40} + b_{31} - 2h_{21}(a_{21} + b_{12}) + (2h_{12}h_{21} - h_{31})(a_{11} + 2b_{02}), & \cdots.
\end{align*}
\] (2.9)

Suppose the conditions in Lemma 2.2 hold, and further assume system (1.2) has two elementary centers \(C_1\) and \(C_2\), surrounded by \(L_0\) and \(L_0\), respectively. Let \(h_{c_i} \equiv H(C_i), i = 1, 2\). By Han [11], one knows that for \(0 < h - h_{c_2} \ll 1\), \(M(h, \delta)\) has an expansion of the form,

\[
M(h, \delta) = \sum_{l \geq 0} b_l (h - h_{c_2})^{l+1}.
\] (2.10)

The coefficients can be obtained by using the programs in [32], which show that

\[
b_0 = k_0(p_x + q_y)(C_2, \delta),
\] (2.11)

where \(k_0\) is a constant.

For \(\hat{m} = 1\), in the process of computing \(c_i(\delta)\) for \(i = 1, 2, 4, 5, 6\), the necessary step is to get \(\tilde{r}_{0,0}, \tilde{r}_{1,0}, \tilde{r}_{2,0}, \tilde{r}_{4,0}, \tilde{r}_{5,0}, \tilde{r}_{0,1}\) and \(\tilde{r}_{1,1}\). To obtain more coefficients in the expansions of the three Melnikov functions, the values of \(\tilde{r}_{i,j}\) in (2.2) and \(r_{i,j}^{(1)}\) in (2.5) need to be obtained first. The results given in Yang et al. [20] show the relationship between \(\tilde{r}_{i,j}\) and \(r_{i,j}^{(1)}\).

**Lemma 2.3.** ([20]) Let (1.3) and (1.4) hold. Then for \(\tilde{r}_{i,j}\) in (2.2) and \(r_{i,j}^{(1)}\) in (2.5), we have

\[
r_{i,2j+1}^{(1)} = \tilde{r}_{2i,j}, \quad i \geq 0, \quad j \geq 0.
\] (2.12)

In this paper, \(\frac{\partial M}{\partial h}\) is needed to obtain more coefficients in the expansion of the Melnikov function. In 1994, Han [33] proved the following result.

**Lemma 2.4.** For the Melnikov function \(M(h, \delta)\), we have

\[
\frac{\partial M}{\partial h} = \oint_{L_h} (p_x + q_y) dt.
\]

In the next section, we will use the above lemmas to derive the formulas of the coefficients in the expansions of the three Melnikov functions \(M, \tilde{M}\) and \(M^*\).

### 3. Main results

In the case that \(H(x, y) = 0\) defines a double homoclinic loop \(L_0^h\) of cuspidal type, except the coefficients of the terms \(h^i\) for \(i \geq 2\), we give the formulas for all the other coefficients in the expansions of \(M(h, \delta), \tilde{M}(h, \delta)\) and \(M^*(h, \delta)\). Especially, if system (1.1) is centrally symmetric, under some conditions we present the formulas of the coefficients of \(h^i\) for \(i \geq 2\).

We first consider the coefficients in the expansions of \(M(h, \delta)\) and \(\tilde{M}(h, \delta)\)
Lemma 3.1. Let (1.3) and (1.4) hold and $L_0^*$ be a double homoclinic loop of cuspidal type as shown in Fig. 1(a). The expansions of $M(h, \delta)$ and $\tilde{M}(h, \delta)$ in (1.5) are given below:

$$M(h, \delta) = c_0 + \sum_{r=0}^{k-2} c_{r+1} |h|^{\frac{r+1}{2}} + \frac{1}{2} c_{\frac{1}{2}} h \ln |h|$$

$$+ \sum_{i \geq 0} \left( c_{\frac{3}{2} + k + i} |h|^{\frac{3}{2} + \frac{1}{2} + i} + \sum_{r=\frac{1}{2}}^{k-2} c_{r+2 + k + i} |h|^{\frac{r+1}{2} + \frac{1}{2} + i} \right),$$

$$\tilde{M}(h, \delta) = \tilde{c}_0 + \sum_{r=0}^{k-2} (-1)^r c_{r+1} |h|^{\frac{r+1}{2} + \frac{1}{2} + \frac{1}{2}} + (-1)^{\frac{1}{2} + 1} c_{\frac{3}{2}} h \ln |h|$$

$$+ \sum_{i \geq 0} \left( \tilde{c}_{\frac{3}{2} + k + i} |h|^{\frac{3}{2} + \frac{1}{2} + i} + (-1)^r c_{r+2 + k + i} |h|^{\frac{r+1}{2} + \frac{1}{2} + i} \right),$$

for $0 < -h \ll 1$, where $c_j, j = 0, 1, \cdots, \frac{3}{2} k$ and $\tilde{c}_0, \tilde{c}_{\frac{1}{2} + 1}$ are given in (2.8), and

$$c_{r+2 + k + i} = (-1)^r \tilde{A}_r \sum_{m+j=i} \tilde{r}_{mk+r+j} \alpha^*_{mk+r+j} \beta^*_{mk+r}, \; r = \frac{k}{2} - 1, \cdots, k - 2,$$

$$c_{k+1 + r + k + i} = (-1)^{i+1} \tilde{A}_r \sum_{m+j=i+1} \tilde{r}_{mk+r+j} \alpha^*_{mk+r+j} \beta^*_{mk+r}, \; r = 0, 1, \cdots, \frac{k}{2} - 2,$$

$$c_{\frac{1}{2} + k + i} = -\frac{1}{2k} \sum_{m+j=i+1} \tilde{r}_{mk+\frac{k}{2} - 1,j} \alpha^*_{mk+\frac{k}{2} - 1,j} \beta^*_{mk+\frac{k}{2} - 1}.$$  (3.2)

Proof. By (2.3) and (2.2) we know that

$$\alpha^*_{mk+r,j} = 1, \; \beta^*_{mk+r} = 1, \; \text{for} \; m = 0, \; j = 0,$$

$$L_{1, r}^*(h) = \tilde{r}_{r,0} + \sum_{m+j \geq 1} \tilde{r}_{mk+r+j} \alpha^*_{mk+r+j} \beta^*_{mk+r} h^{j+m},$$
where \(0 \leq r \leq k - 1\). Then, we have

\[
|h|^{\frac{1}{2} + \frac{1}{2}} \sum_{r=0}^{k-2} \tilde{A}_r I_{1,r}^* (h)|h|^{\frac{r}{2}} = \sum_{r=0}^{\frac{k}{2} - 2} \tilde{A}_r \tilde{r}_{r,0} |h|^{\frac{r}{2} + \frac{1}{2}} + \sum_{r=\frac{k}{2}}^{k-2} \tilde{A}_r \tilde{r}_{r,0} |h|^{\frac{r}{2} + \frac{1}{2}} + \sum_{r=0}^{k-2} \tilde{A}_r \tilde{r}_{r,0} |h|^{\frac{r}{2} + \frac{1}{2}} + \sum_{r=\frac{k}{2}}^{k-2} \tilde{A}_r \tilde{r}_{r,0} |h|^{\frac{r}{2} + \frac{1}{2}} \]

\[
+ \sum_{r=0}^{k-2} \tilde{A}_r \sum_{m+j=i \geq 1 \atop m,j \geq 0} \tilde{r}_{mk+r,j} \alpha_{mk+r,j}^* \beta_{mk+r}^* (-1)^i |h|^{\frac{r}{2} + \frac{1}{2} + \frac{1}{2} + i}
\]

\[
+ \sum_{r=0}^{\frac{k}{2} - 2} \sum_{r=0}^{k-2} \tilde{A}_r \sum_{m+j=i \geq 1 \atop m,j \geq 0} \tilde{r}_{mk+r,j} \alpha_{mk+r,j}^* \beta_{mk+r}^* (-1)^i |h|^{\frac{r}{2} + \frac{1}{2} + \frac{1}{2} + i}
\]

and

\[
\frac{h \ln |h|}{2k} J_{1, \frac{k}{2} - 1}^* = \frac{h \ln |h|}{2k} \tilde{r}_{\frac{k}{2} - 1,0} + \frac{1}{2k} \sum_{i \geq 1 \atop m+j=i \atop m,j \geq 0} \tilde{r}_{mk+r,j} \alpha_{mk+r,j}^* \beta_{mk+r}^* (-1)^i |h|^{\frac{r}{2} + \frac{1}{2} + \frac{1}{2} + i} \]

Thus, by (2.1), we get

\[
M(h, \delta) = c_0 + \sum_{r=0}^{\frac{k}{2} - 2} \tilde{A}_r \tilde{r}_{r,0} |h|^{\frac{r}{2} + \frac{1}{2}} - \frac{h \ln |h|}{2k} \tilde{r}_{\frac{k}{2} - 1,0} + \frac{1}{2k} \sum_{i \geq 1 \atop m+j=i \atop m,j \geq 0} \tilde{r}_{mk+r,j} \alpha_{mk+r,j}^* \beta_{mk+r}^* (-1)^i |h|^{\frac{r}{2} + \frac{1}{2} + \frac{1}{2} + i}
\]

\[
+ \sum_{r=0}^{k-2} \tilde{A}_r \sum_{m+j=i \geq 1 \atop m,j \geq 0} \tilde{r}_{mk+r,j} \alpha_{mk+r,j}^* \beta_{mk+r}^* (-1)^i |h|^{\frac{r}{2} + \frac{1}{2} + \frac{1}{2} + i}
\]

\[
+ \sum_{r=0}^{\frac{k}{2} - 2} \sum_{r=0}^{k-2} \tilde{A}_r \sum_{m+j=i \geq 1 \atop m,j \geq 0} \tilde{r}_{mk+r,j} \alpha_{mk+r,j}^* \beta_{mk+r}^* (-1)^i |h|^{\frac{r}{2} + \frac{1}{2} + \frac{1}{2} + i}
\]

\[
- \frac{1}{2k} \sum_{m+j=i \geq 1 \atop m,j \geq 0} \tilde{r}_{mk+r,j} \alpha_{mk+r,j}^* \beta_{mk+r}^* (-1)^i + 2 \ln |h| \right).
\]

35
It is easy to see that (3.2) hold. Similarly, a direct computation using the formula of \( \hat{M} \) in (2.1) yields the expansion of \( \hat{M} \) in (3.1). The proof is complete. 

Next we study the coefficients in the expansion of \( M^*(h, \delta) \).

**Lemma 3.2.** Let (1.3) and (1.4) hold and \( L_0^* \) be a double homoclinic loop of cuspidal type. For \( 0 < h \ll 1 \), if \( r_1 \) is not an integer (i.e., \( \hat{m} \) is odd), the expansion of \( M^*(h, \delta) \) in (1.5) has the form,

\[
M^*(h, \delta) = c_0^* + \sum_{r=0}^{\frac{k}{4}-1} c_{r+1}^* h^{\frac{1}{2} + \frac{1}{2} + \frac{2r}{k}} + \sum_{i \geq 0} \left( c_{\frac{k}{2} + 1 + (\frac{k}{2} + 1)i}^* h^{i+1} \right) + \sum_{r=\frac{k}{2}}^{\frac{k}{4}-1} c_{r+2+(\frac{k}{2}+1)i}^* h^{\frac{1}{2} + \frac{1}{2} + \frac{2r}{k} + i} + \sum_{r=0}^{\frac{k}{2}+r+2+(\frac{k}{2}+1)i} c_{\frac{k}{2} + r+2+(\frac{k}{2}+1)i}^* h^{\frac{1}{2} + \frac{1}{2} + \frac{2r}{k} + i+1},
\]

where \( c_0^* \), \( c_{r+1}^* \) and \( c_{\frac{k}{2}+1}^* \) are given in [17] and [20] respectively, with

\[
c_0^* = c_0 + \tilde{c}_0, \quad c_{r+1}^* = 2\tilde{A}_r \tilde{A}_{2r} 0 = \frac{2\tilde{A}_r}{A_{2r}} c_{2r+1}, \quad r = 0, \cdots, \frac{k}{4} - 1,
\]

\[
c_{\frac{k}{2}+1}^* = \int_{L_0^*} \left( p_x + q_y - \sum_{i=0}^{\frac{k}{2}-2} \tilde{b}_i 0 x^i \right) dt + \sum_{i=1}^{\frac{k}{2}-1} O_1(c_i),
\]

and

\[
c_{r+2+(\frac{k}{2}+1)i}^* = (-1)^i \frac{2\tilde{A}_r}{A_{2r}} c_{2r+2+ki}, \quad r = \frac{k}{4}, \cdots, \frac{k}{2} - 1,
\]

\[
c_{\frac{k}{2}+r+2+(\frac{k}{2}+1)i}^* = (-1)^{i+1} \frac{2\tilde{A}_r}{A_{2r}} c_{k+1+2r+ki}, \quad r = 0, \cdots, \frac{k}{4} - 1.
\]

If \( r_1 \) is an integer (i.e., \( \hat{m} \) is even), for \( 0 < h \ll 1 \), \( M^*(h, \delta) \) in (1.5) has the expansion of the form,

\[
M^*(h, \delta) = c_0^* + \sum_{r=0}^{\frac{k}{2} - \frac{3}{2}} c_{r+1}^* h^{\frac{1}{2} + \frac{1}{2} + \frac{2r}{k}} + c_{\frac{k}{2} + 1}^* h \ln h
\]

\[
+ \sum_{i \geq 0} \left( c_{\frac{k}{2} + \frac{1}{2} + (\frac{k}{2} + 1)i}^* h^{i+1} \right) + \sum_{r=\frac{k}{4} + \frac{1}{2}}^{\frac{k}{2} - 1} c_{r+2+(\frac{k}{2}+1)i}^* h^{\frac{1}{2} + \frac{1}{2} + \frac{2r}{k} + i} + \sum_{r=0}^{\frac{k}{2}+r+2+(\frac{k}{2}+1)i} c_{\frac{k}{2} + r+2+(\frac{k}{2}+1)i}^* h^{\frac{1}{2} + \frac{1}{2} + \frac{2r}{k} + i+1} + c_{\frac{k}{2} + \frac{1}{2} + (\frac{k}{2} + 1)i}^* h^{i+2} \ln h,
\]

36
where

\[
c_0^* = c_0 + \tilde{c}_0, \quad c_{r+1}^* = 2\tilde{A}_r \tilde{r}_{2r,0} = \frac{2\tilde{A}_r}{\tilde{A}_{2r}} c_{2r+1}, \quad r = 0, \ldots, \frac{k}{4}, \quad c_{k/2}^* = 2c_1^*.
\]

\[
c_{r+1}^* = \oint_{L_0^*} \left( p_x + q_y - \sum_{i=0}^{\frac{k}{2}-1} \tilde{b}_{i,0} x^i \right) dt + \sum_{i=1}^{\frac{k}{2}} O(c_i),
\]

are given in [20], and

\[
c_{r+2 + (\frac{k}{2} + 1)i}^* = \frac{2\tilde{A}_r}{\tilde{A}_{2r}} (-1)^i c_{2r+2+ki}, \quad r = \frac{k}{4}, \ldots, \frac{k}{2} - 1,
\]

\[
c_{\frac{k}{2} + r+2 + (\frac{k}{2} + 1)i}^* = \frac{2\tilde{A}_r}{\tilde{A}_{2r}} (-1)^{i+1} c_{k/2+1+2r+ki}, \quad r = 0, \ldots, \frac{k}{4}, \frac{3}{2},
\]

\[
c_{\frac{k}{3} + k/2 + (\frac{k}{2} + 1)i}^* = 2c_{\frac{k}{3} + k+ki}.
\]

**Proof.** For any \( k = 2\hat{m} + 2, \hat{m} \in N^* \) and \( 0 \leq r \leq \frac{k}{2} - 1 \), by (2.3) and (2.6) we obtain that

\[
\alpha_{mk+r,2j+1}^* = \frac{\frac{3}{2} k \cdot \frac{5}{2} k \cdots \frac{2j+1}{2} k}{(\frac{3}{2} k + mk + 2r + 1)(\frac{5}{2} k + mk + 2r + 1) \cdots (\frac{2j+1}{2} k + mk + 2r + 1)} = \alpha_{mk+2r,j}^*,
\]

for \( m \geq 0, j \geq 1 \),

\[
\alpha_{mk+r,2j+1}^* = \alpha_{mk+2r,j}^* = 1, \quad \text{for } m \geq 0, j = 0,
\]

and

\[
\tilde{\beta}_{mk+r} = \beta_{mk+2r}^* = 1, \quad \text{for } m = 0,
\]

\[
\tilde{\beta}_{mk+r} = \frac{(-1)^m (mk+2r+1-k)(mk+2r+1-2k) \cdots (mk+2r+1-(m-1)k)(mk+2r+1-mk)}{(mk+2r+1+\frac{1}{2} k)(mk+2r+1-\frac{1}{2} k) \cdots (mk+2r+1-\frac{2m-3}{2} k)}
\]

\[
\quad \quad = \frac{(-1)^m ((m-1)k+2r+1)((m-2)k+2r+1) \cdots (k+2r+1)(2r+1)}{(\frac{2m+1}{2} k + 2r + 1)(\frac{2m-1}{2} k + 2r + 1) \cdots (\frac{3}{2} k + 2r + 1)}
\]

\[
= \beta_{mk+2r}, \quad \text{for } m \geq 1.
\]

Further, by (2.5) and (2.12) we get the relation between \( J_{1,r}^* (h) \) and \( I_{1,2r}^* (h) \) as follows:

\[
J_{1,r}^* (h) = \sum_{m \geq 0, \quad j \geq 1 \text{ odd}} r_{mk+r, j}^{(1)} \tilde{\alpha}_{mk+r, j}^* \tilde{\beta}_{mk+r}^* h^{m+\frac{j}{2}}
\]

\[
= \sum_{m \geq 0, \quad j \geq 1 \text{ odd}} r_{mk+2r, i-1} \tilde{\alpha}_{mk+r, j} \tilde{\beta}_{mk+r} h^{m+\frac{j}{2}}
\]
\begin{align}
  &= \sum_{m \geq 0, j \geq 0} \tilde{r}_{m k + 2 r, j} \tilde{\alpha}_{m k + r, 2 j + 1} \tilde{\beta}_{m k + 2 r, j} h^{m + j} \\
  &= \sum_{m \geq 0, j \geq 0} \tilde{r}_{m k + 2 r, j} \alpha^*_{m k + 2 r, j} \beta^*_{m k + 2 r, j} h^{m + j} \\
  &= I_{1, 2 r}^* (h).
\end{align}

Then, by (2.4) and (2.2) we can write \( M^*(h, \delta) \) as

\[
M^*(h, \delta) = \varphi^*(h, \delta) + 2 h^{\frac{1}{2} + \frac{1}{k}} \sum_{r = 0}^{\frac{k - 1}{2}} \tilde{A}_r h^{\frac{2}{k} r} I_{1, 2 r}^*(h)
\]

\[
= \varphi^*(h, \delta) + 2 \sum_{r = 0}^{\frac{k - 1}{2}} \tilde{A}_r \tilde{r}_{2 r, 0} h^{\frac{1}{2} + \frac{2 r}{k}} + 2 \sum_{r = \frac{k}{4}}^{\frac{k - 1}{2}} \tilde{A}_r \tilde{r}_{2 r, 0} h^{\frac{1}{2} + \frac{2 r}{k}} + i
\]

\[
+ 2 \sum_{r = \frac{k}{4}}^{\frac{k - 1}{2}} \tilde{A}_r \sum_{m + j = i \atop m, j \geq 0} \tilde{r}_{m k + 2 r, j} \alpha^*_{m k + 2 r, j} \beta^*_{m k + 2 r, j} h^{\frac{1}{2} + \frac{2 r}{k} + i}
\]

\[
= \varphi^*(h, \delta) + 2 \sum_{r = 0}^{\frac{k - 1}{2}} \tilde{A}_r \tilde{r}_{2 r, 0} h^{\frac{1}{2} + \frac{2 r}{k}}
\]

\[
+ 2 \left( \sum_{i \geq 0} \left( \sum_{r = \frac{k}{4}}^{\frac{k - 1}{2}} \tilde{A}_r \sum_{m + j = i \atop m, j \geq 0} \tilde{r}_{m k + 2 r, j} \alpha^*_{m k + 2 r, j} \beta^*_{m k + 2 r, j} h^{\frac{1}{2} + \frac{2 r}{k} + i} + \sum_{r = 0}^{\frac{k - 1}{2}} \tilde{A}_r \sum_{m + j = i + 1 \atop m, j \geq 0} \tilde{r}_{m k + 2 r, j} \alpha^*_{m k + 2 r, j} \beta^*_{m k + 2 r, j} h^{\frac{1}{2} + \frac{2 r}{k} + i + 1} \right) \right),
\]

if \( r_1 \) is not an integer. This shows that in this case \( M^*(h, \delta) \) can be written in the form of (3.3), where

\[
c_{r+1}^* = 2 \tilde{A}_r \tilde{r}_{2 r, 0}, \quad r = 0, \ldots, \frac{k}{4} - 1,
\]

\[
c_{r+2+(\frac{k}{2}+1)i}^* = 2 \tilde{A}_r \sum_{m + j = i \atop m, j \geq 0} \tilde{r}_{m k + 2 r, j} \alpha^*_{m k + 2 r, j} \beta^*_{m k + 2 r}, \quad r = \frac{k}{4}, \ldots, \frac{k}{2} - 1,
\]

38
\[ c_{\frac{k}{2}+r+2+(k+1)i} = 2\tilde{A}_r \sum_{m+j=i+1 \atop m,j \geq 0} \tilde{r}_{mk+2r,j}^* \alpha_{mk+2r,j}^* \beta_{mk+2r,j}^* + 2 \sum_{r=0}^{k-1} \tilde{A}_r h \frac{2r}{k} \tilde{I}_{1,2r}^*(h) \quad r = 0, \ldots, \frac{k}{4} - 1. \tag{3.8} \]

Comparing the above formulas with that in (3.2) leads to (3.4).

If \( r_1 \) is an integer, by (2.2), (2.4) and (3.7) we have

\[
M^*(h, \delta) = \varphi^*(h, \delta) - \frac{h \ln h}{k} I_{1,2r_1}^*(h) + 2h^{\frac{1}{4} + \frac{2r}{k}} \tilde{I}_{1,2r_1}^*(h)
\]

\[
= \varphi^*(h, \delta) - \frac{h \ln h}{k} \sum_{i \geq 0} ^{\frac{k}{2} - \frac{3}{2}} \tilde{A}_r \tilde{r}_{2r,0} h^{\frac{1}{4} + \frac{1}{2} + \frac{2r}{k}} + 2 \sum_{r=1}^{\frac{k}{2} - \frac{1}{2}} \tilde{A}_r \tilde{r}_{2r,0} h^{\frac{1}{4} + \frac{1}{2} + \frac{2r}{k}}
\]

\[
+ 2 \sum_{r=0}^{\frac{k}{2} - \frac{1}{2}} \tilde{A}_r \sum_{i \geq 1} \tilde{r}_{mk+2r,j}^* \alpha_{mk+2r,j}^* \beta_{mk+2r,j}^* h^{\frac{1}{4} + \frac{1}{2} + \frac{2r}{k} + i}
\]

\[
+ 2 \sum_{r=\frac{k}{4} + \frac{1}{2}}^{\frac{k}{4} - \frac{3}{2}} \tilde{A}_r \sum_{m+j=i \geq 1} \tilde{r}_{mk+2r,j}^* \alpha_{mk+2r,j}^* \beta_{mk+2r,j}^* h^{\frac{1}{4} + \frac{1}{2} + \frac{2r}{k} + i}
\]

\[
= \varphi^*(0, \delta) + 2 \sum_{r=0}^{\frac{k}{2} - \frac{3}{2}} \tilde{A}_r \tilde{r}_{2r,0} h^{\frac{1}{4} + \frac{1}{2} + \frac{2r}{k}} - \frac{h \ln h}{k} \tilde{r}_{2r_1,0}
\]

\[
+ \sum_{i \geq 0} \left( c_{\frac{k}{4} + \frac{1}{2} + (k+1)i}^* h^{i+1} + 2 \sum_{r=\frac{k}{4} + \frac{1}{2}}^{\frac{k}{2} - \frac{1}{2}} \sum_{m+j=i \geq 1} \tilde{A}_r \tilde{r}_{mk+2r,j}^* \alpha_{mk+2r,j}^* \beta_{mk+2r,j}^* h^{\frac{1}{4} + \frac{1}{2} + \frac{2r}{k} + i} \right)
\]

\[
+ 2 \sum_{r=0}^{\frac{k}{4} - \frac{3}{2}} \sum_{m+j=i \geq 1} \tilde{r}_{mk+2r,j}^* \alpha_{mk+2r,j}^* \beta_{mk+2r,j}^* h^{\frac{1}{4} + \frac{1}{2} + \frac{2r}{k} + i + 1}
\]

\[
- \frac{h \ln h}{k} \sum_{m+j=i \geq 1} \tilde{r}_{mk+2r,j}^* \alpha_{mk+2r,j}^* \beta_{mk+2r,j}^* h^{i+1},
\]

which indicates that in this case \( M^*(h, \delta) \) can be written in the form of (3.5), where
Theorem 3.2. Let (1.4) hold with \( k = 4 \) and \( L_0^+ \) be a double homoclinic loop of cuspidal type. For the Melnikov functions \( M(h, \delta) \), \( \tilde{M}(h, \delta) \) and \( M^*(h, \delta) \) in (1.5), we have

\[
M(h, \delta) = c_0 + c_1 |h|^\frac{3}{4} + c_2 h \ln |h| \\
+ \sum_{i \geq 1} \left( c_{4i-1} h^i + c_{4i} |h|^\frac{4i+1}{4} + c_{4i+1} |h|^\frac{4i+3}{4} + c_{4i+2} h^{i+1} \ln |h| \right), \quad 0 < -h \ll 1,
\]

\[
\tilde{M}(h, \delta) = \tilde{c}_0 + c_1 |h|^\frac{3}{4} - c_2 h \ln |h| \\
+ \sum_{i \geq 1} \left( \tilde{c}_{4i-1} h^i + c_{4i} |h|^\frac{4i+1}{4} + c_{4i+1} |h|^\frac{4i+3}{4} - c_{4i+2} h^{i+1} \ln |h| \right), \quad 0 < -h \ll 1,
\]

\[
M^*(h, \delta) = c_0^* + c_1^* |h|^\frac{3}{4} + \sum_{i \geq 1} \left( c_{3i-1} h^i + c_{3i} h^{\frac{4i+1}{4}} + c_{3i+1} h^{\frac{4i+3}{4}} \right), \quad 0 < h \ll 1,
\]

(3.9)

where \( c_0, \tilde{c}_0, c_1, c_2, c_0^*, c_1^* \) are given in [17] (or see (2.8) and Lemma 3.2), and for \( i \geq 1 \)

\[
c_{4i} = (-1)^{i-1} \tilde{A}_2 \sum_{m+j=i-1, m,j \geq 0} \tilde{r}_{4m+2,j} a_{4m+2,j}^* b_{4m+2,j}^*,
\]

\[
c_{4i+1} = (-1)^i \tilde{A}_0 \sum_{m+j=i, m,j \geq 0} \tilde{r}_{4m,j} a_{4m,j}^* b_{4m,j}^*,
\]

\[
c_{4i+2} = -\frac{1}{8} \sum_{m+j=i, m,j \geq 0} \tilde{r}_{4m+1,j} a_{4m+1,j}^* b_{4m+1,j}^*,
\]

(3.10)

\[
c_{3i}^* = (-1)^{i-1} \frac{2A_1}{A_2} c_{4i}, \quad c_{3i+1}^* = (-1)^i \frac{2A_0}{A_0} c_{4i+1}.
\]
For $i \geq 1$, we give a Maple program (see Appendix A), by which the coefficients $c_{4i}$, $c_{4i+1}$, $c_{4i+2}$ in (3.10) can be obtained as follows:

$$
c_8 = \tilde{A}_2 \left( \frac{1}{3} \tilde{r}_{6,0} - \frac{2}{3} \tilde{r}_{2,1} \right), \quad c_9 = \tilde{A}_0 \left( \frac{5}{77} \tilde{r}_{8,0} - \frac{6}{77} \tilde{r}_{4,1} + \frac{60}{77} \tilde{r}_{0,2} \right),
$$

$$
c_{10} = -\frac{1}{64} \left( \tilde{r}_{9,0} - \tilde{r}_{5,1} + 5 \tilde{r}_{1,2} \right), \quad c_{12} = \tilde{A}_2 \left( \frac{7}{39} \tilde{r}_{10,0} - \frac{2}{13} \tilde{r}_{6,1} + \frac{20}{39} \tilde{r}_{2,2} \right),
$$

$$
c_{13} = -\tilde{A}_0 \left( -\frac{3}{77} \tilde{r}_{12,0} + \frac{2}{77} \tilde{r}_{8,1} - \frac{4}{77} \tilde{r}_{4,2} + \frac{8}{11} \tilde{r}_{0,3} \right),
$$

$$
c_{14} = -\frac{1}{8} \left( -\frac{5}{64} \tilde{r}_{13,0} + \frac{3}{64} \tilde{r}_{9,1} - \frac{5}{64} \tilde{r}_{5,2} + \frac{35}{64} \tilde{r}_{1,3} \right),
$$

$$
c_{16} = -\tilde{A}_2 \left( -\frac{77}{663} \tilde{r}_{14,0} + \frac{14}{221} \tilde{r}_{10,1} - \frac{20}{221} \tilde{r}_{6,2} + \frac{280}{663} \tilde{r}_{2,3} \right),
$$

$$
c_{17} = \tilde{A}_0 \left( \frac{39}{1463} \tilde{r}_{16,0} - \frac{18}{1463} \tilde{r}_{12,1} + \frac{20}{1463} \tilde{r}_{8,2} - \frac{8}{209} \tilde{r}_{4,3} + \frac{144}{209} \tilde{r}_{0,4} \right),
$$

$$
c_{18} = -\frac{1}{8} \left( \frac{7}{128} \tilde{r}_{17,0} - \frac{3}{128} \tilde{r}_{13,1} + \frac{3}{128} \tilde{r}_{9,2} - \frac{7}{128} \tilde{r}_{5,3} + \frac{63}{128} \tilde{r}_{1,4} \right),
$$

$$
c_{20} = \tilde{A}_2 \left( \frac{55}{663} \tilde{r}_{18,0} - \frac{22}{663} \tilde{r}_{14,1} + \frac{20}{663} \tilde{r}_{10,2} - \frac{40}{663} \tilde{r}_{6,3} + \frac{80}{221} \tilde{r}_{2,4} \right),
$$

$$
c_{21} = -\tilde{A}_0 \left( -\frac{663}{33649} \tilde{r}_{20,0} + \frac{234}{33649} \tilde{r}_{16,1} - \frac{180}{33649} \tilde{r}_{12,2} + \frac{40}{4807} \tilde{r}_{8,3} h^5
\right.
\left. + \frac{144}{4807} \tilde{r}_{4,4} + \frac{288}{437} \tilde{r}_{0,5} \right),
$$

$$
c_{22} = -\frac{1}{4096} \left( -21 \tilde{r}_{21,0} + 7 \tilde{r}_{17,1} - 5 \tilde{r}_{13,2} + 7 \tilde{r}_{9,3} - 21 \tilde{r}_{5,4} + 231 \tilde{r}_{1,5} \right),
$$

$$
c_{24} = \tilde{A}_2 \left( -\frac{352}{1105} \tilde{r}_{2,5} + \frac{48}{1105} \tilde{r}_{6,4} - \frac{56}{3315} \tilde{r}_{10,3} + \frac{44}{3315} \tilde{r}_{4,2}
\right.
\left. - \frac{22}{1105} \tilde{r}_{18,1} + \frac{209}{3315} \tilde{r}_{22,0} \right), \quad \cdots \cdots \ . \quad (3.11)
$$

Now we suppose that the equation $H(x, y) = 0$ defines a double homoclinic loop of cuspidal type, denoted by $L_0^*$, and system (1.1) is centrally symmetric. We will give the coefficients of $h^i$ for $i \geq 2$. In this case, we have $\tilde{M}(h, \delta) = M(h, \delta)$. By (3.1) and Lemma 3.2, the expansions of $M(h, \delta)$ and $M^*(h, \delta)$ are given by

$$
M(h, \delta) = c_0 + \sum_{i=0}^{\frac{k-3}{2}} c_{2i+1} |h|^{\frac{1}{2} + \frac{1}{2} + \frac{2}{k} + \frac{1}{2} + \frac{1}{2} + \frac{k}{2}} + c_{\frac{k}{2}} h \ln |h|
$$

41
\[ M^\ast (h, \delta) = c_0^\ast + \sum_{i \geq 0} \left( c_{\frac{k}{2}+1+k_i} h^{i+1} + \sum_{r=\frac{k}{2}+\frac{1}{2}}^{\frac{k}{2}+1} c_{2r+2+ki} |h|^\frac{1}{2} + \frac{2r}{3} + i \right) + \sum_{r=0}^{\frac{k}{2}-\frac{3}{2}} c_{k+1+2r+ki} |h|^\frac{1}{2} + \frac{2r}{3} + i + c_{\frac{3}{2}k+ki} h^{2+i} \ln |h| \),

\[
\begin{align*}
M^\ast (h, \delta) &= c_0^\ast + \sum_{i \geq 0} \left( c_{\frac{k}{2}+1+k_i} h^{i+1} + \sum_{r=\frac{k}{2}+\frac{1}{2}}^{\frac{k}{2}+1} \frac{2\tilde{A}_r}{A_{2r}}(-1)^i c_{2r+2+ki} h^{\frac{1}{2} + \frac{1}{2} + \frac{2r}{3} + i} \\
&+ \sum_{r=0}^{\frac{k}{2}-\frac{3}{2}} \frac{2\tilde{A}_r}{A_{2r}}(-1)^{i+1} c_{k+1+2r+ki} h^{\frac{1}{2} + \frac{2r}{3} + \frac{2r}{3} + i} + 2c_{\frac{3}{2}k+ki} h^{2+i} \ln |h| \right),
\end{align*}
\]

if \( \tilde{m} \) is even, and

\[
\begin{align*}
M^\ast (h, \delta) &= c_0^\ast + \sum_{i \geq 0} \left( c_{\frac{k}{2}+1+k_i} h^{i+1} + \sum_{r=\frac{k}{2}+\frac{1}{2}}^{\frac{k}{2}+1} c_{2r+2+ki} |h|^\frac{1}{2} + \frac{1}{2} + \frac{2r}{3} + i \right) + \sum_{r=0}^{\frac{k}{2}-\frac{3}{2}} c_{k+1+2r+ki} |h|^\frac{1}{2} + \frac{1}{2} + \frac{2r}{3} + i + c_{\frac{3}{2}k+ki} h^{2+i} \ln |h| \),
\end{align*}
\]

if \( \tilde{m} \) is odd.

It should be noted that the coefficients of \( h^i, i \geq 2 \) are not given in Lemmas 3.1 and 3.2. They are given in the following theorem.

**Theorem 3.1.** Consider centrally symmetric system (1.1). Let (1.4) hold and \( L_0^\ast \), defined by \( H(x, y) = 0 \), be a double homoclinic loop of cuspidal type. For convenience, let \( P_0(x, y, \delta) = p(x, y, \delta) \), \( Q_0(x, y, \delta) = q(x, y, \delta) \). Suppose there exist analytic functions \( P_s(x, y, \delta) \) and \( Q_s(x, y, \delta) \) for \( s = 1, 2, \ldots, n \) and \( (x, y) \in U \) with
\[ U = U_1 \cup U_2, \quad U_1 = \bigcup_{h_2 \leq h \leq 0} (L_h \cup \tilde{L}_h), \quad U_2 = \bigcup_{0 \leq h \leq h_0} L^*_h, \quad h_0 > 0, \]

such that the following equality holds

\[ (P_{s-1})_x + (Q_{s-1})_y = H_x P_s(x, y, \delta) + H_y (x, y) Q_s(x, y, \delta), \quad (3.14) \]

for

\[ b_0 = \cdots = b_{s-1} = 0, \quad c_{\frac{1}{2}k+ki} = 0, \]

\[ c_{k+1+2r+k\ast i} = 0, \quad r = 0, \cdots, \frac{k}{4} - \frac{3}{2}, \]

\[ c_{2r+2+k\ast(i+1)} = 0, \quad r = \frac{k}{4} + \frac{1}{2}, \cdots, \frac{k}{2} - 1, \quad (3.15) \]

if \( \hat{m} \) is even, or

\[ b_0 = \cdots = b_{s-1} = 0, \]

\[ c_{k+1+2r+k\ast i} = 0, \quad r = 0, \cdots, \frac{k}{4} - 1, \]

\[ c_{2r+2+k\ast(i+1)} = 0, \quad r = \frac{k}{4}, \cdots, \frac{k}{2} - 1, \]

if \( \hat{m} \) is odd, where \( i = -1, 0, 1, \cdots, s-2 \).

Suppose \( P_s(x, y, \delta) \) and \( Q_s(x, y, \delta) \) can be written as the form

\[ P_s(x, y, \delta) = \sum_{i+j \geq 0 \atop \text{i+j is odd}} d_{ij}^s x^i y^j, \quad Q_s(x, y, \delta) = \sum_{i+j \geq 0 \atop \text{i+j is odd}} b_{ij}^s x^i y^j, \quad (3.16) \]

for \((x, y)\) near the origin. Then,

\[ c_{\frac{k}{2}+1+kn} = \frac{1}{(n+1)!} c_{\frac{k}{2}+1}^{[n]} + \sum_{r=0}^{\frac{k}{2}+1} O_1(c_{kn+1+2r}), \quad (3.17) \]

\[ c_{\frac{k}{2}+\frac{3}{2}+(\frac{k}{2}+1)n} = \frac{1}{(n+1)!} c_{\frac{k}{2}+\frac{3}{2}}^{[n]} + \sum_{r=0}^{\frac{k}{2}+\frac{3}{2}} O_1(c_{kn+1+2r}), \]

if \( \hat{m} \) is even, or

\[ c_{\frac{k}{2}+1+kn} = \frac{1}{(n+1)!} c_{\frac{k}{2}+1}^{[n]} + \sum_{r=0}^{\frac{k}{2}-1} O_1(c_{kn+1+2r}), \quad (3.18) \]

\[ c_{\frac{k}{2}+1+(\frac{k}{2}+1)n} = \frac{1}{(n+1)!} c_{\frac{k}{2}+1}^{[n]} + \sum_{r=0}^{\frac{k}{2}-1} O_1(c_{kn+1+2r}), \]
if $\hat{m}$ is odd, where

$$c_{\frac{k+1}{2}}^{[n]} = \oint_{L_0} [(P_n)x + (Q_n)y - \sum_{i=0}^{\frac{k-1}{2}} \tilde{b}_{i,0}^{[n]} x^i] \, dt,$$

$$c_{\frac{k+2}{2}}^{[n]} = 2 \oint_{L_0} [(P_n)x + (Q_n)y - \sum_{i=0}^{\frac{k-2}{2}} \tilde{b}_{i,0}^{[n]} x^i] \, dt,$$

$$c_{\frac{k+3}{2}}^{[n]} = 2 \oint_{L_0} [(P_n)x + (Q_n)y - \sum_{i=0}^{\frac{k-1}{2}} \tilde{b}_{i,0}^{[n]} x^i] \, dt,$$

and $\tilde{b}_{i,0}^{[n]}$ satisfies (2.9), with $a_{i,j}$ and $b_{i,j}$ replaced by $a_{i,j}^{[n]}$ and $b_{i,j}^{[n]}$, respectively.

**Proof.** If $\hat{m}$ is even and (3.15) holds, by induction we can get

$$\frac{\partial^n M}{\partial h^n} = n! c_{\frac{k+1}{2}+k(n-1)}^{[1]}(k,r) c_{kn+1+2r} h^{\frac{2r+1}{2} + \frac{1}{2}}$$

$$+ (n+1)! c_{\frac{k+1}{2}+kn} h \ln |h| + [\eta_{[2]}^{1}] c_{\frac{k}{2}+kn} + (n+1)! c_{\frac{k}{2}+kn} h + O(|h|^{1+\frac{3}{2}}),$$

where $\eta_{[1]}^{(1)}(k,r)$ is determined by $k$ and $r$, $\eta_{[2]}^{2}$ is a constant, and

$$\frac{\partial^n M}{\partial h^n} = \oint_{L_h} Q_n \, dx - P_n \, dy \equiv M_n(h, \delta).$$

So, $M_n(h, \delta)$ can be written as

$$M_n(h, \delta) = c_0^{[n]} + \sum_{r=0}^{\frac{k}{2}+\frac{1}{2}} c_{2r+1}^{[n]} |h|^{\frac{2r+1}{2} + \frac{1}{2}} + c_{\frac{k}{2}}^{[n]} h \ln |h|$$

$$+ c_{\frac{k}{2}+1}^{[n]} h + \sum_{r=\frac{k}{2}+1}^{\frac{k-1}{2}} c_{2r+2}^{[n]} |h|^{\frac{2r+1}{2} + \frac{1}{2}}$$

$$+ \sum_{r=0}^{\frac{k}{2}+1+2r} c_{k+1+2r}^{[n]} |h|^{\frac{2r+1}{2} + \frac{3}{2}} + c_{\frac{3}{2}k}^{[n]} h^2 \ln |h| + O(h^2).$$

Further, by (2.7) and (2.8), we know that
\[ c_{\frac{k}{2}+1}^{[n]} = c_{\frac{k}{2}+1}^{[n]} + \sum_{r=0}^{\frac{k}{4} - \frac{1}{2}} O_1(c_{kn+2r}). \]

Comparing (3.19) and (3.20), we obtain that

\[ c_{\frac{k}{2}+1}^{[n]} = \eta_n^{[2]} c_{\frac{k}{2}+kn} + (n+1)! c_{\frac{k}{2}+1+kn}, \]

which yields the formula of \( c_{\frac{k}{2}+1+kn} \) in (3.17). The formula of \( c_{\frac{k}{2}+\frac{3}{2}+(\frac{k}{2}+1)n} \) can be obtained similarly.

If \( \hat{m} \) is odd, the proof is similar to the above. This ends the proof. \( \square \)

Especially, for \( k = 4 \), by (3.13) we directly get the expansions of \( M \) and \( M^* \) as follows:

\[
M(h, \delta) = c_0 + c_1 |h|^\frac{3}{4} + \sum_{i \geq 1} \left( c_{4i-1} h^i + c_{4i} |h|^{\frac{4i+1}{4}} + c_{4i+1} |h|^{\frac{4i+3}{4}} \right),
\]

\[
M^*(h, \delta) = 2c_0 + \frac{2\tilde{A}_0}{A_0} c_1 h^\frac{3}{4} + \sum_{i \geq 1} \left( c_{3i-1} h^i + (-1)^{i-1} \frac{2\tilde{A}_1}{A_2} c_{4i} h^{\frac{4i+1}{4}} + (-1)^{i} \frac{2\tilde{A}_0}{A_0} c_{4i+1} h^{\frac{4i+3}{4}} \right),
\]

where \( \tilde{A}_0 > 0, \tilde{A}_1 < 0, \tilde{A}_0 < 0, \tilde{A}_2 > 0, c_{4i} \) and \( c_{4i+1} \) for \( i \geq 1 \) satisfy (3.10), and

\[
c_3 = \int_{L_0} [(p_x + q_y) - (a_{1,0} + b_{0,1})] dt + O_1(c_1), \]

(3.22)

\[
c_3^* = 2 \int_{L_0} [(p_x + q_y) - (a_{1,0} + b_{0,1})] dt + O_1(c_1).
\]

Suppose the conditions in Theorem 3.1 hold with \( k = 4 \). Then, when

\[ b_0 = b_1 = \cdots = b_{n-1} = 0, c_{4i+1} = c_{4i+4} = 0, \quad i = 0, 1, \ldots, n - 1, \]

we have

\[
c_{4n+3} = \frac{1}{(n+1)!} c_3^{[n]} + O_1(c_{4n+1}), \quad c_{3n+2}^{*} = \frac{2}{(n+1)!} c_2^{*[n]} + O_1(c_{4n+1}),
\]

(3.23)

where

\[
c_3^{[n]} = \int_{L_0} [(P_n)_x + (Q_n)_y - (a_{10}^{[n]} + b_{01}^{[n]})] dt,
\]

\[
c_2^{*[n]} = 2 \int_{L_0} [(P_n)_x + (Q_n)_y - (a_{10}^{[n]} + b_{01}^{[n]})] dt.
\]
Theorem 3.2. If there exists $\delta_0$ such that

$$c_0(\delta_0) = c_1(\delta_0) = 0,$$

$$c_{4i-1}(\delta_0) = c_{4i}(\delta_0) = c_{4i+1}(\delta_0) = 0, \quad i = 1, 2, \cdots, \tilde{k} - 1, \quad \tilde{k} \geq 2,$$

$$c_{4\tilde{k}-1}(\delta_0) = 0, \quad c_{4\tilde{k}}(\delta_0) = 0, \quad c_{4\tilde{k}+1}(\delta_0) \neq 0,$$

$$b_0(\delta_0) = b_1(\delta_0) = \cdots = b_{\tilde{k}-1}(\delta_0) = 0, \quad b_j(\delta_0) \neq 0, \quad j \geq \tilde{k} - 1,$$

$$\operatorname{rank} \left. \frac{\partial (c_0, c_1, c_3, c_4, c_5, c_7, \cdots, c_{4\tilde{k}}, b_0, b_1, \cdots, b_{\tilde{k}-1})}{\partial \delta} \right|_{\delta = \delta_0} = 3\tilde{k} + 1 + j, \quad (3.25)$$

and

$$b_j(\delta_0)c_{4\tilde{k}+1}(\delta_0) < 0 \quad (\text{or } b_j(\delta_0)c_{4\tilde{k}+1}(\delta_0) > 0),$$

then, system (1.1) can have $8\tilde{k} + 2j + 4$ (or $8\tilde{k} + 2j + 2$) limit cycles for some $(\varepsilon, \delta)$ near $(0, \delta_0)$.

Proof. Under the conditions (3.24), we have

$$M(h, \delta_0) = c_{4\tilde{k}+1}(\delta_0) |h|^{\frac{4\tilde{k}+3}{2}} + O(h^{\tilde{k}+1}), \quad 0 < -h < 1,$$

$$M(h, \delta_0) = b_j(\delta_0)(h - h_{c_2})^{\tilde{k}+1} + O((h - h_{c_2})^{\tilde{k}+2}), \quad 0 < h - h_{c_2} < 1.$$ 

Thus, if $b_j(\delta_0)c_{4\tilde{k}+1}(\delta_0) < 0$, then there exist $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that

$$M(-\varepsilon_1, \delta_0)M(h_{c_2} + \varepsilon_2, \delta_0) < 0,$$

which means that $M(h, \delta_0)$ has a simple zero $h_0^*$ with $h_0^* \in (h_{c_2}, 0)$. If $b_j(\delta_0)c_{4\tilde{k}+1}(\delta_0) > 0$, we can not find a simple zero of $M(h, \delta_0)$ in $(h_{c_2}, 0)$.

Without loss of generality, suppose $(-1)^{\tilde{k}}c_{4\tilde{k}+1}(\delta_0) > 0$. By (3.25), $c_0, c_1, c_3, c_4, c_5, c_7, \cdots, c_{4\tilde{k}}$ and $b_0, b_1, \cdots, b_{\tilde{k}-1}$ can be taken as free parameters. Let them satisfy

$$0 < -c_0 \ll c_1 \ll |c_3| \ll |c_4| \ll |c_5| \ll |c_7| \ll |c_8| \ll |c_9| \ll \cdots,$$

$$\ll |c_{4\tilde{k}-5}| \ll |c_{4\tilde{k}-4}| \ll |c_{4\tilde{k}-3}| \ll |c_{4\tilde{k}-1}| \ll |c_{4\tilde{k}}| \ll |c_{4\tilde{k}+1}(\delta_0)|,$$

and

$$0 < |b_0| \ll |b_1| \ll \cdots \ll |b_{\tilde{k}-1}| \ll |b_j(\delta_0)|,$$

with

$$c_{4i-1} > 0, \quad (-1)^i c_{4i} < 0, \quad i = 1, 2, \cdots, \tilde{k},$$

$$(-1)^i c_{4\tilde{k}+1} > 0, \quad i = 1, 2, \cdots, \tilde{k} - 1,$$

$$b_i b_{j-1} < 0, \quad i = 1, 2, 3, \cdots, j - 1, \quad b_{j-1} b_j(\delta_0) < 0.$$
which ensure that $M(h, \delta)$ has $3\tilde{k} + 1$ simple zeros near $h = 0$ and $j$ simple zeros near $h_{c_2}$, $M^*(h, \delta)$ has $2\tilde{k}$ simple zeros near $h = 0$. If $b_{\tilde{k}-1}(\delta_0)c_{4\tilde{k}+1}(\delta_0) < 0$, $M(h, \delta)$ has another simple zero near $h_{c_2}^*$.  

By taking $\delta$ sufficiently small, for system (1.1), we can obtain $3\tilde{k} + 1$ limit cycles respectively near $L_0$ and $\tilde{L}_0$, $j$ limit cycles respectively near $C_1$ and $C_2$, and $2\tilde{k}$ limit cycles near $L_0^*$ for some $(\epsilon, \delta)$ near $(0, \delta_0)$. Further, if $b_j(\delta_0)c_{4\tilde{k}+1}(\delta_0) < 0$, then system (1.1) has another 2 limit cycles, one of which surrounds the $j$ limit cycles near $C_2$ and is surrounded by the $3\tilde{k} + 1$ limit cycles near $L_0$, the other one surrounds the $j$ limit cycles near $C_1$ and is surrounded by the $3\tilde{k} + 1$ limit cycles near $\tilde{L}_0$.

Thus, if $b_{\tilde{k}}(\delta_0)c_{4\tilde{k}+1}(\delta_0) < 0$ (or $b_j(\delta_0)c_{4\tilde{k}+1}(\delta_0) > 0$), for some $(\epsilon, \delta)$ near $(0, \delta_0)$ system (1.1) can have $8\tilde{k} + 2j + 4$ (or $8\tilde{k} + 2j + 2$) limit cycles. □

**Theorem 3.3.** Suppose there exists $\delta_0$ such that

\[

c_0(\delta_0) = c_1(\delta_0) = 0,
\]

\[
c_{4i-1}(\delta_0) = c_{4i}(\delta_0) = c_{4i+1}(\delta_0) = 0, \quad i = 1, 2, \ldots, \tilde{k} - 1, \quad \tilde{k} \geq 2,
\]

\[
b_0(\delta_0) = b_1(\delta_0) = \cdots = b_{j-1}(\delta_0) = 0, \quad b_j(\delta_0) \neq 0, \quad j \geq \tilde{k} - 1.
\]

(1) If

\[
\text{rank} \frac{\partial (c_0, c_1, c_3, c_4, c_5, c_7, \ldots, c_{4\tilde{k}-1}, b_0, b_1, \ldots, b_{j-1})}{\partial \delta} \bigg|_{\delta=\delta_0} = 3\tilde{k} + j,
\]

and

\[
b_j(\delta_0)c_{4\tilde{k}}(\delta_0) < 0 \quad \text{(or \ } b_j(\delta_0)c_{4\tilde{k}}(\delta_0) > 0),
\]

then, system (1.1) can have $8\tilde{k} + 2j + 2$ (or $8\tilde{k} + 2j + 2$) limit cycles for some $(\epsilon, \delta)$ near $(0, \delta_0)$.

(2) If

\[
\text{rank} \frac{\partial (c_0, c_1, c_3, c_4, c_5, c_7, \ldots, c_{4\tilde{k}-3}, b_0, b_1, \ldots, b_{j-1})}{\partial \delta} \bigg|_{\delta=\delta_0} = 3\tilde{k} + j - 1,
\]

and

\[
(-1)^\tilde{k} b_j(\delta_0)c_{4\tilde{k}-1}(\delta_0) < 0 \quad \text{(or \ } (-1)^\tilde{k} b_j(\delta_0)c_{4\tilde{k}-1}(\delta_0) > 0),
\]

then, system (1.1) can have $8\tilde{k} + 2j - 1$ (or $8\tilde{k} + 2j - 3$) limit cycles for some $(\epsilon, \delta)$ near $(0, \delta_0)$.

47
4. An application

Consider the system

\[ \dot{x} = y, \quad \dot{y} = x^3(1 - x^2) + \varepsilon \sum_{i=0}^{\hat{n}} a_i x^{2i} y, \quad 11 \leq \hat{n} \leq 23, \quad (4.1) \]

which has the Hamiltonian function,

\[ H(x, y) = \frac{1}{2} y^2 - \frac{1}{4} x^4 + \frac{1}{6} x^6. \]

The phase portrait of system \((4.1)\)\(_{\varepsilon = 0}\) is shown in Fig. 1 (a).

We apply the formulas derived in the previous section to obtain the coefficients in \(M(h, \delta)\) and \(M^*(h, \delta)\). By \((2.8)\), we have

\[ c_0 = \sum_{i=0}^{\hat{n}} a_i \oint_{L_0} x^{2i} y \, dx = \sum_{i=0}^{\hat{n}} 2a_i \int_{0}^{\sqrt{\frac{\varepsilon}{2}}} x^{2i+2} \left( \frac{1}{2} - \frac{1}{3} x^2 \right) \, dx, \]

\[ c_3 = \oint_{L_0} (\sum_{i=0}^{\hat{n}} a_i x^{2i} - a_0) \, dt = \sum_{i=1}^{\hat{n}} 2a_i \int_{0}^{\sqrt{\frac{\varepsilon}{2}}} x^{2i-2} \left( \frac{1}{2} - \frac{1}{3} x^2 \right) \, dx. \]  

\[ (4.2) \]

Further, by \((2.8), (3.11)\) and the programs in [16], we can obtain the coefficients \(c_1\) and \(c_{4i}, c_{4i+1}\) for \(i = 1, 2 \cdots, 5\). Since their expressions are long, we put them in Appendix A.

With a transformation of variables, \(u = \sqrt{2}(x - 1)\), \(v = y\), and a time scaling \(t \rightarrow \sqrt{2}t\), system \((4.1)\) becomes

\[ \dot{u} = v, \quad \dot{v} = \frac{\sqrt{2}}{2} \left( \frac{u}{\sqrt{2}} + 1 \right)^3 \left( 1 - \left( \frac{u}{\sqrt{2}} + 1 \right)^3 \right) + \frac{\sqrt{2}}{2} \varepsilon \sum_{i=0}^{\hat{n}} a_i \left( \frac{u}{\sqrt{2}} + 1 \right)^{2i} v. \]

When \(\varepsilon = 0\), the above system has a Hamiltonian function,

\[ H_1(u, v) = \frac{1}{2} (v^2 + u^2) + \frac{1}{48} u^6 + \frac{1}{8} \sqrt{2} u^5 + \frac{9}{16} u^4 + \frac{7}{12} \sqrt{2} u^3. \]

We consider the case \(\hat{n} = 23\). Using the programs in [32], we can get the coefficients \(b_i(\delta)\) in \((2.10)\) (we put them in Appendix A since they are long).

From the equations \(b_0 = c_1 = c_4 = 0\) we get

\[ a_0 = 0, \quad a_1 = 0, \quad a_2 = -\sum_{i=3}^{23} a_i. \]  

\[ (4.3) \]

Then, by the formulas in [18], we obtain \(P_1(x, y, \delta)\) and \(Q_1(x, y, \delta)\) as follows:
\[
P_1(x, y, \delta)_{|b_0=c_1=c_4=0} = \sum_{i \geq 0}^{23} a_i x^{2i} - x^3 (1 - x^2) \bigg|_{b_0=c_1=c_4=0} = \sum_{j=3}^{23} \sum_{i=j}^{23} a_i x^{2j-5}, \quad Q_1(x, y, \delta) = 0,
\]
such that (3.14) holds for \(s = 1\). Thus, by (3.23) and (4.3), we obtain
\[
c_7_{|b_0=c_1=c_4=0} = \oint_{L_0} [(P_1)_x - (P_1)_x \big|_{x=0}] dt
\]
\[
= \sqrt{3} \pi \left( 3a_4 + \frac{27}{4} a_5 + \frac{405}{32} a_6 + \frac{2835}{128} a_7 + \frac{76545}{2048} a_8 \right.
+ \frac{505197}{8192} a_9 + \frac{6567561}{65536} a_{10}
+ \frac{42220035}{262144} a_{11} + \frac{2153221785}{8388608} a_{12} + \frac{13637071305}{33554432} a_{13} + \frac{171827098443}{268435456} a_{14}
+ \frac{1077824526597}{1073741824} a_{15} + \frac{26945613164925}{68719476736} a_{16} + \frac{16789189741225}{68719476736} a_{17}
+ \frac{2086656439266225}{549755813888} a_{18} + \frac{12937269923450595}{2199023255552} a_{19}
+ \frac{1280789722421608905}{140737488355328} a_{20}
+ \frac{7910760050251113825}{562949953421312} a_{21} + \frac{97566040619763737175}{4503599627370496} a_{22}
+ \frac{600801408026966171025}{18014398509481984} a_{23} \bigg).
\]
Based on (4.3) we solve the equations \(b_1 = c_5 = c_8 = 0\) for \(a_3, a_4\) and \(a_5\), and get
\[
a_3 = 0,
\]
\[
a_4 = \frac{7}{5} a_6 + \frac{16}{5} a_7 + \frac{27}{5} a_8 + 8 a_9 + 11 a_{10} + \frac{72}{5} a_{11} + \frac{91}{5} a_{12} + \frac{112}{5} a_{13}
+ 27 a_{14} + 32 a_{15} + \frac{187}{5} a_{16} + \frac{216}{5} a_{17} + \frac{247}{5} a_{18} + 56 a_{19}
+ 63 a_{20} + \frac{352}{5} a_{21} + \frac{391}{5} a_{22} + \frac{432}{5} a_{23},
\]
\[
a_5 = -\frac{252}{5} a_{18} - \frac{12}{5} a_6 - \frac{21}{5} a_7 - \frac{32}{5} a_8 - 9 a_9 - 12 a_{10} - \frac{77}{5} a_{11} - \frac{96}{5} a_{12}
- \frac{117}{5} a_{13} - 28 a_{14} - 33 a_{15} - \frac{192}{5} a_{16} - \frac{221}{5} a_{17} - \frac{396}{5} a_{22}
- 57 a_{19} - 64 a_{20} - \frac{357}{5} a_{21} - \frac{437}{5} a_{23}.
\]
Substituting (4.4) into (4.3), we get \(a_2 = 0\).
Similarly, for \( b_0 = b_1 = c_1 = c_4 = c_5 = c_8 = 0 \), we can find

\[
P_2(x, y, \delta) = \frac{(P_1)_x}{x^3(x^2 - 1)}, \quad Q_2(x, y) = 0,
\]
such that (3.14) holds with \( s = 2 \). Then, under the conditions (4.4) and (4.3), i.e., \( b_0 = b_1 = c_1 = c_4 = c_5 = c_8 = 0 \), we have

\[
c_{11}|_{b_0=b_1=c_1=c_4=c_5=c_8=0} = \oint \left( (P_2)_x - (P_2)_x |_{x=0} \right) dt
\]

\[
= \sqrt{3}\pi \left( 27a_7 + \frac{405}{4}a_8 + \frac{8505}{32}a_9 + \frac{76545}{128}a_{10} + \frac{2525985}{2048}a_{11} \right.
\]

\[
+ \frac{19702683}{8192}a_{12} + \frac{295540245}{65536}a_{13}
\]

\[
+ \frac{2153221785}{262144}a_{14} + \frac{122733641745}{8388608}a_{15} + \frac{859135492215}{33554432}a_{16} + \frac{11856069792657}{268435456}a_{17}
\]

\[
+ \frac{80836839494775}{1073741824}a_{18} + \frac{2182594666358925}{17179869184}a_{19}
\]

\[
+ \frac{14606595074863575}{68719476736}a_{20} + \frac{194059048851758925}{549755813888}a_{21}
\]

\[
+ \frac{1280789722421608905}{2199023255552}a_{22} + \frac{134482920854268935025}{140737488355328}a_{23} \right). \]

Similarly, for

\[
b_0 = \cdots = b_{s-1} = 0, \quad c_{4i+5} = c_{4i+8} = 0, \quad i = -1, 0, \cdots, s - 2,
\]
we can find \( P_s(x, y, \delta) \) and \( Q_s(x, y, \delta) \) as follows:

\[
P_s(x, y, \delta) = \frac{(P_{s-1})_x}{x^3(x^2 - 1)}, \quad Q_s(x, y, \delta) = 0, \quad s = 3, 4, 5
\]
such that (3.14) holds. Similarly, we obtain

\[
c_{15}|_{b_0=b_1=b_2=c_1=c_4=c_5=c_8=c_9=c_{12}=0} = \oint \left( (P_3)_x - (P_3)_x |_{x=0} \right) dt
\]

\[
= \sqrt{3}\pi \left( \frac{8505}{4}a_{11} + \frac{405}{32}a_{10} + \frac{229635}{128}a_{12} + \frac{2525985}{2048}a_{13} + \frac{98513415}{1073741824}a_{14} \right.
\]

\[
+ \frac{886620735}{8192}a_{15} + \frac{15072552495}{65536}a_{16} + \frac{122733641745}{262144}a_{17} + \frac{7732219429935}{8388608}a_{18}
\]

\[
+ \frac{59280348962835}{33554432}a_{19} + \frac{889205234442525}{268435456}a_{20} + \frac{6547783999076775}{1073741824}a_{21}
\]

\[
+ \left. \cdots + \frac{1280789722421608905}{2199023255552}a_{22} + \frac{134482920854268935025}{140737488355328}a_{23} \right). \]
\[= \oint_{L_0} \left[ (P_4)_x - (P_4)_x |_{x=0} \right] dr \]
\[= \sqrt{3} \pi \left( \frac{229635}{4} a_{14} + 8505 a_{13} + \frac{7577955}{32} a_{15} + \frac{98513415}{128} a_{16} + \frac{4433103675}{2048} a_{17} \right. \]
\[+ \frac{45217657485}{8192} a_{18} + \frac{859135492215}{65536} a_{19} + \frac{7732219429935}{262144} a_{20} + \frac{533523140665515}{8388608} a_{21} \]
\[+ \left. \frac{4446026172212625}{3354432} a_{22} + \frac{72025623989844525}{268435456} a_{23} \right) , \]
\[= \oint_{L_0} \left[ (P_3)_x - (P_3)_x |_{x=0} \right] dr \]
\[= \sqrt{3} \pi \left( \frac{7577955}{4} a_{17} + 229635 a_{16} + \frac{295540245}{32} a_{18} + \frac{4433103675}{128} a_{19} \right. \]
\[+ \frac{226088287425}{2048} a_{20} + \frac{2577406476645}{8192} a_{21} \]
\[+ \frac{54125536009545}{65536} a_{22} + \frac{533523140665515}{262144} a_{23} \). \]

For \( \delta_0 = (a_0, a_1, \cdots, a_{23}) \) satisfying
\[a_0 = a_1 = \cdots = a_{10} = 0, \]
\[a_{11} = \frac{112970457}{219152384} a_{23}, a_{12} = \frac{41796853983}{2848980992} a_{23}, a_{13} = \frac{-217348128711}{1424490496} a_{23}, \]
\[a_{14} = \frac{3044761832889}{7122452480} a_{23}, a_{15} = -\frac{5582471316303}{14244904960} a_{23}, a_{16} = \frac{-420059945607}{1294991360} a_{23}, \]
\[a_{17} = \frac{693245743852039}{596149272576} a_{23}, a_{18} = -\frac{1472075464589699}{1117779886080} a_{23}, a_{19} = \frac{3639524693975}{4299153408} a_{23}, \]
\[a_{20} = -\frac{16534088582225}{48466237248} a_{23}, a_{21} = -\frac{3174058863809}{36349677936} a_{23}, a_{22} = -\frac{5620913}{416988} a_{23}, \]
we have
\[b_0(\delta_0) = b_1(\delta_0) = \cdots = b_4(\delta_0) = 0, \]
\[c_0(\delta_0) = c_1(\delta_0) = c_2(\delta_0) = c_{4i-1}(\delta_0) = c_{4i}(\delta_0) = c_{4i+1}(\delta_0), i = 1, 2, \cdots, 5, \ c_{23}(\delta_0) = 0, \]
and
\[b_5(\delta_0) = \frac{3254293}{150994944} \sqrt{2} a_{23}, \ c_{24}(\delta_0) = \frac{2215107}{8320} \tilde{A}_{2} a_{23}. \]
Note that $b_5(\delta_0)c_{24}(\delta_0) > 0$ if $a_{23} \neq 0$. By Theorem 3.3 (1) with $\bar{k} = 6$, $j = 5$, for some $(\varepsilon, \delta)$ near $(0, \delta_0)$, system (4.1) has 58 limit cycles. This proves that $H(46, 5) \geq 58$.

Similarly, for $\hat{n} = 22, 18, 14$, we can find $\hat{\delta}_0^{[\hat{n}]}$ such that (3.26) and (3.28) hold with $(\bar{k}, j) = (6, 5), (5, 4)$ and $(4, 3)$, respectively, and $(-1)^{\hat{k}}c_{4\bar{k}+1}(\delta_0)b_j(\delta_0) > 0$ if $a_{\hat{n}} \neq 0$. Thus, by Theorem 3.3 (2), for some $(\varepsilon, \delta)$ near $(0, \delta_0^{[\hat{n}]}), \text{system (4.1) has } 8\bar{k} + 2j - 3 \text{ limit cycles which means that}$

$$H(44, 5) \geq 55, \ H(36, 5) \geq 45, \ H(28, 5) \geq 35.$$  

For $\hat{n} = 21, 17, 13,$ (or $\hat{n} = 20, 16, 12$), we can find $\delta_0^{[\hat{n}]}$ such that (3.24) and (3.25) hold with $(\bar{k}, j) = (5, 5), (4, 4)$ and $(3, 3)$ (or $(\bar{k}, j) = (5, 4), (4, 3)$ and $(3, 2)$), respectively, and $c_{4\bar{k}+1}(\delta_0)b_j(\delta_0) > 0$ if $a_{\hat{n}} \neq 0$. By Theorem 3.2, for some $(\varepsilon, \delta)$ near $(0, \delta_0^{[\hat{n}]})$, system (4.1) has $8\bar{k} + 2j + 2$ limit cycles which means that

$$H(42, 5) \geq 52, \ H(34, 5) \geq 42, \ H(26, 5) \geq 32,$$
$$H(40, 5) \geq 50, \ H(32, 5) \geq 40, \ H(24, 5) \geq 30.$$

For $\hat{n} = 19, 15, 11$, we can find $\delta_0^{[\hat{n}]}$ such that (3.26) and (3.27) hold with $(\bar{k}, j) = (5, 4), (4, 3)$ and $(3, 2)$, respectively, and $(-1)^{\bar{k}}c_{4\bar{k}}(\delta_0)b_j(\delta_0) > 0$ if $a_{\hat{n}} \neq 0$. Thus, by Theorem 3.3 (1), for some $(\varepsilon, \delta)$ near $(0, \delta_0^{[\hat{n}]})$, system (4.1) has $8\bar{k} + 2j$ limit cycles which means that

$$H(38, 5) \geq 48, \ H(30, 5) \geq 38, \ H(22, 5) \geq 28.$$

Summarizing the above results gives the following theorem.

**Theorem 4.1.** For the lower bounds of $H(n, 5)$ with $n = 2\hat{n}$ ($11 \leq \hat{n} \leq 23$), we have

$$H(2\hat{n}, 5) \geq \frac{5\hat{n} - 1}{2}, \ \hat{n} = 13, 17, 21, \ H(2\hat{n}, 5) \geq \frac{5\hat{n} + 1}{2}, \ \hat{n} = 11, 15, 19.$$  

For system (1.6), by Theorem 4.1 we directly have

$$H(2\hat{n} + 1, 5) \geq H(2\hat{n}, 5), \ 11 \leq \hat{n} \leq 23,$$
which shows the lower bound of $H(n, 5)$ for $22 \leq n \leq 46$ odd.

**Appendix A**

The following Maple program can be used to calculate the coefficients $c_{4i}, c_{4i+1}, c_{4i+2}$ in (3.10) for $i \geq 1$.

```maple
restart; with(LinearAlgebra): i := 6:
for r from 0 to 3 do
    beta[4*0+r] := 1:
end do;
```
> for m from 1 to i do
> beta[4*m+r]:=beta[4*(m-1)+r]*(-1)*((m-1)*4+r+1)/((2*m+1)*2+r+1):
> od:
> od:
> for r from 0 to 3 do
> form from 0 to i do
> alpha[m*4+r,0]:=1:
> for j from 1 to i do
> alpha[4*m+r,j]:=alpha[4*m+r,j-1]*((2*j+1)*4/2)/((2*j+1)*4/2+(4*m+r)+1):
> od:
> od:
> od:
> od:
> for n from 1 to i do
> c[4*n+2]:=0:c[4*n+1]:=0:c[4*(n+1)]:=0:
> c[4]:=tilA[2]*tilr[2,0]*alpha[2,0]*beta[2]:
> for m from 0 to n do
> c[4*n+2]:=c[4*n+2]-1/8*tilr[4*m+1,n-m]*alpha[4*m+1,n-m]*beta[4*m+1]:
> c[4*n+1]:=c[4*n+1]+(-1)^n*tilA[0]*tilr[4*m,n-m]*alpha[4*m,n-m]*beta[4*m]:
> c[4*(n+1)]:=c[4*(n+1)]+(-1)^n*tilA[2]*tilr[4*m+2,n-m]*alpha[4*m+2,n-m]*beta[4*m+2]:
> od:
> od:

For system (4.1), the following are the formulas of some coefficients in the expansion of $M(h, \delta)$ for $0 < -h \ll 1$.

\[ c_1 = 4 \tilde{A}_0 a_0, \quad c_4 = 4 \tilde{A}_2 (4 a_0 + 8 a_1), \quad c_5 = \tilde{A}_0 \left( \frac{10}{7} a_0 + \frac{40}{21} a_1 + \frac{16}{7} a_2 \right), \]

\[ c_8 = \tilde{A}_2 \left( \frac{770}{81} a_0 + \frac{308}{27} a_1 + \frac{112}{9} a_2 + \frac{32}{3} a_3 \right), \]

\[ c_9 \approx \tilde{A}_0 \left( \frac{1105}{198} a_0 + \frac{4420}{693} a_1 + \frac{520}{77} a_2 + \frac{480}{77} a_3 + \frac{320}{77} a_4 \right), \]

\[ c_{12} = \tilde{A}_2 \left( \frac{33649}{70} a_0 + \frac{168245}{3159} a_1 + \frac{58520}{1053} a_2 + \frac{6160}{117} a_3 + \frac{4928}{117} a_4 + \frac{896}{39} a_5 \right), \]

\[ c_{13} \approx \tilde{A}_0 \left( \frac{32045}{972} a_0 + \frac{32045}{891} a_1 + \frac{11050}{297} a_2 + \frac{3536}{99} a_3 + \frac{7072}{231} a_4 + \frac{1664}{77} a_5 + \frac{768}{77} a_6 \right), \]

\[ c_{16} = \tilde{A}_2 \left( \frac{5215595}{16524} a_0 + \frac{36509165}{107406} a_1 + \frac{2086238}{5967} a_2 + \frac{672980}{1989} a_3 + \frac{5383840}{17901} a_4 \\
+ \frac{468160}{663} a_5 + \frac{98560}{663} a_6 + \frac{39424}{663} a_7 \right), \]

\[ c_{17} \approx \tilde{A}_0 \left( \frac{243061325}{1034208} a_0 + \frac{48612265}{193914} a_1 + \frac{1185665}{4617} a_2 + \frac{128180}{513} a_3 + \frac{1281800}{5643} a_4 \\
+ \frac{353600}{1881} a_5 + \frac{28288}{209} a_6 + \frac{113152}{1463} a_7 + \frac{39936}{1463} a_8 \right), \]

\[ c_{20} \approx \tilde{A}_2 \left( \frac{1505816785}{629856} a_0 + \frac{1505816785}{594864} a_1 + \frac{32038655}{12393} a_2 + \frac{10431190}{4131} a_3 + \frac{41724760}{17901} a_4 \\
+ \frac{53}{3} a_5 + \frac{40}{3} a_6 + \frac{11}{3} a_7 + \frac{2}{3} a_8 \right). \]
For system (4.1), the following are the formulas of some coefficients in the expansion of $M(h, \delta)$ for $0 < h - h_{c2} \ll 1$.

\begin{align*}
b_0 &= \sqrt{2\pi} \sum_{i=0}^{23} a_i, \\
b_1 &= \frac{\pi}{24} \sqrt{2} (14785 a_0 + 1465 a_1 - 959 a_2 + 25 a_3 + 385 a_4 - 455 a_5 + 385 a_6 \\
&\quad + 9241 a_7 + 35905 a_8 + 93625 a_9 + 199105 a_{10} + 372505 a_{11} + 637441 a_{12} \\
&\quad + 1020985 a_{13} + 1553665 a_{14} + 2269465 a_{15} + 3205825 a_{16} + 4403641 a_{17} \\
&\quad + 5907265 a_{18} + 7764505 a_{19} + 10026625 a_{20} + 12748345 a_{21} \\
&\quad + 15987841 a_{22} + 19806745 a_{23}), \\
b_2 &= \frac{\pi}{1728} \sqrt{2} (2867893 a_0 + 246841 a_1 - 112763 a_2 - 6311 a_3 + 25333 a_4 \\
&\quad - 9191 a_5 - 7931 a_6 + 17017 a_7 - 19019 a_8 + 17017 a_9 + 1015045 a_{10} \\
&\quad + 5409817 a_{11} + 18217717 a_{12} + 48322393 a_{13} + 11009221 a_{14} \\
&\quad + 224748601 a_{15} + 423228085 a_{16} + 747633337 a_{17} + 1254177925 a_{18} \\
&\quad + 2015881945 a_{19} + 3125599477 a_{20} + 4699294873 a_{21} + 6879567877 a_{22} \\
&\quad + 9839427577 a_{23}), \\
\cdots.
\end{align*}

References


