Complex Dynamics of Predator–Prey Systems with Allee Effect

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In this paper, we apply bifurcation theory to consider four predator–prey systems which include the Allee effect, and show that the species having a strong Allee effect may affect their predation and hence extinction risk. It is shown that the models with the Allee effect exhibit more complex dynamical behaviors compared with that without the Allee effect. In particular, two models with no Allee effect do not have Hopf bifurcation, but can have Hopf bifurcation with the Allee effect; and one model, which does not have Bogdanov–Takens bifurcation if no Allee effect is involved, can have Bogdanov–Takens bifurcation of codimension two. Especially, for one model with Holling type II functional response of the predator to the prey, the Allee effect not only completely changes the stability of the equilibrium at the origin, but also changes the supercritical Hopf bifurcation arising from an interior equilibrium to subcritical Hopf bifurcation with very limited parameter values to yield unstable limit cycles, and further increases the system’s stability.

Keywords: Predator–prey system; Allee effect; Hopf bifurcation; Bogdanov–Takens bifurcation; normal form; the simplest normal form; limit cycle.

1. Introduction

The dynamics of a population is greatly affected by its interaction with other populations. There exist many kinds of interaction among populations, such as competition, predation, parasitism and mutualism. The predator–prey interaction is one of the most fundamental interactions and one of the most fascinating interactions to investigate. A lot of attention has been paid by many researchers to model these various kinds of interactions. Since the first predator–prey model was independently proposed by Lotka [1924] and Volterra [1926], the construction of the predator–prey models and the study on the population dynamics have remained a dominant branch in theoretical and mathematical ecology (e.g. see Freedman [1980] and references therein). After this pioneering work of Lotka and Volterra, predator–prey models with different kinds of prey-dependent functional response were studied extensively [Arditi & Ginzburg, 1992; Holling, 1959; Huang et al., 2014; Murray, 2002]. A well-known generalized Gause predator–prey model is described by

\[
\dot{x} = rx \left(1 - \frac{x}{K}\right) - yp(x),
\]

\[
\dot{y} = y[cq(x) - d],
\]

where dot denotes differentiation with respect to time t, x and y represent the population densities of
prey and predator, respectively. The logistic growth function \( rx(1 - \frac{x}{K}) \) is a typical function to describe the specific growth rate of the prey in the absence of predators, where the positive parameters \( r \) and \( K \) stand respectively for the prey’s intrinsic growth rate and the carrying capacity of the prey. The positive parameters \( c \) and \( d \) denote the conversion rate of the prey to the predator and the predator death rate, respectively. \( p(\cdot) \) is a functional response function, which reflects the capture ability of the predator to the prey, and \( q(\cdot) \) describes how predator converts the consumed prey into the growth of predators. There are a lot of models developed using different functions \( p \) and \( q \). Functional response is the key component in the predator–prey relationship, characterizing the rate of prey consumption by an average predator.

In prey-dependent functional response, the consumption rate is the function of prey density only, denoted by \( p(x) \). However, later a lot of observations indicate that on large temporal and spatial scales, the predator may appear to search, compete or share for food and thus the functional response should depend on both the prey and predator. For example, in the case of perfect sharing, the predator-dependent functional response takes the ratio-dependent form in which functional response depends on the ratio of prey to predator, usually represented as \( p(\frac{x}{y}) \), which is supported by many field and laboratory observations, and is extensively applied in studying predator–prey models \[\text{[Arditi \\& Ginzburg, 1989]}\]. The qualitative investigation of ratio-dependent predator–prey models has shown that these models provide much richer and more reasonable dynamics than their traditional counterparts, and do not exhibit the paradox of enrichment \[\text{[Kuang \\& Beretta, 1993; Kuang, 1998; Xiao \\& Ruan, 2000; Liu et al., 2001; Li \\& Kuang, 2007]}\].

In the past studies, \( q(\cdot) \) mainly takes three typical forms: (A) \( q(x) = p(x) \) used in most predator–prey models \[\text{[Holling, 1966]}\], (B) \( q(\frac{x}{y}) = p(\frac{x}{y}) \) used to represent more models in their performance to fit observed data \[\text{[Abrams \\& Ginzburg, 2004]}\], and (C) \( q(\frac{x}{y}) = s(1 - \frac{1}{K_y}) \) used to analyze the general effect of harvesting \[\text{[Kao et al., 2002]}\]. Combining these different types of \( p \) and \( q \) functions yield many different predator–prey models to describe different situations. Nine predator–prey models with different response functions have been studied in \[\text{[Zeng \\& Yu, 2013; Zeng et al., 2020; Jiang et al., 2021]}\], described as

\[
A_1: \begin{align*}
\dot{x} &= rx \left(1 - \frac{x}{K}\right) - mxy, \\
\dot{y} &= y(\alpha cx - d);
\end{align*}
\]

\[
A_2: \begin{align*}
\dot{x} &= rx \left(1 - \frac{x}{K}\right) - mxy, \\
\dot{y} &= y \left(\frac{mcx}{a + x} - d\right);
\end{align*}
\]

\[
A_3: \begin{align*}
\dot{x} &= rx \left(1 - \frac{x}{K}\right) - \frac{mxy}{ax^2 + bx + 1}, \\
\dot{y} &= y \left(\frac{mcx^2}{ax^2 + bx + 1} - d\right);
\end{align*}
\]

\[
B_1: \begin{align*}
\dot{x} &= rx \left(1 - \frac{x}{K}\right) - mxy, \\
\dot{y} &= mcx - dy;
\end{align*}
\]

\[
B_2: \begin{align*}
\dot{x} &= rx \left(1 - \frac{x}{K}\right) - \frac{mxy}{x + ay}, \\
\dot{y} &= y \left(\frac{mcx}{x + ay} - d\right);
\end{align*}
\]

\[
B_3: \begin{align*}
\dot{x} &= rx \left(1 - \frac{x}{K}\right) - \frac{mxy}{ax^2 + bx + y}, \\
\dot{y} &= y \left(\frac{mcx^2}{ax^2 + bx + y} - d\right);
\end{align*}
\]

\[
C_1: \begin{align*}
\dot{x} &= rx \left(1 - \frac{x}{K}\right) - mxy, \\
\dot{y} &= sy \left(1 - \frac{y}{K_y}\right);
\end{align*}
\]

\[
C_2: \begin{align*}
\dot{x} &= rx \left(1 - \frac{x}{K}\right) - \frac{mxy}{a + x}, \\
\dot{y} &= sy \left(1 - \frac{y}{K_y}\right);
\end{align*}
\]

\[
C_3: \begin{align*}
\dot{x} &= rx \left(1 - \frac{x}{K}\right) - \frac{mxy}{ax^2 + bx + 1}, \\
\dot{y} &= sy \left(1 - \frac{y}{K_y}\right);
\end{align*}
\]
where different types of the functional response of the predator to the prey, including the Lotka–Volterra type, Holling type II and generalized Holling type III are used, while the function describing how the predator converts the consumed prey into the growth of predator is taken either as the same function of the predator to the prey, or that depending upon the ratio of the prey to the predator, or the ratio of predator to the prey. All parameters take positive real values, except $b > -2 \sqrt{\gamma}$. Dynamical properties including positivity of solutions, stability and bifurcation of equilibria and Hopf bifurcation are given in models $A_1$ through $B_4$, and in model $B_5$, Bogdanov–Takens bifurcation for models $A_1$ and $B_1$ are studied in [Zeng et al. 2021].

The development of the traditional predator–prey models is based on the concept that when a population is at a low density, the fitness of an individual species is high due to greater availability of resources; but as the population increases, the competition between individuals for resources increases and individual fitness declines. Thus, in the study of most classical predator–prey models, the effect of cooperation is neglected and it is assumed that the growth of prey population reaches its maximum at low densities and declines as population increases. However, the phenomenon called the Allee effect was discovered [Allee 1931] because there exist many cooperative biological species that suffer a reduction in fitness at low population size due to lack of conspecifics. It was observed that species crashes to extinction if its population experiences a negative growth below a certain threshold level when the Allee effect is strong enough. The studies of the Allee effect have shown that it induces complex dynamics in predator–prey systems, for example, see [Verma & Misra 2014] and references therein. It was shown in these studies that inclusion of the Allee effect in ratio-dependent predator–prey models may reduce possible sustained oscillations of species and yield richer complex dynamical behaviors, causing an increase for the basin of attraction of extinction state and thus increasing the possibility of extinction of species.

An interesting phenomenon called prey refuge may decrease the risk of extinction of species by decreasing the predation risk. Many investigations have paid attention to the effect of prey refuge on the dynamics of predator–prey models without the Allee effect. It has been shown in most of these studies that prey refuge increases the equilibrium density of the prey and has a stabilizing effect on the predator–prey interaction. However, under a restricted set of conditions, refuge may have a destabilizing effect on the system dynamics [Ma et al. 2004, Verma & Misra 2018]. It is also found that the refuge which protects a constant number of prey has more stabilizing effect than the refuge which protects a constant proportion of prey. Recently, the combined effect of prey refuge with the Allee effect has been extensively studied by Verma and Misra [2018]. It is found that if prey refuge is less than the Allee threshold, the incorporation of prey refuge increases the threshold values of the predation rate and conversion efficiency at which unconditional extinction occurs. In addition, it is shown that if the prey refuge is greater than the Allee threshold, unconditional extinction may not be possible. Moreover, the study reveals that at a critical value of prey refuge, which is greater than the Allee threshold but less than the carrying capacity of prey population, the system undergoes a cusp bifurcation and exhibits complex dynamical behaviors.

In this paper, we will investigate the dynamical properties between the new models with the Allee effect and the models without the Allee effect. In particular, we will show new dynamical behavior due to the Allee effect. It will be noted that the dynamical analysis becomes much involved even for general stability and bifurcation analysis on equilibria. With the Allee effect added, the logistic growth function becomes

$$r x \left(1 - \frac{x}{K}\right) \left(x - e\right).$$

We assume that the Allee threshold is far from the carrying capacity so that the parameter $e$ is assumed to be in the interval $(0, \frac{K}{2})$ [Verma & Misra 2018]. Then, applying a dimensionalless process to the resulting equations, with $\tau = r K t$, $x = K X$, $D = \frac{D}{K}$, $E = \frac{E}{K}$ and $S = \frac{S}{K}$ (for simplicity, we still use dot to indicate the differentiation with the new time $\tau$), we obtain the following nine
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dimensionless models:

System A:
\[
\begin{align*}
X &= X(1 - X)(X - E) - XY, \\
Y &= Y(CX - D), \\
&\quad Y = \frac{my}{rK}, \quad C = \frac{mc}{r}.
\end{align*}
\]

System B:
\[
\begin{align*}
X &= X(1 - X)(X - E) - \frac{XY}{A + X}, \\
Y &= Y \left(\frac{CX}{A + X} - D\right), \\
&\quad Y = \frac{my}{rK^2}, \quad A = \frac{a}{K}, \quad C = \frac{mc}{rK};
\end{align*}
\]

System C:
\[
\begin{align*}
X &= X(1 - X)(X - E) - \frac{X^2Y}{AX^2 + BX + 1}, \\
Y &= Y \left(\frac{CX^2}{AX^2 + BX + 1} - D\right), \\
&\quad Y = \frac{my}{rK^2}, \quad A = K^2a, \quad B = Kb, \quad C = \frac{mc}{rK};
\end{align*}
\]

System D:
\[
\begin{align*}
X &= X(1 - X)(X - E) - \frac{MY}{X + Y}, \\
Y &= Y \left(\frac{CX}{X + Y} - D\right), \\
&\quad Y = \frac{my}{rK}, \quad M = \frac{m}{rK};
\end{align*}
\]

System E:
\[
\begin{align*}
X &= X(1 - X)(X - E) - \frac{MXY}{X + Y}, \\
Y &= Y \left(\frac{CX^2}{X + Y} - D\right), \\
&\quad Y = \frac{my}{rK}, \quad M = \frac{m}{rK};
\end{align*}
\]

System F:
\[
\begin{align*}
X &= X(1 - X)(X - E) - \frac{MY}{X + Y}, \\
Y &= Y \left(\frac{CX}{X + Y} - D\right), \\
&\quad Y = \frac{my}{rK}, \quad M = \frac{m}{rK};
\end{align*}
\]

where $0 < E < \frac{1}{4}$, and $\frac{d^2Y}{dX^2} > -2$ for systems $A_{ii}$ and $C_{ii}$, while $B > -2$ for system $B_{ii}$. 

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In this paper, we will consider the four models $A_i$, $A_{ii}$, $B_i$ and $B_{ii}$. It was shown in [Jiang & Yu 2017] that without the Allee effect, all the four models have no B-T bifurcation, and only models $A_i$ and $B_i$ have codimension-one Hopf bifurcation. In this paper, besides bifurcation and stability analysis, we will show that with the Allee effect, the model $A_i$ also has codimension-one Hopf bifurcation, and the models $A_{ii}$, $A_{ii}$ and $B_{ii}$ still do not have B-T bifurcation, but the model $B_{ii}$ has codimension-two B-T bifurcation. Moreover, it is shown that with the Allee effect, the trivial equilibrium solution — the origin $(0,0)$ in the model $B_i$ becomes a stable node for all parameter values, while that in the model $B_{ii}$ without the Allee effect has very complex dynamics around it [Xiao & Yuan 2018]. Also, it is interesting to note that for the model $B_{ii}$, the supercritical Hopf bifurcation generated under no Allee effect is changed to a subcritical Hopf bifurcation under the Allee effect, and thus the stable limit cycle becomes unstable, leading to the stable domain of the trajectories being increased.

Before giving detailed discussions on the four models one by one, we prove the positivity of the solutions of the models. Moreover, we also show that these solutions are bounded.

**Theorem 1.1.** The solutions of the systems $A_i$, $A_{ii}$, $B_i$ and $B_{ii}$ are positive provided the initial conditions are positive. Moreover, these solutions are bounded and eventually attracted to a trapping region.

**Proof.** Since the proofs for the four models are similar, we do not repeat all of them, but only show the proof for model $A_i$.

Using the method of constant variations, we can write the general solution of system $A_i$ with the initial values $X(0)$ and $Y(0)$ as

$$X(\tau) = X(0) \exp \left\{ \int_0^{\tau} \left[ (1 - X(s)) \times (X(s) - E) - Y(s) \right] ds \right\},$$

$$Y(\tau) = Y(0) \exp \left\{ \int_0^{\tau} \left[ CX(s) - D \right] ds \right\},$$

which clearly indicates that $X(\tau) > 0$, $Y(\tau) > 0$ for any $\tau > 0$, if $X(0) > 0$ and $Y(0) > 0$.

Next, we prove that these solutions are bounded and eventually attracted to a right triangle trapping region $\Omega_{\Lambda_i}$, defined by

$$\Omega_{\Lambda_i} = \{(X,Y)|X > 0, Y > 0, Y < C \max \left\{ \frac{1 - E}{D}, 1 \right\} - CX \}.$$  

which is bounded by the X-axis, the Y-axis and the straight line: $Y + CX = C \max \left\{ \frac{1 - E}{D}, 1 \right\}$. Now, consider the line L, defined by

$$L: Y + CX = C \geq C \max \left\{ \frac{1 - E}{D}, 1 \right\}.$$ 

which implies that the line L is above the trapping region $\Omega_{\Lambda_i}$.

To prove that all trajectories are attracted into the trapping region $\Omega_{\Lambda_i}$, we construct the function

$$F = \frac{1}{D}(Y + CX).$$

Since both $X$-axis and $Y$-axis are invariant, we only need to prove that $\frac{dF}{d\tau} < 0$ along the trajectories on the line L. Simple calculation shows that the transversality condition is given by

$$\frac{dF}{d\tau} = \frac{C}{D}X(1 - X)(X - E) - Y.$$ 

Consider the planar curve on the $X-Y$ plane, defined by the equation

$$S: \frac{C}{D}X(1 - X)(X - E) - Y = 0,$$

which passes through the $X$-axis at the points: $(0,0)$, $(E,0)$ and $(1,0)$, as shown in Fig. [1(b)] where the red curve is $S$, the green line is L, and the blue curve denotes the tangent line T to $S$, passing through the point $(1,0)$ on $S$.

It is easy to see that $\frac{dF}{d\tau} > 0 (< 0)$ in the region below (above) the curve $S$, and it can be shown that the slope of the tangent line $T$ at the point $(1,0)$ is $-\frac{C}{D}(1-E)$. Thus, $\frac{dF}{d\tau} < 0$ on the line L. In fact, it is easy to see from Fig. [1(b)] that $\frac{dF}{d\tau} < 0$ for $0 < X < E$ and $X > 1$, and the line $X = 1$ can serve as a part of the boundary of $\Omega_{\Lambda_i}$ [see Fig. [1(b)]]. But for $E < X < 1$, in the region bounded by the curve $S$ and the $X$-axis, $\frac{dF}{d\tau} > 0$. That is why the line L must enclose the curve S. More precisely, we may find the line L which is just tangent to the curve S, but its expression is more involved.
In the next four sections, we will respectively study boundedness of system B and Allee effect for a convenient comparison. Simulations are presented in Sec. 5, and finally conclusion is drawn in Sec. 6.

The rest of the paper is organized as follows. In the next four sections, we will respectively study the models $A_i$, $A_ii$, $B_i$, and $B_ii$ on the property of solutions, equilibrium solutions and their stability, and bifurcations from the equilibrium solutions. At the beginning of each section, we will summarize the results of the model without the Allee effect, taken from Jiang & Yu [2017], in order to give a comparison with those results obtained for the model with the Allee effect. Simulations are presented in Sec. 5 and finally conclusion is drawn in Sec. 6.

2. Dynamics and Bifurcations of System $A_i$

Now we consider the system $A_i$. First, we list the existing results for the system without the Allee effect in order to give a comparison.

2.1. The results for system $A_i$ without the Allee effect

The system without the Allee effect is described by

$$\begin{align*}
\dot{X} &= X(1 - X) - XY, \\
\dot{Y} &= Y(CX - D),
\end{align*}$$

which has three equilibrium solutions. The solutions and their stability Jiang & Yu [2017] are given below:

$$E_0 = (0, 0), \quad \text{Saddle},$$

$$E_1 = (1, 0), \quad \text{GAS for } D \frac{C}{C} \geq 1,$$

$$E_3 = \left( \frac{D}{C}, 1 - \frac{D}{C} \right), \quad \text{GAS for } 0 < \frac{D}{C} < 1,$$

$$E_i \quad \text{for } \frac{D}{C} \leq 0,$$

where GAS stands for Globally Asymptotically Stable. The notation on the numbers of equilibria follows that for the system with the Allee effect in order to have an easy comparison. It is clear that the system $A_i$ without the Allee effect does not have complex dynamical behaviors, which has either a GAS equilibrium $E_1$ or a GAS equilibrium $E_3$, depending upon the ratio $\frac{D}{C}$. The bifurcation diagram is shown in Fig. 2(a), which is placed together with the bifurcation diagram for system $A_ii$ with the Allee effect for a convenient comparison.

2.2. Stability and bifurcations of equilibria of system $A_i$ with the Allee effect

Now we turn to system $A_i$ with the Allee effect, which is given by

$$\begin{align*}
A_i : \begin{cases}
\dot{X} &= X(1 - X)(X - E) - XY, \\
\dot{Y} &= Y(CX - D).
\end{cases}
\end{align*}$$
The equilibrium solutions of this system can be easily obtained by setting \( X = Y = 0 \), as given by

\[ \begin{align*}
E_0 & : (X_0, Y_0) = (0, 0), \\
E_1 & : (X_1, Y_1) = (1, 0), \\
E_2 & : (X_2, Y_2) = (E, 0), \\
E_3 & : (X_3, Y_3) = \left( \frac{D}{C} \left( 1 - \frac{D}{C} \right) \left( \frac{D}{C} - E \right) \right).
\end{align*} \]  

(12)

The equilibrium solutions \( E_0, E_1 \) and \( E_2 \) exist for positive parameter values, while \( E_3 \) exists for \( E < \frac{D}{C} < 1 \). It is seen that system \( \Lambda_1 \) with the Allee effect has one more equilibrium \( E_2 \) than that of the system without the Allee effect.

For \( \Lambda_1 \) system with the Allee effect, we have the following theorem for the stability and bifurcations of these equilibrium solutions.

**Theorem 2.1.** For system \( \Lambda_1 \), \( E_0 \) is a stable node and globally asymptotically stable for \( \frac{D}{C} < E \); \( E_1 \) is a stable node when \( \frac{D}{C} > 1 \) and a saddle point when \( \frac{D}{C} < 1 \); \( E_2 \) is a saddle point when \( \frac{D}{C} > E \), and an unstable node when \( \frac{D}{C} < E \); \( E_3 \) exists for \( E < \frac{D}{C} < 1 \), and it is a stable focus for \( \frac{1 - D}{D} < \frac{C}{D} < 1 \) and an unstable focus for \( E < \frac{D}{C} < \frac{1 - D}{D} \). Hopf bifurcation occurs from \( E_3 \) at the critical point \( \frac{|B_1|^2}{E_3} = \frac{1 + D}{2} \). The system has no B–T bifurcation.

**Proof.** The stability of these equilibrium solutions is determined from the Jacobian matrix of the system, given by

\[ J(X, Y) = \begin{bmatrix}
-3X^2 + 2(1 + E)X - Y & -X \\
CY & CX - D
\end{bmatrix} \]  

(13)

Evaluating the Jacobian \( J \) on the equilibrium \( E_0 \) yields two eigenvalues: \(-E < 0 \) and \(-D < 0 \), indicating that \( E_0 \) is a stable node. In fact, the corresponding two eigenvectors are along the \( X \)-axis and the \( Y \)-axis since the \( X \)- and \( Y \)-axes are invariant.

Similarly, evaluating the Jacobian \( J \) on \( E_1 \), we obtain two eigenvalues: \(-1 - E < 0 \) and \( C - D \). So \( E_1 \) is a stable node when \( \frac{D}{C} > 1 \) and a saddle point when \( \frac{D}{C} < 1 \). Next, evaluating the Jacobian \( J \) on \( E_2 \), we obtain two eigenvalues: \( E(1 - E) > 0 \) and \( CE - D \). So \( E_2 \) is a saddle point when \( \frac{D}{C} > E \) and an unstable node when \( \frac{D}{C} < E \).

For the equilibrium \( E_3 \) which exists for \( E < \frac{D}{C} < 1 \), we have the Jacobian matrix:

\[ J_3 = \begin{bmatrix}
\frac{D}{C} \left( 1 + E \right) & \frac{2D}{C} - \frac{D}{C} \\
C \left( 1 - \frac{D}{C} \right) - \frac{D}{C} & 0
\end{bmatrix}. \]

(14)

which yields the determinant and trace as

\[ \det(J_3) = D \left( 1 - \frac{D}{C} \right) \left( \frac{D}{C} - E \right) > 0, \]

and \( E_3 \) exists,

\[ \text{Tr}(J_3) = \frac{D}{C} \left( 1 - \frac{D - 2D}{C} \right). \]

Hence, \( E_3 \) is stable (unstable) if \( \text{Tr}(J_3) < 0 \) (\( > 0 \)), i.e. if \( \frac{1 + D}{2} < \frac{D}{C} < 1 \) (\( E < \frac{D}{C} < \frac{1 - D}{2} \)). Since \( \det(J_3) > 0 \), B–T bifurcation is not possible.

At the critical point, \( \frac{|B_1|^2}{E_3} = \frac{1 + D}{2} \), \( J_3 \) has a purely imaginary pair. Moreover, a simple calculation shows that the transversality condition is given by

\[ T_{\text{trans}} = \frac{1}{2} \left( \frac{d}{|B_1|^2} \right) \left( \frac{1 + D}{2} \right) \]

(15)

implying that Hopf bifurcation occurs from \( E_3 \) at the critical point \( E_3 = \frac{|B_1|^2}{E_3} \).

To prove that the equilibrium \( E_0 \) is globally asymptotically stable for \( \frac{D}{C} < E \), note that the equilibria \( E_0, E_1, \) and \( E_2 \) are all boundary equilibria, while \( E_3 \) is an interior point if it exists. Since for \( \frac{D}{C} < E \), \( E_3 \) does not exist, and \( E_1 \) and \( E_2 \) are unstable, so the only stable equilibrium is \( E_0 \) on the boundary of the trapping region. Thus, all trajectories would converge to \( E_0 \).

**2.3. Codimension of Hopf bifurcation**

Next, we prove that the Hopf bifurcation is supercritical and bifurcating limit cycles are stable. Moreover, we show that the codimension of the Hopf bifurcation is one. We have the following theorem.

**Theorem 2.2.** For the system \( \Lambda_1 \), Hopf bifurcation occurs from \( E_3 \) at the critical point \( \frac{|B_1|^2}{E_3} = \frac{1 + D}{2} \).
and it is supercritical with stable bifurcating limit cycles. Moreover, the codimension of Hopf bifurcation is one.

Proof. We apply normal form theory to find the focus value. To achieve this, without loss of generality, we may use the critical value \( \frac{D}{C} = \frac{[E]}{[E]} \) to let \( D = \frac{C(1+E)}{1+D} \) and then introduce the affine transformation:

\[
X = \frac{1}{2} + x_1, \quad Y = \frac{1}{4}(1 - E)^2 - \frac{2\omega_c}{1 + E}x_2,
\]

where \( \omega_c = \frac{1}{2\sqrt{2}}(1 - E)\sqrt{C(1 + E)} \), into system (11) to obtain

\[
\dot{x}_1 = \omega_c x_2 - \frac{1 + E}{2} x_1^3 + \frac{2\omega_c}{1 + E} x_1 x_2 - x_1^2 \\
\dot{x}_2 = -\omega_c x_1 + C x_1 x_2 \equiv g(x_1, x_2).
\]

Then, we may either apply the Maple program in [13] or directly use the following formula to get the first-order focus value, evaluated at \((x_1, x_2) = (0, 0)\):

\[
v_1 = \frac{1}{16} \left( \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_1 \partial x_2} + \frac{\partial^2 g}{\partial x_1^2} \frac{\partial^2 g}{\partial x_2^2} \right) \\
- \frac{1}{16\omega_c} \left[ \frac{\partial^2 f}{\partial x_1 \partial x_2} \left( \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} \right) \\
- \frac{\partial^2 f}{\partial x_1^2} \left( \frac{\partial^2 g}{\partial x_1^2} + \frac{\partial^2 g}{\partial x_2^2} \right) \\
- \frac{\partial^2 f}{\partial x_2^2} \left( \frac{\partial^2 g}{\partial x_1^2} + \frac{\partial^2 g}{\partial x_2^2} \right) \right) \\
- \frac{6}{16} + \frac{1}{16\omega_c} \left( \frac{2\omega_c}{1 + E} (1 + E) \right) \\
- \frac{1}{4}
\]

which holds negative for any positive parameter values, indicating that the codimension of Hopf bifurcation is one, and it is supercritical with stable bifurcating limit cycles for \( \frac{D}{C} \in (E, \frac{[E]}{1+D}) \).

2.4. Bifurcation diagram and simulation

Based on Theorem 2.4, we choose \( X \) as the state variable and \( \frac{D}{C} \) as the bifurcation parameter to obtain the bifurcation diagram, as shown in Fig. 2(b). It can be seen from this figure that there exist bistable states (\( E_0, E_1 \)) for \( \frac{D}{C} > 1 \), and states (\( E_0, E_1 \)) for \( \frac{D}{C} < 1 \). There is a transcritical bifurcation between \( E_0 \) and \( E_1 \) at \( \frac{D}{C} = 1 \), and a Hopf bifurcation from \( E_0 \) at \( \frac{D}{C} = \frac{[E]}{1+D} \). T and H denote the transcritical bifurcation and Hopf bifurcation, respectively.

Now we present simulations for the model A, since the bifurcation diagrams in Fig. 2 have shown the difference between the models without the Allee effect and with the Allee effect, we shall only present the results for the model with the Allee effect, which display more complex dynamical behaviors. We take the following parameter values for simulation:

\[
E = 0.4, \quad \frac{D}{C} = 0.2, 0.68, 0.8, 1.2.
\]

In simulation we take \( C = 1 \) for simplicity. The Hopf critical point is at \( \frac{D}{C} = 0.7 \). Hence, the equilibrium \( E_1 \) only exists for \( \frac{D}{C} \in (0.1, 1) \) and it is stable for \( 0.7 < \frac{D}{C} < 1 \) and unstable for \( 0.4 < \frac{D}{C} < 0.7 \). Therefore, the bistable phenomenon exists for \( \frac{D}{C} > 0.7 \), for which two stable equilibria \( E_0 \) and \( E_1 \) coexist for \( 0.7 < \frac{D}{C} < 1 \), and two stable equilibria \( E_0 \) and \( E_1 \) coexist for \( \frac{D}{C} > 1 \). Moreover, since the Hopf bifurcation is supercritical, yielding stable limit cycles, another bistable phenomenon involving the stable equilibrium \( E_0 \) and a stable limit cycle near the Hopf critical point for \( \frac{D}{C} = \frac{[E]}{1+D} \) can also occur. Note that the unstable equilibrium \( E_2 \) is \((0,4,0)\) is between the two stable equilibria \( E_0 \) and \( E_1 \). So at the bistable situations, choosing different initial points may converge to different equilibria. The simulations are given in Fig. 3 indicating an excellent agreement with the analytical predictions which are shown in the bifurcation diagram [see Fig. 2(b)].

Remark 2.3. Comparing bifurcation diagram in Fig. 2(b) (with the Allee effect) with that in Fig. 2(a) (without the Allee effect) we can see that the Allee effect has a great impact on the dynamical behaviors of the system. First of all, note that the model without the Allee effect does not exhibit complex behaviors, it either has a globally asymptotically stable equilibrium \( E_1 \) if \( \frac{D}{C} > 1 \) or a globally asymptotically stable equilibrium \( E_0 \) if \( \frac{D}{C} < 1 \). Only a transcritical bifurcation occurs between \( E_1 \) and \( E_1 \) at \( \frac{D}{C} = 1 \), see Fig. 2(a). However, for the model with the Allee effect, except for no change
on the equilibrium $E_1$, we have the following interesting observations.

(1) A new unstable equilibrium $E_3 = (E, 0)$ emerges due to the Allee effect, which causes the stability change of the equilibrium $E_0$ from unstable to stable. In other words, due to the Allee effect, solutions near $E_0$ now converge to $E_0$, implying that a strong Allee effect can cause species to extinct, in particular when $\frac{D}{C} < E$, as shown in Figs. 2 and 3(b).

(2) The positive (interior) equilibrium $E_3$ now exists only for $E < \frac{D}{C} < 1$, while $E_0$ is globally asymptotically stable for $0 < \frac{D}{C} < E$. Moreover, $E_3$ is only stable for $\frac{A+D}{X} < \frac{D}{C} < 1$. This implies that the Allee effect destabilizes the positive equilibrium for intermediate values of $\frac{D}{C}$, see Figs. 2 and 3(b).

(3) Complex dynamics including bistable phenomena occurs. One kind of the bistable phenomena includes two stable equilibria: $E_0$ and $E_2$ for slightly larger $\frac{D}{C} \in \left(\frac{A+D}{X}, 1\right)$, and $E_0$ and $E_1$ for $\frac{D}{C} > 1$. Solutions converge to either $E_0$ or $E_2$ for the former, and $E_0$ or $E_1$ for the latter, depending upon the initial conditions. These two bistable phenomena are shown in Figs. 2(a) and 3(b). The other kind of bistable phenomenon, due to Hopf bifurcation, involves the stable equilibrium $E_0$ and a stable limit cycle for slightly larger $\frac{D}{C} \in \left(E, \frac{A+D}{X}\right)$, and solutions converge to either the equilibrium $E_0$ or the stable limit cycle, depending on the initial condition, see Figs. 2(b) and 3(b).

(4) Note that the Hopf critical point is always the midpoint of the interval $\frac{D}{C} \in \left(E, 1\right)$ for which $E_3$ exists. As $E \to 0$, the Hopf bifurcation exists as long as $E > 0$. It shows a discontinuity (jumping) from the case $E = 0$ to the case $E > 0$, because when $E = 0$, the model does not have the term $(X - E)$ on the first equation. Thus, as $E \to 0$, the unstable $E_3$ coincides with $E_0$ which then becomes unstable; while as $E$ goes to its maximal value $\frac{D}{C}$, $E_0$ becomes more stable, and the Hopf critical point on $E_3$ moves towards $\left(\frac{D}{C}\right)_H = \frac{1}{2}$, and it becomes less stable.

3. Dynamics and Bifurcations of System $A_{ii}$

In this section, we consider the model $A_{ii}$. Similarly, we first summarize the existing results on this system without the Allee effect.

3.1. The results for the system $A_{ii}$ without the Allee effect

The model $A_{ii}$ without the Allee effect is described by

$$\dot{X} = X(1 - X) - \frac{XY}{A + X}
$$

(17)

$$\dot{Y} = Y \left( \frac{CX}{A + X} - D \right),$$

which has three equilibrium solutions. The solutions and their stability [Jiang & Yu 2013] are given
Fig. 3. Simulated phase portraits for the model $A_i$ with $E = 0.4$, $C = 1$: (a) $D = 0.2$, converging to $E_0$, (b) $D = 0.68$, converging to $E_0$ or a stable limit cycle, (c) $D = 0.8$, converging to $E_0$ or $E_3$ and (d) $D = 1.2$, converging to $E_0$ or $E_1$.

below:

$E_0 = (0, 0)$, Saddle,

$E_1 = (1, 0)$, GAS for $A \geq \max \left\{ 0, \frac{C}{D} - 1 \right\}$,

$E_3 = \left( \frac{AD}{C-D} \frac{AC-(A+1)D}{(C-D)^2} \right)$, GAS for $0 < A < \frac{C}{D} - 1$.

(18)

It is seen that like system $A_i$ without the Allee effect, system $A_{ii}$ without the Allee effect does not have complex dynamical behaviors, which has either a GAS equilibrium $E_1$ or a GAS equilibrium $E_3$ depending on the value of $\frac{C}{D} - 1$. The bifurcation diagram is depicted in Fig. 3(a), where it is assumed $C > D$; otherwise, $E_3$ does not exist and $E_1$ is globally asymptotically stable.

3.2. Stability and bifurcations of equilibria of system $A_{ii}$ with the Allee effect

The model $A_{ii}$ with the Allee effect is given by

$$A_{ii}:
\begin{cases}
\dot{X} = X(1-X)(X-E) - \frac{XY}{A+X}, \\
\dot{Y} = Y \left( \frac{CX}{A+X} - D \right).
\end{cases}
$$

(19)
The equilibrium solutions of this system can be easily found by setting $\dot{X} = \dot{Y} = 0$ as:

- $E_0 : (X_0, Y_0) = (0, 0)$,
- $E_1 : (X_1, Y_1) = (1, 0)$,
- $E_2 : (X_2, Y_2) = (E, 0)$,
- $E_3 : (X_3, Y_3)$, where $X_3 = \frac{AD}{C - D}$,

$$Y_3 = (1 - X_3)|(X_3 - E)|(A + X_3).$$

The equilibrium solutions $E_0$, $E_1$ and $E_2$ exist for positive parameter values, while $E_3$ exists for

$$C > D, \quad \frac{(C - D)E}{D} < A < \frac{C - D}{D}, \quad (20)$$

since $E < X_3 < 1$ due to $Y_3 > 0$.

Theorem 3.1. For system $A_0$, $E_0$ is a stable node; $E_1$ is either a stable node if $A > \max\{0, \frac{C}{D} - 1\}$ or a saddle point if $0 < A < \frac{C}{D} - 1$; $E_2$ is unstable (either a saddle point or an unstable node). $E_3$ exists when the condition (20) is satisfied, and is asymptotically stable for $A_H < A < \frac{C - D}{D}$, where:

$$A_H = \frac{C - D}{2D(2C + D)}[(1 + E)(C + D)$$

$$+ \sqrt{(1 - E)^2(C + D)^2 + 4EC^2}], \quad (22)$$

Hopf bifurcation occurs from $E_3$ at the critical point $A_H$. There does not exist $B$-$T$ bifurcation.

Proof. Similarly, with the Jacobian of the system, given by

$$J(X, Y) = \begin{bmatrix}
    -E + 2(1 + E)X - 3X^2 & -AV \frac{X}{A + X} \\
    \frac{ACY}{(A + X)^2} & CX \frac{X}{A + X} - D
\end{bmatrix},$$

we can determine the stability of these equilibrium solutions.

Evaluating the Jacobian $J$ on the equilibrium $E_0$, we have two negative eigenvalues: $-E$ and $-D$, implying that $E_0$ is a stable node. Again note that the $X$- and $Y$-axes are invariant.

Similarly, evaluating the Jacobian $J$ on $E_1$ yields two eigenvalues: $-(1 - E) < 0$ and $\frac{AD}{D} - D$.

So $E_1$ is a stable node if $A > \max\{0, \frac{C}{D} - 1\}$ and a saddle point if $0 < A < \frac{C}{D} - 1$. Next, evaluating the Jacobian $J$ on $E_2$, we obtain two eigenvalues: $E(1 - E) > 0$ and $\frac{AD}{D} - D$. Thus, $E_2$ is a saddle point if $A > \max\{0, \frac{C}{D} - 1\}E$, and an unstable node if $0 < A < \frac{C}{D} - 1$. 

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Obviously, \( \det(J) > 0 \) for existing \( E_0 \), therefore, \( E_0 \) is asymptotically stable if \( \Tr(J) < 0 \). Let
\[
F_{A_0} = D(2C + D)A^2 - (1 + E)(C^2 - D^2)A + E(C - D)^2, \tag{23}
\]
we have
\[
A_0 = \frac{(C - D)E}{D} = \frac{C - D}{2D(2C + D)}(C + D) - E(3C + D) \geq \sqrt{(1 - E)^2(C + D)^2 + 4EC^2} \tag{26}
\]
which is a quadratic polynomial in \( A \) with the discriminant,
\[
\Delta_{A_0} = (1 + E)^2(C^2 - D^2)^2 - 4ED(2C + D)(C - D)^2 = (C - D)^2[(1 - E)^2(C + D)^2 + 4EC^2] > 0.
\]
Thus, \( P = 0 \) has two real positive roots:
\[
A_\pm = \frac{C - D}{2D(2C + D)}[(1 + E)(C + D) \pm \sqrt{(1 - E)^2(C + D)^2 + 4EC^2}], \tag{25}
\]
Due to the existence condition (21), it requires that at least one of \( A_\pm \) must belong to the interval \( (\frac{C - D}{2D(2C + D)}, \frac{-C}{2D}) \). A simple calculation shows that
\[
A_+ - \frac{(C - D)E}{D} < \frac{C - D}{2D(2C + D)}, \quad \text{and so } A_- \notin (\frac{C - D}{2D(2C + D)}, \frac{-C}{2D}). \tag{28}
\]
Next, we prove \( A_+ < \frac{C - D}{2D(2C + D)} \). To show \( A_+ > \frac{C - D}{2D(2C + D)} \), we notice that \( (C + D) - E(3C + D) > 0 \) for \( \frac{C}{2D} > E \), so the first equation in (28) implies \( A_+ > \frac{C - D}{2D} \), while for \( \frac{C}{2D} < E < \frac{1}{2} \), we have \( (C + D) - E(3C + D) \leq 0 \), so the second equation in (28) again leads to \( A_+ > \frac{C - D}{2D} \). To prove \( A_+ < \frac{C - D}{2D} \), we can similarly obtain that
\[
A_+ - \frac{C - D}{2D} > 0.
\]
Next, we prove \( A_+ \leq \frac{C - D}{2D(2C + D)} \). To show \( A_+ > \frac{C - D}{2D} \), we notice that \( (C + D) - E(3C + D) > 0 \) for \( \frac{C}{2D} > E \), so the first equation in (28) implies \( A_+ > \frac{C - D}{2D} \), while for \( \frac{C}{2D} < E < \frac{1}{2} \), we have \( (C + D) - E(3C + D) \leq 0 \), so the second equation in (28) again leads to \( A_+ > \frac{C - D}{2D} \). To prove \( A_+ < \frac{C - D}{2D} \), we can similarly obtain that
\[
A_+ = \frac{C - D}{2D(2C + D)} \sqrt{(1 - E)^2(C + D)^2 + 4EC^2} - [(1 - E)(C + D) + 2C]
\]
which clearly indicates that \( A_+ < \frac{C - D}{2D} \). Hence, \( A_+ \in (\frac{C - D}{2D(2C + D)}, \frac{C}{2D}) \) for \( 0 < E < \frac{1}{2} \).

Let \( A_H = A_+ \). Then, concluding the above discussions we know that the equilibrium \( E_0 \) exists for \( \frac{C - D}{2D(2C + D)} < A < \frac{C}{2D} \), and it is asymptotically stable for \( A_H < A < \frac{C}{2D} \), and unstable for \( \frac{C - D}{2D(2C + D)} < A < A_H \). Further, we can use (23) to find the transversality condition as follows:
\[
T_{\text{trans}} = \frac{1}{2} \frac{d\Tr(J)}{dA} \bigg|_{A=A_H} = \frac{D}{2C(C - D)^2} [2D(2C + D)A_H - (1 + E)(C^2 - D^2)]
\]
\[
= \frac{D}{2C(C - D)} \sqrt{(1 - E)^2(C + D)^2 + 4EC^2} < 0,
\]

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implying that Hopf bifurcation occurs from the equilibrium $E_3$ at the critical point $A = A_H$. Note that B-T bifurcation cannot happen since $\det(J_3) > 0$ when $E_3$ exists.

### 3.3. Codimension of the Hopf bifurcation

Now we further study the Hopf bifurcation and determine the codimension of the Hopf bifurcation. We have the following theorem.

**Theorem 3.2.** For system $A$, Hopf bifurcation occurs from $E_3$ at the critical point $A = A_H$. The Hopf bifurcation is supercritical and bifurcating limit cycles are stable. Moreover, the codimension of Hopf bifurcation is one.

**Proof.** In order to determine the stability of bifurcating limit cycles, similar to the analysis for the system $A$, we apply the method of normal forms. To have an affine transformation for system $[39]$ at the critical point $A = A_H$, since $F_{A_0}$ in $[40]$ is a quadratic polynomial in $A$, we use $E$, instead of $A$, to solve $F_{A_0} = 0$ for the convenience in normal form computation. Thus, solving $F_{A_0} = 0$ for $E$

$$E_H = A[D(2C + D)A - (C^2 - D^2)] / ([C - D][C + D]A - (C - D))$$

In order to have $E_H \in (0, \frac{1}{2})$, the following conditions must hold:

$E_H > 0 \Rightarrow A < \frac{C - D}{C + D} A_1$ or $A > \frac{C^2 - D^2}{D(2C + D)} \triangleq A_2$

and

$$E_H < \frac{1}{2} \Leftrightarrow 2D(2C + D)A^2 - 3(C^2 - D^2)A + (C - D)^2 < 0$$

$A_0 \triangleq \frac{(C - D)[3(C + D) - \sqrt{8C^2 + (C + D)^2}]}{4D(2C + D)} < A$

$$< \frac{(C - D)[3(C + D) + \sqrt{8C^2 + (C + D)^2}]}{4D(2C + D)} \triangleq A_p.$$  

It is easy to prove that $0 < A_0 < A_1 < A_2 < A_p < \frac{\sqrt{8C^2}}{2D}$. Moreover, it follows from the existence condition of $E_3$ that

$$\frac{(C - D)E}{D} < A < \frac{C - D}{D} \Rightarrow E_H < \frac{AD}{C - D} \Rightarrow \frac{C - D}{C + D} < A < \frac{C - D}{D}$$

Hence, the required constraints on $A$ are given by

$$A \in (A_2, A_p),$$

for which $0 < E_H < \frac{1}{2}$.

To find the normal form (or focus value) of the system associated with the Hopf bifurcation, we multiply the equations in $[39]$ by $A + X$ (i.e. applying a time rescaling $\tau \rightarrow (A + X)\tau$) for convenience, and then apply the following transformation,

$$X = \frac{AD}{C - D} + x_1, \quad Y = \left(1 - \frac{AD}{C - D}\right)\left(\frac{AD}{C - D} - \frac{E}{A + \frac{AD}{C - D}}\right) - \frac{\omega_x(C - D)}{AD} x_2,$$

where

$$\omega_x = \frac{AC(C - D - AD)}{C - D} \sqrt{\frac{AD}{[C - D][C(A + D) - (C - D)]}} > 0, \quad \text{for} \quad A \in (A_2, A_p).$$
Applying the above transformation, we have the following new system:

\[
\begin{align*}
\dot{x}_1 &= \omega x_2 - \frac{AD[A^2(C^2 + CD + D^2) - A(C - D)(C + 2D) + (C - D)^2]x_1^2}{(C - D)[A(C + D) - (C - D)]} \\
&+ \frac{(C - D)x_1\omega x_2}{AD} - \frac{A^2(C^2 + 2CD + D^2) - A(C - D)(C + 3D) + (C - D)^2}{(C - D)[A(C + D) - (C - D)]} x_1^2 - x_1^3,
\end{align*}
\]

\[
\dot{x}_2 = -\omega x_1 + (C - D)x_1 x_2.
\]

Now, applying the Maple program [Yu 2019b] to the above system we obtain the following first-order focus value:

\[
v_1 = -\frac{1}{8(C - D)[A(C + D) - (C - D)]} \times \left[ (2C^2 + 5CD + 5D^2)x^2 - (C - D) \right] \times (2C + 7D) + 2(C - D)^2,
\]

where the denominator in \(v_1\) is positive for \(A \in (A_2, A_3)\), and the term in the square bracket is a quadratic polynomial in \(A\), which has the discriminant,

\[
\Delta = -3(C - D)^2(2C + 3D)(2C - D) < 0, \quad (C > D),
\]

implying that the term in the square bracket is positive and so \(v_1 < 0\). Therefore, the codimension of the Hopf bifurcation is one, and it is supercritical, yielding stable limit cycles.

3.4. Bifurcation diagram and simulation

The bifurcation diagram of system \(A_\text{ii} \) for \(C > D\) is shown in Fig. 3(b).

Because the simulations for this model \(A_\text{ii} \) are similar to that of the model \(A_\text{i} \) (see Fig. 3(a), we only present the phase portraits near the Hopf critical point, see Fig. 4. For simulation, taking \(C = 1\), \(D = 0.6\), \(E = 0.4\), we have

\[
0.2667 \approx \frac{(C - D)E}{D} < A_{\text{ii}} = \frac{2}{195}(28 + \sqrt{394})
\]

\[
\approx 0.4908 < \frac{C - D}{E} \approx 0.6667.
\]

The simulation for \(A = 0.48 < A_{\text{ii}}\) is depicted in Fig. 3(b) demonstrating the bistable phenomenon: trajectories either converge to the stable equilibrium \(E_0\) or to a stable limit cycle, depending upon the initial conditions. This again shows an excellent agreement with the analytical prediction.

Remark 3.3. Comparing the bifurcation diagram in Fig. 3(b) having the Allee effect with that in Fig. 3(a) without the Allee effect shows the great impact of the Allee effect on the dynamics of system \(A_\text{ii} \). Since system \(A_\text{ii} \) is quite similar to system \(A_\text{i} \), Remark 3.3 on the dynamics \(A_\text{i} \) can be applied here, as long as the three critical values in Fig. 3(b), \(E = E_0\), \(\frac{C - D}{E} = E\), and 1 are changed to \(A = \frac{C - D}{E} = A_{\text{ii}}\) and \(\frac{C - D}{E} \), respectively. The bistable phenomenon with one stable equilibrium and one stable limit cycle is shown in Fig. 3(b).

4. Dynamics and Bifurcations of System \(B_\text{i} \)

We first summarize the results for the model without the Allee effect.
4.1. The results for model $B_i$ without the Allee effect

The model without the Allee effect is described by

\[ \dot{X} = X(1-X) - MX, \]

\[ \dot{Y} = X - DY. \]

which has only two equilibrium solutions, given below with stability [Jiang & Yu 2017]:

\[ E_0 = (0, 0), \quad \text{GAS for } M \geq 1, \]

\[ E_3 = \left( 1 - M, \frac{1 - M}{D} \right), \quad \text{GAS for } 0 < M < 1, \]

which clearly shows that the system does not have complex dynamical behaviors, but either a GAS equilibrium $E_0$ or a GAS equilibrium $E_3$ depending on whether $M \geq 1$ or $M < 1$. The bifurcation diagram is shown in Fig. 6(a).

4.2. Stability and bifurcation of equilibria of system $B_i$ with the Allee effect

The model $B_i$ with the Allee effect is described by

\[ \dot{X} = X(1-X)(X-E) - MX \equiv h_1(X), \]

\[ \dot{Y} = X - DY \equiv h_2(X, Y), \]

which also has only two equilibrium solutions, given below:

\[ E_0 : (X_0, Y_0) = (0, 0), \]

\[ E_3 : (X_3, Y_3) = \left( X_3, \frac{1}{D} X_3 \right), \]

where $X_3$ is determined from the following quadratic polynomial equation,

\[ F_1 = (1 - X_3)(X_3 - E) - M = -[X_3^2 - (1 + E)X_3 + E + M] = 0. \]

The quadratic polynomial has two solutions:

\[ X_3 = \frac{1 + E \pm \sqrt{\Delta_{B_i}}}{2}, \]

where $\Delta_{B_i} = (1 + E)^2 - 4(E + M) = (1 - E)^2 - 4M$. (33)

$\Delta_{B_i} \geq 0$ is needed for $X_3$ to be real positive, which yields

\[ 0 < M < \frac{(1 - E)^2}{4} \equiv M_u. \]

It is easy to see that $E < X_3 < 1$ under the condition $E \in (E, 1)$. When $M = M_u$, $F = 0$ has a unique solution: $X_3 = \frac{1 + E}{2} \in (E, 1)$, which actually defines a saddle-node bifurcation point. For $M > M_u$, $E_3$ does not exist. More precisely, define the equilibria:

\[ E_{3\pm} = \left( X_{3\pm}, \frac{1}{D} X_{3\pm} \right). \]

Fig. 6. Bifurcation diagrams for system $B_i$: (a) without the Allee effect and (b) with the Allee effect.
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For stability and bifurcation of the equilibria, we have the following theorem.

**Theorem 4.1.** For the system $B_i$, $E_0$ is a stable node; $E_{3-}$ is a saddle point, while $E_{3+}$ is a stable focus. There is no Hopf bifurcation, nor B–T bifurcation.

**Proof.** Note that equation $\dot{Y}$ is decoupled from equation $X$, and that $\frac{\partial h_0}{\partial Y} = -D < 0$, which clearly shows that there is no Hopf bifurcation, nor B–T bifurcation. For stability of the equilibria, we only need to consider the equation $X$, i.e. the function $h_1(X)$. It is easy to obtain

$$h_1'(X) = -E + 2(1 + E)X - 3X^2 - M.$$ 

So $h_1'(0) = -(E + M) < 0$ indicates that $E_0$ is a stable node.

Next, evaluating $h_1'(X_{3\pm})$ yields

$$h_1'(X_{3\pm}) = \frac{1}{2} \sqrt{(1 - E)^2 - 4M}$$

and so $h_1'(X_{3\pm}) > 0$, implying that $E_{3-}$ is a saddle point; while $h_1'(X_{3\pm}) < 0$, indicating that $E_{3+}$ is a stable node. Moreover, at the turning point $M = M_0$, $g_1'(X_{3\pm}) = 0$, we know that $M_0$ is a saddle-node bifurcation point.

**4.3. Bifurcation diagram and simulation**

The bifurcation diagram is shown in Fig. 6(b).

The model $B_i$ does not have Hopf bifurcation, but exhibits bistable phenomenon involving two stable equilibria, $E_0$ and $E_{3\pm}$, which does not appear in the model without the Allee effect. To show the bistable phenomenon, we choose $D = 0.5$, $E = 0.25$ and $M = 0.10 < \frac{1 - E}{2} = 0.140625$. The simulation is given in Fig. 7, which shows that trajectories either converge to the stable equilibrium $E_0$ or to the equilibrium $E_{3\pm}$ depending on the initial conditions. Since the solution $X_1$ is independent of $D$ (only depending upon $E$ and $M$), and $Y_1 = X_1^2$ which yields $\frac{\partial Y_1}{\partial X} = \frac{X_1}{X_1}$, we see that $Y_1$ is monotonically increasing as $D$ is decreasing regardless the value $X_1$.

**Remark 4.2.** It is seen from Figs. 6 and 7 that although this system does not have Hopf bifurcation, nor B–T bifurcation, the Allee effect does have an impact on the dynamics. In particular, the transcritical bifurcation is changed to a saddle-node bifurcation, and the unstable part of the equilibrium $E_0$ without the Allee effect becomes stable with the Allee effect, which indicates an increase of extinction risk of species. Moreover, note that the equilibrium $E_{3\pm}$ exists with smaller interval in the parameter $M$ for the system with the Allee effect [see Fig. 6(b)], compared to that without the Allee effect. Also note that the $X$-axis is no longer invariant for $B_i$ system, as seen in Fig. 7 regardless whether the system has or does not have the Allee effect. However, the system still has the positivity property.

5. Dynamics and Bifurcations of System $B_i$

Finally, we consider system $B_i$. Again, we first list the existing results for the system without the Allee effect.

5.1. **The results for model $B_i$ without the Allee effect**

The system without the Allee effect is described by

$$\dot{X} = X(1 - X) \frac{MY}{X + Y},$$

$$\dot{Y} = Y \left( \frac{CX}{X + Y} - D \right).$$  \hspace{1cm} (36)
which has three equilibrium solutions given below together with stability [Jiang & Yu 2017].

\[ E_0 = (0, 0), \quad E_1 = (1, 0), \quad E_3 = \left( X_3, \left( \frac{C}{D} - 1 \right) X_3 \right). \]

- Stable or unstable or coexistence of stable and unstable sectors;
- GAS (a node) for \( 0 < C \leq D \), and a saddle point for \( C > D \);
- GAS for \( 0 < M < \frac{C}{C - D} \), \( C \geq D + 1 \),
- LAS for \( 0 < M < \frac{C(\text{C} - D) - D^2}{C^2 - D^2} \), \( D < C < D + 1 \),
- Supercritical Hopf bifurcation at \( M = M_B = \frac{C(\text{C} - D) - D^2}{C^2 - D^2} \),

where \( X_3 = 1 - M(1 - \frac{1}{D}) \). It should be noted that unlike the previous systems, the dynamics near the equilibrium \( E_0 \) in the first quadrant is very complex, and has been studied in detail by Xiao and Ruan [2001] using the blow-up technique. It was shown that in a neighborhood of \( E_0 \), there can exist various types of topological structures including the parabolic orbits, the elliptic orbits, the hyperbolic orbits, and any combination of them. In particular, \( E_0 \) can be asymptotically stable or unstable, or even coexistence of stable and unstable sectors.

For a comparison, the bifurcation diagrams for system (36) without the Allee effect are given in Figs. 8(a) and 8(b), in which only two cases \( 0 \leq D < C < D + 1 \) are shown, since the equilibrium \( E_3 \) is globally asymptotically stable for \( 0 < C \leq D \).

### 5.2. Stability and bifurcation of equilibria of system \( B_u \) with the Allee effect

The model \( B_u \) with the Allee effect is given by

\[
\begin{align*}
\dot{X} &= X(1 - X)(X - E) - \frac{MY}{X + Y}, \\
\dot{Y} &= Y\left( \frac{CX}{X + Y} - D \right). 
\end{align*}
\]

(38)

The system \( B_u \) has four equilibrium solutions:

\[
\begin{align*}
E_0 &: (X_0, Y_0) = (0, 0), \\
E_1 &: (X_1, Y_1) = (1, 0), \\
E_2 &: (X_2, Y_2) = (E, 0), \\
E_3 &: (X_3, Y_3) = \left( X_3, \left( \frac{C}{D} - 1 \right) X_3 \right). 
\end{align*}
\]

(39)

Define

\[ E_{3\pm} = \left( X_{3\pm}, \frac{C - D}{D}X_{3\pm} \right). \]

(43)

For the stability of the equilibrium solutions, we have the following two theorems. The first one
is for $E_0$. It should be noted that system (38) is not well defined at $E_0$. But since the $X$-axis and $Y$-axis are invariant, and so $\lim_{X \to 0^+} Y = \lim_{X \to 0^-} Y = 0$ provided that $X > 0$, $Y > 0$. Thus, for system (38) we define that $\dot{X} = Y = 0$ when $X = Y = 0$. However, the stability analysis of this equilibrium is not straightforward. We will again apply the blow-up technique to give a complete analysis on the stability of this equilibrium and show that, unlike $E_0$ of the system without the Allee effect, $E_0$ of the system with the Allee effect is actually a stable node. For convenience, define the region in the first quadrant (including the $X$-axis and $Y$-axis) in the $X$-$Y$ plane as $I$ and the interior of the first quadrant as $I^+$. Theorem 5.1. For system $B_{1i}$, $E_0$ is a stable node in $I^+$. Proof. First, it is easy to see from the equations in system (38) that both the $X$-axis and the $Y$-axis are invariant, and near $E_0$ trajectories converge to $E_0$ along the two axes. Thus, we only need to consider the solution trajectories in $I^+$. In order to apply the blow-up technique, we introduce the time rescaling $d\tau = (X + Y)dt_1$ into (38) such that system (38) is equivalent to the following system in $I^+$ (where the dot is now used to indicate differentiation with respect to $t_1$),

$$B_{1i}: \begin{cases} \dot{X} = X(1 - X)(X - E)(X + Y) - MXY \\ = -MXY - EX(X + Y) \\ + X^2(X + Y)(1 + E - X) \\ \equiv X_2(X, Y), \\ \dot{Y} = CYX - DY(X + Y) \\ \equiv Y_2(X, Y), \end{cases}$$

(44)

where $X_2$ and $Y_2$ represent second-degree homogeneous polynomials in $X$ and $Y$, and $\Phi(X, Y) = X^2(X + Y)(1 + E - X)$. It is obvious that $E_0$ is an isolated critical point of higher order for the system (44). It is easy to see that system (38) is analytic in a neighborhood of $E_0$. According to Theorem 3.10 in [Zhang et al., 1991], any orbit of (44) tending to $E_0$ must tend it spirally or along a fixed direction, which is determined by the characteristic equation of system (38). Since the solution of (44) is restricted in $I^+$, it is impossible to have the possibility that the orbit of (38) tending to $E_0$ spirally, and the only possibility is tending to $E_0$ along a fixed direction. Introduce the polar coordinates $X = r \cos \theta$, $Y = r \sin \theta$ into (38), and define

$$G(\theta) = \cos \theta Y_2(\cos \theta, \sin \theta) - \sin \theta X_2(\cos \theta, \sin \theta)$$

$$= \sin \theta \cos \theta(C + E - D) \cos \theta$$

$$+ (M + E - D) \sin \theta],$$

$$H(\theta) = \sin \theta Y_2(\cos \theta, \sin \theta) + \cos \theta X_2(\cos \theta, \sin \theta)$$

$$= -\sin^2 \theta(D \sin \theta + (D - C) \cos \theta)$$

$$-\cos^2 \theta(E \cos \theta + (M + E) \sin \theta),$$

(45)

where $\theta \in (0, \pi)$. The characteristic equation of (38) takes the form $G(\theta) = 0$, which clearly shows that this equation either has a real root $\theta$ or $G(\theta) \equiv 0$ in $\theta \in (0, \pi)$. In order to apply Theorems 3.1-3.3 in [Zhang et al., 1991], based on $G(\theta)$, we consider the following four cases:

(i) $C + E - D = 0$, $M + E - D \neq 0$, for which we have

$$G(\theta) = \sin^2 \theta \cos \theta(M + E - D) \neq 0$$

for $\theta \in \left(0, \frac{\pi}{2}\right)$. (ii) $C + E - D \neq 0$, $M + E - D = 0$, which yields

$$G(\theta) = \sin \theta \cos^2 \theta(C + E - D) \neq 0$$

for $\theta \in \left(0, \frac{\pi}{2}\right)$. (iii) $(C + E - D)(M + E - D) \neq 0$, for which we obtain

$$G(\theta) = \sin \theta \cos^2 \theta(M + E - D)$$

$$\times \left[\tan \theta + \frac{C + E - D}{M + E - D}\right],$$

which further gives two subcases:

(iii-a) $(C + E - D)(M + E - D) > 0$, leading to either $C > D - E$, $M > D - E$, or $C < D - E$, $M < D - E (D > E)$, both yield

$$G(\theta) \neq 0 \text{ for } \theta \in \left(0, \frac{\pi}{2}\right).$$
(iii-b) $(C + E - D)(M + E - D) < 0$ (i.e. $C < D < E$ or $M < D < E < C$, $D > E$), for which we have a unique solution $\theta_i$ such that

$$G(\theta_i) = 0,$$

where

$$\theta_i = \tan^{-1}\left(-\frac{C + E - D}{M + E - D}\right) \in \left(0, \frac{\pi}{2}\right).$$


(iv) $(C + E - D) = (M + E - D) = 0$ which yields $C = M = D = E$ ($D > E$) under which $G(\theta) \equiv 0$. This is a singular case. Using the Briot–Bouquet transformation $Y = uX$ (and so $u > 0$ due to $X, Y > 0$), (43) can be changed to Zhang et al. [1991]

$$\frac{du}{dX} = -n\Phi(X, u) X_2(1, u) + X\Phi(X, u),$$

because

$$E + 2Eu + Du^2 - (1 + E)(u^2 + X + (1 + u)^2 X^2 > 0$$

for $0 < X \ll 1$,

due to $D > E$. This shows that the differential equation (43) has a unique solution passing through the point $(0, u^*)$, $u^* > 0$, implying that on the $(X, Y)$-plane, for any $\theta_i' \in (0, \frac{\pi}{2})$, there exists a unique trajectory tending to $E_0$ along the direction $\theta = \theta_i'$. Next, we investigate the direction that the orbit moves along $\theta = \theta_i$ (for the case (iii-b)) and $\theta = \theta_i'$ (for the case (iv)). To achieve this, using the polar coordinates, the system (43) can be transformed to (neglecting higher order terms)

$$\dot{r} = \frac{rH(\theta)}{\sin \theta + \cos \theta}, \quad \dot{\theta} = \frac{G(\theta)}{\sin \theta + \cos \theta} \quad (47)$$

Then, for the case (iii-b), using $G(\theta) = 0$ and the unique solution $\tan \theta_i = -\frac{C + E - D}{M + E - D}$, we obtain

$$H(\theta_i) = -\sin^2 \theta_i [D \sin \theta_i + (D - C) \cos \theta_i]$$

$$- \cos^2 \theta_i [D \sin \theta_i + (D - C) \cos \theta_i]$$

$$= -\frac{u(1 + u)(E + 1 - X)}{u(D-E) + (1 + u)(1 - X)(E-X)} \equiv g(u, X), \quad \text{(46)}$$

where

$$X_2(1, u) = \frac{\Phi(X, u)}{X^2} = -(Du + E) < 0,$$

$$\Phi(X, u) = \Phi(X, u) = (1 + u)(1 + E - X).$$

Hence, according to Zhang et al. [1991], on the $(X, Y)$-plane, there exists a direction $\theta = \theta_i'$ along which an orbit of (43) tends to $E_0$ if and only if there is a solution curve of (43) passing through the point $(0, u^*)$ on the $(X, Y)$-plane, with $u^* = \tan \theta_i'$. For $0 < X \ll 1$ and $u > 0$, we have the continuous function $g(u, X) > 0$ and the continuous function,

$$0 \leq \frac{(D - C)(M - CE)}{M + E - D} \cos \theta_i$$

for $C < D - E < M$, we have $M + E - D > 0$ and $(D - C)(M - CE) > E(M - C) > 0$; while for $M < D - E < C$ ($D > E$), we have $M + E - D < 0$ and $(C - D)M + CE > E(C - M) > 0$, hence, the orbit moves towards $E_0$ in the direction $\theta = \theta_i'$ for the case (iii-b). For the case (iv), it is known that $\theta_i' \in (0, \frac{\pi}{2})$. Using the condition $C = M = D = E$ ($D > E$), we obtain

$$H(\theta_i') = -\sin^2 \theta_i' [D \sin \theta_i' + E \cos \theta_i']$$

$$- \cos^2 \theta_i' [E \sin \theta_i' + D \cos \theta_i']$$

$$= [D - C] \sin \theta_i' \cos \theta_i'$$

$$< 0 \quad \text{for } \theta_i' \in \left(0, \frac{\pi}{2}\right).$$

Summarizing the above discussions we have shown that including the $X$-axis and the $Y$-axis, orbits move towards $E_0$ along the characteristic direction

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in the neighborhood of $E_0$ in the first quadrant of the $(X,Y)$-plane as $t \to +\infty$.

The proof of Theorem 3.1 is complete. ■

Remark 5.2. The Allee effect has significant influence on the equilibrium $E_0$. Without the Allee effect, it has been shown that as the parameter varied, the dynamics of the system around $E_0$ in the first quadrant of the phase portrait can be very complex [Xiao & Ruan [300]]. It can be stable or unstable, and even for the same set of parameter values, it can be stable or unstable depending upon the initial values. However, for system (38) with the Allee effect, $E_0$ becomes always stable, at least locally.

$$
M_H = \frac{C}{2(C - D)(2C + D)^2}[(2C + D)[2D(C - D) + C(1 - E)^2] - C^2(1 + E)^2 + C(1 + E)\sqrt{C^2(1 + E)^2 + D(2C + D)[(1 - E)^2 - 4(C - D)]}.
$$

Proof. To study the stability of the equilibria, we calculate the Jacobian of system (38), given by

$$
J(X,Y) = \begin{bmatrix}
-X + 2(1 + E)X - 3X^2 - \frac{MY^2}{(X + Y)^2} & -\frac{MX^2}{(X + Y)^2} \\
CY^2 & Cy^2
\end{bmatrix}
$$

Evaluating the $J$ on the equilibrium $E_1$, we have two eigenvalues $-(1 - E) < 0$ and $C - D$. Hence, $E_1$ is a stable node if $C < D$ or a saddle point if $C > D$. For the equilibrium $E_2$, we obtain two eigenvalues $E(1 - E) > 0$ and $C - D$, which implies that $E_2$ is a saddle point if $C < D$ or an unstable node if $C > D$.

For the positive equilibria $E_{3+}$, we compute the $J$ on $E_{3+}$ to obtain the determinant:

$$
\det(J(E_{3+})) = \left. \frac{D(C - D)}{2C} \sqrt{(1 - E)^2 - 4M \left(1 - \frac{D}{C}\right)} \right| \times \left[ (1 - E)^2 - 4M \left(1 - \frac{D}{C}\right) \pm (1 + E) \right],
$$

yielding $\det(J(E_{3+})) < 0$ and $\det(J(E_{1+})) > 0$. Thus, the equilibrium $E_{3+}$ is a saddle point, and the stability of $E_{3+}$ is determined by the sign of $\text{Tr}(J(E_{3+}))$: $E_{3+}$ is asymptotically stable if $\text{Tr}(J(E_{3+})) < 0$ and unstable if $\text{Tr}(J(E_{3+})) > 0$, which again indicates an increase of extinction risk of species.

Theorem 5.3. For system $B_3$, $E_1$ is either a stable node if $C < D$, or a saddle point if $C > D$; $E_2$ is unstable (either a saddle point when $C < D$, or an unstable node when $C > D$). $E_{3+}$ exist for $0 < M < M_n$, and $E_3$ is always a saddle point. $E_{3+}$ is a stable focus for $C \geq D + \frac{(1 - E)^2}{4}$, and for $D < C < D + \frac{(1 - E)^2}{4}$, if $0 < M < M_H$; $E_{3+}$ is an unstable focus for $D < C < D + \frac{(1 - E)^2}{4}$ if $M_H < M < M_n$. Hopf bifurcation occurs from $E_{3+}$ at the critical point $M = M_H$, where $M_H$ is given by

$$
M_H = \frac{C}{2(C - D)(2C + D)^2}[(2C + D)[2D(C - D) + C(1 - E)^2] - C^2(1 + E)^2 + C(1 + E)\sqrt{C^2(1 + E)^2 + D(2C + D)[(1 - E)^2 - 4(C - D)]}.
$$

A direct computation shows that

$$
\text{Tr}(J(E_{1+})) = -\frac{1}{2C}f_2,
$$

where $f_2$ is given by

$$
f_2 = 2D(C - D)(C - M) + C(1 - E)^2 - 4M(C - D)] + C(1 + E)\sqrt{C(1 - E)^2 - 4M(C - D)},
$$

for which $0 < M < M_n$ and so the second and third terms in $f_2$ are positive, implying that $f_2 \geq 0 \Leftrightarrow \text{Tr}(J(E_{1+})) \leq 0$. In the following, we discuss the sign of $f_2$, which has several cases.

(I) $0 < M \leq C$ for which $f_2 > 0 \Leftrightarrow \text{Tr}(J(E_{1+})) < 0$ so $E_{1+}$ is asymptotically stable. Comparing $M$ with the limiting value $M_n$, we have two subcases:

(1α) If $C \geq M_n$, i.e. $C \geq D + \frac{(1 - E)^2}{4}$, then $E_{1+}$ is asymptotically stable for $0 < M < M_n$. 

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Let's denote $E_3^+$ as the positive equilibrium of the system. We have the following results:

1. When $C > D + \frac{(1-E)^2}{4}$, $E_3^+$ is asymptotically stable.

2. When $C < D + \frac{(1-E)^2}{4}$, one of the following holds:
   - $E_3^+$ is unstable for $M = M_H$.
   - Hopf bifurcation occurs from $E_3^+$ at the critical point $M = M_H$.

3. Codimension of the Hopf Bifurcation

Next, we consider the cubic polynomial $f_3$ defined by $f_3 = 16C^3(C-D)^2(1+E)^2C^2(1+E)^2 - D(2C+D)[(1-E)^2 - 4(C-D)]^2$. The discriminant of $f_3$ is given by

$$
\Delta_{f_3} = 16C^4(C-D)^2(1+E)^2C^2(1+E)^2 - D(2C+D)[(1-E)^2 - 4(C-D)]^2 > 0
$$

So $f_3 = 0$ has two real solutions. Moreover, we have

$$
f_{3|u=M} = \frac{D(C(1+E)^2 + 2D(C-D))}{2(C-D)(2C+D)} > 0,
$$
due to $C < D + \frac{(1-E)^2}{4}$, and

$$
f_{3|u=M} = \frac{D^2C^2}{4}[(1-E)^2 - 4(C-D)]^2 < 0.
$$

Since $f_3$ is a quadratic polynomial in $M$ with a negative coefficient of the term $M^2$ (implying that $f_3$ has a maximum), it must have a unique positive root $M = M_H \in (M^*, M_b)$, which is the bigger one of the two real solutions. This solution yields $f_2 = 0$ and so $\mathcal{V}(f(E_3^+)) = 0$, indicating that $M_H$ is a Hopf critical point. Note that the other real root of $f_3$ does not satisfy $f_3 = 0$ and so it is not a Hopf critical point.

Summarizing the above discussions, we have the following results:

1. When $C \geq D + \frac{(1-E)^2}{4}$, $E_3^+$ is asymptotically stable for $0 < M < M_b$.

2. When $C < D + \frac{(1-E)^2}{4}$, one of the following holds:
   - $E_3^+$ is asymptotically stable for $0 < M < M_H$;
   - $E_3^+$ is unstable for $M_H < M < M_b$;
   - Hopf bifurcation occurs from $E_3^+$ at the critical point $M = M_H$.

5.3. Codimension of the Hopf Bifurcation

Next, we further study the Hopf bifurcation and determine the codimension of the bifurcation. We have the following result.

Theorem 5.4. For system $B_3$, Hopf bifurcation occurs from $E_3^+$ at the critical point $M = M_H$, and the bifurcation is subcritical with codimension one, yielding a family of unstable limit cycles.

Proof. Again we apply normal form theory to find the focus values to determine the stability of limit cycles. To achieve this and make it simpler in normal form computation, we solve the equation in (55) for $C$, instead of solving for $X_3$, to obtain (noticing
that in the following $X_3 = X_{3*}.$

\[
C = \frac{MD}{M - (1 - X_3)(X_3 - E)} > 0
\]
\[
\Rightarrow M > (1 - X_3)(X_3 - E) > 0,
\]
\[
(E < X_3 < 1)
\]

and then solving $\text{Tr}(J(E))$ for $M$ yields the Hopf critical point, given by

\[
M_H = \frac{(1 - X_3)(X_3 - E)[D + (X_3 - E)(1 - X_3)]}{-3X_3^2 + 2(1 + E)X_3 - E} > 0.
\]

which requires

\[-3X_3^2 + 2(1 + E)X_3 - E > 0,
\]

and so

\[E < X_3 < \frac{1}{3}(1 + E + \sqrt{1 - E + E^2}) < 1.
\]

Then, we multiply the vector field of (53) by $X + Y$ for convenience (equivalent to using a time scaling, $\tau \rightarrow (X + Y)\tau$), and apply the following transformation at the critical point $M = M_H$.

\[
X = X_3 + x_1,
\]
\[
Y = \frac{C - D}{D}X_3 + \frac{D[(2X_3 - E)(1 - X_3) - X_3(X_3 - E)^2]}{D(1 + E)X_3 + 2X_3^2(1 - X_3)(X_3 - E)}x_1
\]
\[-\omega_x[(2X_3 - E)(1 - X_3) - X_3(X_3 - E)]X_3[D - (1 + E)X_3 + 2X_3^2(1 - X_3)(X_3 - E)]^{-2x_2},
\]

where

\[
\omega_x = \frac{X_3^2[D(2X_3 - E)(1 - X_3) - X_3(X_3 - E)^2][-3X_3^2 + 2(1 + E)X_3 - E]}{[D + X_3(2X_3 - 1 - E)]}
\]

In order to have $\omega_x > 0$, we need $X_3 > \frac{1}{3}(1 + E)$.

Since $M_H < M_0$, and noticing from (53) that $X_3 = \frac{1}{2}(1 + E)$ at $M = M_0$, we have $X_3 = X_{3*} > \frac{1}{2}(1 + E)$ at $M = M_H$. Therefore, we finally obtain

\[
\frac{1}{2}(1 + E) < X_3 < \frac{1}{3}(1 + E + \sqrt{1 - E + E^2}),
\]
\[
0 < E < \frac{1}{2}. \quad (54)
\]

Now, substituting the transformation (53) with the time scaling, we obtain

\[
\dot{v}_1 = G_1
\]

where

\[
G_1 = D(1 + E)(1 - X_3)(X_3 - E) + G_2,
\]
\[
G_2 = (1 - X_3)(X_3 - E)G_3,
\]
\[
G_3 = (1 - E)^2[10X_3 - 1 - E] - (1 - X_3)(X_3 - E)[24X_3 - 3 - 3E].
\]

In the following, we prove $G_2 > 0$ and so $v_1 > 0$ for $X_3$ in the interval given in (53). First we show $G_3 > 0$.

To simplify the analysis, we let $X_3 = \frac{1}{2}(1 + E) + G$ and thus (53) is changed to

\[
0 < G < G_n = \frac{1}{6}(2\sqrt{1 - E + E^2} - (1 + E)) < \frac{1}{6}, \quad \text{for } E \in \left(0, \frac{1}{2}\right). \quad (55)
\]
under which $G_3$ becomes

$$G_3 = 24G^3 + 9(1 + E)G^2 + 4(1 - E)^2 G + \frac{7}{4}(1 + E)(1 - E)^2 > 0.$$  

To prove $G_2 > 0$ in the region on the $E$-$G$ plane, bounded by $0 < G < G_u$, $0 < E < \frac{1}{2}$, we first show that the function $G_2$ has no extreme points inside the region and then prove that $G_2 > 0$ on the boundaries of the region. Now, $G_2$ is a function of $G$ and $E$, given by

$$G_2(G, E) = \frac{1}{16}(1 + E)(1 - E)^4 \left[ \frac{1}{2}G^2 - \frac{1}{2}(1 + E)(1 - E)^2 + 2G[2(1 - E)^2 - 3G(3E + 8G + 3)] \right]$$

from which we obtain

$$\frac{\partial G_2}{\partial G} = -Gg_1, \quad \frac{\partial G_2}{\partial E} = -\frac{1}{16}(2G + 1 - E)g_2,$$

where

$$g_1 = (1 + E)(1 - E)^2 + 6G[(1 - E)^2 - 2G(3E + 8G + 3)]$$

$$g_2 = (3 + 5E)(1 - E)^2 - 2G[(3 + 5E) \times (1 - E) + 2G(18G - 1 - E)].$$

Now with the help of Maple, eliminating $G$ from the two equations $g_1 = 0$ and $g_2 = 0$ we obtain the solution,

$$G = \frac{(1 - E)(220 + 527E + 236E^2 - 11E^3)}{56 + 77E + 124E^2 + 67E^3} > \frac{421}{238} > G_u, \quad E \in \left(0, \frac{1}{2}\right),$$

and a resultant,

$$R = -(1 - E)(1915E^5 + 677E^4 + 1390E^3 + 5978E^2 + 4615E + 977) < 0, \quad E \in \left(0, \frac{1}{2}\right),$$

which implies that $\frac{\partial G_2}{\partial G} = \frac{\partial G_2}{\partial E} = 0$ do not have solutions for $0 < E < \frac{1}{2}$, $0 < G < G_u$, and so the function $G_2$ does not have extreme points in the region bounded by $0 < E < \frac{1}{2}$ and $0 < G < G_u$.

Next, we prove that $G_2 > 0$ on the boundaries of the region, $0 < E < \frac{1}{2}$, $0 < G < G_u$, which has three line segments and one curve:

$$L_1 : E = 0, \quad 0 < G < \frac{1}{6},$$

$$L_2 : E = \frac{1}{2}, \quad 0 < G < \frac{1}{6}(2 - \sqrt{3}),$$

$$L_3 : G = 0, \quad 0 < E < \frac{1}{2},$$

$$C_1 : G = G_u, \quad 0 < E < \frac{1}{2}.$$  

By a direct computation, we obtain the following results.

On $L_1 : \quad G_2(G, 0) = \frac{1}{16}(2G + 1)(192G^4 - 24G^3 - 4G^2 - 2G + 1)$

$$= \frac{1}{16}(2G + 1)[(1 - 6G)(1 + 4G + 20G^2) + 96G^2 + 192G^4]$$

$> 0, \quad$ for $0 < G < \frac{1}{6}.$

On $L_2 : \quad G_2 \left(G, \frac{1}{2}\right) = \frac{3}{512} \left[ \frac{3}{16}G^2 - \frac{1}{2}G^3 + 27G^4 + 4G^5 \right]$  

$$= \frac{3}{512} \left[ 1 - 20G^3(1 + 20G + 36G^2) + \frac{341}{8}G^2 + 27G^4 + 4G^5 \right]$$

$> 0, \quad$ for $0 < G < \frac{\sqrt{3}}{12}(2 - \sqrt{3}) < \frac{1}{20}.$

On $L_3 : \quad G_2(0, E) = \frac{1}{16}(1 + E)(1 - E)^4 > 0, \quad$ for $0 < E < \frac{1}{2}$.
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On the curve \( C_1 \), we have

\[
G(G_1, E) = \frac{1}{81}(1 + E)[(2E^3 - 7E^2 + 18E^2 - 7E + 2) + 2(E^2 - 4E + 1)(1 + E)\sqrt{1 - E + E^2}]
\]

\[
= \frac{1}{81}(1 + E) \left[ 2 \left( 1 - \frac{7}{4} \right)^2 + \frac{95}{8} E^2 \left( 1 - \frac{28}{95} E \right)^2 \right] + 2(E^2 - 4E + 1)(1 + E)\sqrt{1 - E + E^2}
\]

\[
= \left\{ 2 \left( 1 - \frac{7}{4} \right)^2 + \frac{95}{8} E^2 \left( 1 - \frac{28}{95} E \right)^2 \right\}
\]

Thus, \( G(G_1, E) > 0 \) regardless whether \( E^2 - 4E + 1 \) is positive or negative.

Therefore, by continuity of the function, \( G_2 > 0 \) on the region defined by (54), and so \( v_1 > 0 \) indicating that the codimension of the Hopf bifurcation is one, and it is subcritical with unstable bifurcating limit cycles.

5.4. Bifurcation diagram and simulation

The bifurcation diagrams are shown in Figs. 8(c) and 8(d), where only the cases for \( C > D \) are presented, since for \( C \leq D \), \( E_3 \) is globally asymptotically stable, and \( E_4 \) does not exist.

Remark 5.5. Comparing the bifurcation diagrams in Fig. 8(c), it can be seen that the model \( B_0 \) with the Allee effect and without the Allee effect has similar dynamical behaviors. Both of them have Hopf bifurcation but the bifurcation curve changes from linear to nonlinear. Also note that the model with the Allee effect has less stable interval in \( M \). The big difference is that the Hopf bifurcation for the model without the Allee effect is supercritical with stable limit cycles while that for the model with the Allee effect is subcritical with unstable limit cycles. Thus, the model without the Allee effect might have bistable phenomenon (containing one stable equilibrium and one stable limit cycle) if \( E_0 \) happens to be stable, while the model with the Allee effect does not have such bistable property. But the model with the Allee effect exhibits the bistable phenomenon with two equilibria \( E_0 \) and \( E_{31} \), see Figs. 8(c) and 8(d).

Now we present simulations for model \( B_0 \). To have a comparison, we show simulations for both with and without the Allee effect. For the case without the Allee effect, it is seen from [15] that a supercritical Hopf bifurcation occurs from the critical point \( M_B = \frac{C - B + E^2}{E} \) when \( C \in (D, 1 + D) \) [see Fig. 8(b)]. Let \( D = 0.8 \) and \( C = 1 \). Then \( M_B = 2 \approx 2.2222222, \) and \( \Delta = 5 \). We take three values for simulation: \( M = 3, 24133419, 3.24133420 > M_B \), which implies that the equilibrium \( E_{14} \) is an unstable focus for the three values of \( M \). The simulations given in Figs. 9(a)–9(f) show very different behaviors though the three values are very close to \( M_B \). It is seen that the simulation for \( M = 3.224 \), as depicted in Figs. 9(a) and 9(b), shows a regular oscillation, while that for \( M = 3.24133419 \) shows a recurrence behavior [Zhang et al., 2001; Yu et al., 2013; Yu & Zhang, 2013], see Figs. 8(c) and 8(d), where the trajectory does not touch the equilibrium \( E_0 \) since \( E_0 \) is unstable. However, it is noted that the sufficient conditions given in the above mentioned articles do not include the case as seen in Figs. 8(c) and 8(d), since here the system does not exhibit saddle-node bifurcation or transcritical bifurcation.

It is interesting to see from the simulation, as given in Figs. 8(c) and 8(d), that when \( M \) has a very small increase from 3.24133419 to 3.24133420, the limit cycle (recurrent oscillation) disappears and the trajectory converges to the equilibrium \( E_0 \). According to the analysis in Xiao & Ruan [2001], the above three sets of parameter values belong to the category \( D < C < 1 + D < M_B < M < \) for which the equilibrium \( E_0 \) can be either asymptotically stable or unstable, as shown in Fig. 8(b). It can
be seen from this figure that in the vicinity of the origin trajectories converge to the origin if $\theta > \theta^*$ while diverge from the origin if $\theta < \theta^*$. For the numerical values chosen for this example, we have $\theta^* = \tan^{-1}(0.555) = 29.63^\circ$. This value 0.555 agrees with the slope of the trajectory near the origin, $0.500 \approx 0.55$, see Figs. 9(c) and 9(e). It is expected that there exists a value of $M$ between $M_H$ and $M^*$ and close to $M_H$, at which the oscillation suddenly ceases. But such a critical value $M^*$ cannot be analytically determined. Here, for the system \[ \frac{dX}{dt} = C - X^2 - D \] with $D = 0.8$ and $C = 1$, $M^* \approx 3.241$. So when $M_H < M < M^*$, the trajectory starting from $E_1$ does not touch $E_0$ (the origin), and returns following a route below the blue curve [see Figs. 9(c) and 10] and converges to a stable limit cycle; while when $M > M^*$, the trajectory starting from $E_1$ converges to the origin following a route above the blue curve [see Figs. 9(e) and 10]. The critical value of $M$ and the critical angle $\theta^*$ must be inherently related, yielding the interesting relaxation oscillation. This needs a further study.

It is seen from the bifurcation diagrams in Figs. 8(c) and 8(d) for model B with the Allee effect that there exist two cases: either the system has no Hopf bifurcation if $C \geq D + \frac{1-E^2}{4}$, or Hopf bifurcation occurs from $E_3+$ at the critical point $M = M_H$ if $D < C < D + \frac{1-E^2}{4}$. The codimension of the Hopf bifurcation is one and it is subcritical, and so the bifurcating limit cycle is unstable. For
Fig. 9. Simulations for the $B_i$ model without the Allee effect with $D = 0.8$, $C = 1$, starting from the initial point $(X, Y) = (1.0, 0.00001)$: (a) phase portrait for $M = 3.224$, and (b) time history for $M = 3.224$, showing a regular oscillation; (c) phase portrait for $M = 3.24133419$, and (d) time history for $M = 3.24133419$, showing a recurrent behavior (a slow-fast motion); (e) phase portrait for $M = 3.24133420$, and (f) time history for $M = 3.24133420$, showing convergence to the equilibrium $E_0$. 

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Fig. 10. Phase portrait of system $B_0$ without the Allee effect for the parameters satisfying $D < C < 1 + D < M_H < 592\times \tan^{-1}(\frac{1}{1 + D})$, where the blue radial denotes $\theta = \theta^* = \tan^{-1}(\frac{1}{M - 1})$.

Simulations, we choose $E = 0.4$ and $D = 0.8$. For the case $C > D + \frac{1}{M - 1} H = 0.89$, we take $C = 0.95$ and $M = 0.3 < M_L = 0.57$. The simulated trajectories either converge to the stable node $E_0$ or to the stable node $E_{1+}$, as shown in Fig. 11(a).

For the case $D < C < D + \frac{1}{M - 1} r = 0.89$, we choose $C = 0.82$, yielding $M_L = 3.69$ and $M_H = 3.6086$. We first take $M = 3.58$ under which $E_{1+}$ is a stable focus, with trajectories still either converging to the stable node $E_0$ or to the stable focus $E_{1+}$, as seen from Fig. 11(b). It seems that the unstable limit cycle does not exist for these parameter values. In fact, it is very difficult to identify the parameter values to obtain such unstable limit cycles. We have to search the parameter values of $M$ very close to the Hopf critical point $M_H$ from $M > M_H$ to $M < M_H$. For the above chosen values of $E$, $D$ and $C$, we take four values of $M$, one of them is greater than $M_H$ and the remaining three are less than $M_H$: $M_H = 3.6086$ and


It is easy to show that $E_{1+}$ is an unstable focus for $M = 3.61$, but a stable focus for $M = 3.606$, 3.59253, 3.592, as expected. The simulated phase portraits for the four values of $M$ are shown in Figs. 11(c)–11(f), respectively. It is seen that an unstable limit cycle exists at $M = 3.606$, and in fact for the above chosen values of $E$, $D$ and $C$, unstable limit cycles exist for $3.59253 < M < M_H = 3.6086$.

At $M = 3.59253$, the unstable limit cycle coincides with the unstable homoclinic loop. Then $E_{1+}$ becomes a stable focus without any closed orbits around.

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Fig. 11. Simulated phase portraits of system $B_0$ with $E = 0.4$ and $D = 0.8$: (a) $C = 0.95$ and $M = 0.3$, showing bistable equilibria $E_0$ and $E_{1+}$, with two attracting regions separated by the stable manifolds connecting to the saddle point $E_{1-}$; (b) $C = 0.82$ and $M = 3.58$, also showing bistable equilibria $E_0$ and $E_{1+}$, with two attracting regions separated by the stable manifolds connecting to the saddle point $E_{1-}$; (c) $C = 0.82$ and $M = 3.61$, showing the unstable $E_{1+}$; (d) $C = 0.82$ and $M = 3.606$, showing the stable $E_{1+}$ and an unstable limit cycle; (e) $C = 0.82$ and $M = 3.59253$, showing the stable $E_{1+}$ and the unstable homoclinic loop and (f) $C = 0.82$ and $M = 3.592$, showing the stable $E_{1+}$.
Remark 5.6. It has been shown in [Xiao & Ruan, 2001] that the dynamics around the equilibrium \( E_0 \) of system \( B_{ii} \) without the Allee effect can be very complex, which may be asymptotically stable or unstable, or even including both stable and unstable sectors. However, the equilibrium \( E_0 \) of system \( B_{ii} \) with the Allee effect is always asymptotically stable. This clearly indicates that species having a strong Allee effect may affect their predation and hence extinction risk. Moreover, the \( B_{ii} \) system without the Allee effect has supercritical Hopf bifurcation generating stable limit cycles; while the \( B_{ii} \) system with the Allee effect not only changes the supercritical Hopf bifurcation to subcritical Hopf bifurcation, resulting in that the equilibrium \( E_{3+} \) becomes stable from unstable. Moreover, the unstable limit cycles exist for a very limited parameter values. To see why such a change in the Hopf bifurcation increases the stability of the system, we take a comparison of the solution trajectories shown in Fig. 9 (without the Allee effect) and in Fig. 11 (with the Allee effect). For Fig. 9 without the Allee effect, there exist values of the parameter \( M \) in an interval for which a supercritical Hopf bifurcation occurs and all trajectories converge to the stable limit cycle [see Fig. 9(a)]. Otherwise, the trajectories may converge to \( E_0 \). However, due to the complex behavior of \( E_0 \) without the Allee effect, for most parameter values, trajectories would not converge to \( E_0 \), but are oscillating. For Fig. 11 with the Allee effect, the
supercritical Hopf bifurcation is changed to a subcritical Hopf bifurcation and the bifurcating limit cycle is unstable [see Fig. 11(d)], and the equilibrium $E_{1,\pm}$ becomes stable. In this case, all trajectories converge to the stable node $E_0$ (which is always stable due to the Allee effect), except those starting from initial points inside the unstable limit cycle, which converge to the stable focus $E_{1,\pm}$. Since the parameter values for generating the unstable limit cycle is very limited, it clearly implies that the system becomes more stable, compared to the system without the Allee effect.

5.5. $B$–$T$ bifurcation

The conditions for the model $B_4$ to have a $B$–$T$ bifurcation are $\det(J(E_{1,\pm})) = \text{Tr}(J(E_{1,\pm})) = 0$. It follows from (50) and (51) that the $B$–$T$ bifurcation occurs when

$$C(1 - E)^2 - 4M(C - D) = C - M = 0 \Rightarrow C = M = D + \frac{1}{4}(1 - E)^2, \quad X_3 = \frac{1}{2}(1 + E).$$

Obviously, it is as expected that the $B$–$T$ bifurcation occurs at the turning point $M = M_a$ at which $E_{3,\pm} = E_{1,\pm} = E_4$

$$= \left(\frac{1}{2}(1 + E), \frac{1}{8D}(1 + E)(1 - E)^2\right),$$

where the saddle-node bifurcation coincides with the Hopf bifurcation.

To analyze the $B$–$T$ bifurcation, we need to find the normal form of the $B$–$T$ bifurcation with unfolding. First, we want to determine the codimension of the $B$–$T$ bifurcation and then to obtain the normal form with unfolding. In the following two sections, we will apply the simplest normal form (SNF) theory (e.g. see [Yu & Leung 2012]) to determine the codimension of the $B$–$T$ bifurcation of the system (55) and the parametric simplest normal form (PSNF) theory (e.g. see [Yu & Leung 2010, 2014]) to obtain the normal form of the system (55) with unfolding terms.

5.5.1. The SNF and the codimension for the $B$–$T$ bifurcation

In order to obtain the normal form for the $B$–$T$ bifurcation, we first need to determine the codimension of the system (55). We have the following theorem.

**Theorem 5.7.** For system (55), when $C = M = D + (1 - E)^2$. $B$–$T$ bifurcation occurs from the equilibrium solution $E_3$: $(X, Y) = (u, \frac{D(1 - E)^2}{1 - E^2 + 4D})$.

The codimension of the $B$–$T$ bifurcation is two and no codimension higher than two can happen for the $B$–$T$ bifurcation.

**Proof.** Let

$$C = D + \frac{1}{4}(1 - E)^2 + \mu_1,$$

$$M = D + \frac{1}{4}(1 - E)^2 + \mu_2,$$

where $\mu_1$ and $\mu_2$ are perturbation parameters. Thus, $(\mu_1, \mu_2) = (0, 0)$ defines the $B$–$T$ bifurcation point. Now, we assume $(\mu_1, \mu_2) = (0, 0)$, and apply the simplest normal form theory [Yu 1999, Gazor & Yu 2010, 2014] to determine the codimension of system (55). To achieve this, introducing the following transformation,

$$X = \frac{1}{2}(1 + E) + \frac{D(1 - E)^2}{(1 - E)^2 + 4D} - u + v,$$

$$Y = \frac{1}{8D}(1 + E)(1 - E)^2 + \frac{(1 - E)^4}{4((1 - E)^2 + 4D)} - u,$$

into (55), we obtain the following system:

$$\frac{du}{dt} = f_1(u, v, D, E), \quad \frac{dv}{dt} = f_2(u, v, D, E)$$

where $f_1$ and $f_2$ are rational functions in $u$ and $v$ with coefficients given in terms of $D$ and $E$. Then, we expand the above system around $(u, v) = (0, 0)$ and apply the SNF theory [Yu 1999, Gazor & Yu 2010, 2014] to the expended system, with the following nonlinear transformation truncated up to second order:

$$u = -x_1 - \frac{(1 - E)^2 + 16DE}{4(1 + E)[(1 - E)^2 + 4D]} x_2;$$

$$v = -x_2 - \frac{(1 - E)^2 + 16DE}{2(1 + E)[(1 - E)^2 + 4D]} x_1 x_2 + \frac{8D}{(1 + E)[(1 - E)^2 + 4D]} x_2^2;$$

introduced into (55) to obtain the SNF up to second-order terms:

$$\dot{x}_1 = x_2 + O(|(x_1, x_2)|^3),$$

$$\dot{x}_2 = C_{20}x_1^2 + C_{11}x_1 x_2 + O(|(x_1, x_2)|^3).$$
In this section, we use the PSNF theory to obtain the normal form with unfolding up to second-order terms for the codimension-two B-T bifurcation, and give a summary on the bifurcation analysis. Here, we will obtain the normal form with the unfolding terms expressed in the original system parameters.

To obtain the normal form with unfolding, we introduce the parametric transformation,

\[
C = D + \frac{1}{4} (1 - E)^2 + \mu_1, \\
M = D + \frac{1}{4} (1 - E)^2 + \mu_2,
\]

together with the change of state variables \((u, v, \mu_1, \mu_2, D, E)\), into (58) to obtain

\[
\dot{u} = F_1(u, v, \mu_1, \mu_2, D, E), \\
\dot{v} = F_2(u, v, \mu_1, \mu_2, D, E).
\]

Then, we expand the above system around the point \((u, v, \mu_1, \mu_2) = (0, 0, 0, 0)\) and apply the PSNF theory, with another change of state variables:

\[
u = \frac{2[(1 - E)^2 + 4D^2]}{D^2(1 + E)(1 - E)^3} x_2
\]

Further, introducing the transformation,

\[
x_1 \rightarrow x_1, \quad x_2 \rightarrow x_2 + O\left((x_1, x_2)^2\right) \quad \rightarrow x_2,
\]

into the above equations, we obtain

\[
\dot{x}_2 = C_20 x_2^3 + C_{11} x_1 x_2 + O\left((x_1, x_2)^3\right).
\]

Since \(C_20 C_{11} \neq 0\), the codimension of the B-T bifurcation is two.

### 5.5.2. The PSNF of the B-T bifurcation and bifurcation analysis

In this section, we use the PSNF theory to obtain the normal forms with unfolding up to second-order terms for the codimension-two B-T bifurcation, and give a summary on the bifurcation analysis. Here, we will obtain the normal form with the unfolding terms expressed in the original system parameters.

To obtain the normal form with unfolding, we introduce the parametric transformation,

\[
C = D + \frac{1}{4} (1 - E)^2 + \mu_1, \\
M = D + \frac{1}{4} (1 - E)^2 + \mu_2,
\]

together with the change of state variables \((u, v, \mu_1, \mu_2, D, E)\), into (58) to obtain

\[
\dot{u} = F_1(u, v, \mu_1, \mu_2, D, E), \\
\dot{v} = F_2(u, v, \mu_1, \mu_2, D, E).
\]

Then, we expand the above system around the point \((u, v, \mu_1, \mu_2) = (0, 0, 0, 0)\) and apply the PSNF theory, with another change of state variables:

\[
\dot{x}_1 = 2[(1 - E)^2 + 4D^2] x_1
\]

\[
\dot{x}_2 = C_20 x_2^3 + C_{11} x_1 x_2 + O\left((x_1, x_2)^3\right).
\]
to obtain the standard normal form with unfolding:
\[ x_1 = x_2 + \mathcal{O}((x_1, x_2, \mu_1, \mu_2)^3), \]
\[ x_2 = \beta_1 x_1 + \beta_2 x_2 + x_1^2 + \mathcal{O}((x_1, x_2, \mu_1, \mu_2)^3) \]

Finally, introducing the transformation
\[ x_1 \rightarrow x_1, \quad x_2 \rightarrow x_2 + \mathcal{O}((x_1, x_2, \mu_1, \mu_2)^3) \]
into the above system yields the normal form with unfolding up to second-order terms:
\[ x_1 = x_2, \]
\[ x_2 = \beta_1 x_1 + \beta_2 x_2 + x_1^2 + \mathcal{O}((x_1, x_2, \beta_1, \beta_2)^3), \]
Moreover, three local bifurcations with the representations of the bifurcation curves are given below.

1. Saddle-node bifurcation occurs from the bifurcation curve:

\[ \text{SN} = \{ (\beta_1, \beta_2) \mid \beta_1 = 0 \}. \]

2. Hopf bifurcations occur from the bifurcation curve:

\[ H = \left\{ (\beta_1, \beta_2) \mid \beta_1 = -\frac{D(1 - E)^2}{2((1 - E)^2 + 4D)}; \beta_2 > 0 \right\}, \text{ subcritical.} \]

3. Homoclinic orbits occur from the bifurcation curve:

\[ \text{HL} = \left\{ (\beta_1, \beta_2) \mid \beta_1 = \frac{49}{25} \left( \frac{D(1 - E)^2}{2((1 - E)^2 + 4D)} \right) \beta_2 > 0 \right\}, \text{ unstable.} \]

The bifurcation diagram for the B–T bifurcation is shown in Fig. 12. Note that the Hopf bifurcation and the homoclinic loop bifurcation are unstable. The formulas for bifurcation curves, given in Thm. 5.8, can be expressed in terms of the original perturbation parameters \( \mu_1 \) and \( \mu_2 \) via [17].

In the following, we discuss how to simulate the dynamical phenomena in the above B–T Bifurcation using the original system (38), in particular for the Hopf bifurcation and the homoclinic loop bifurcation. With [25],

\[ C = D + \frac{1}{4}(1 - E)^2 + \mu_1, \]

\[ M = D + \frac{1}{4}(1 - E)^2 + \mu_2, \]

the condition \( D < C < D + \frac{1}{4}(1 - E)^2 \) yields

\[ \mu_1 > \frac{1}{4}(1 - E)^2. \]

Further, to have solutions \( E_{1\pm} \), we need

\[ M < M_u = \frac{C(1 - E)^2}{4(C - D)} \Rightarrow \mu_2 < \frac{D - 4 + (1 - E)^2}{4M_2 + (1 - E)^2}. \]

For simulation, we again take \( E = 0.4, D = 0.8 \) and thus obtain the critical value \( C = M = 0.89 \) at which the positive equilibrium becomes \( E_{1\pm} = (0.7, 0.07875) \). Thus, \( \mu_1 > -0.09 \). For simplicity, choose \( \mu_1 = -0.04 \). Then, \( \mu_2 < -\frac{0.09}{0.07875} = 0.64 \).

We vary \( \mu_2 \) as

\[ \mu_2 \in (0.63, 0.628, 0.62255, 0.62). \]

It is easy to show that for \( \mu_2 = 0.63 \), \( E_{1\pm} \) is an unstable focus, while a stable focus for \( \mu_2 = 0.628, 0.62255 \) and 0.62. Moreover, an unstable limit cycle exists for \( \mu_2 = 0.628 \), the unstable limit cycle coincides with the homoclinic loop at \( \mu_2 = 0.62255 \). When \( \mu_2 < 0.62255 \), the homoclinic loop is broken and \( E_{3\pm} \) is a stable focus. The simulated phase
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Fig. 13. Simulated phase portraits of system $B_r$ for B–T bifurcation with $E = 0.4, D = 0.8$ and $\mu_1 = -0.04$: (a) $\mu_2 = 0.63$, showing the unstable focus $E_{3+}$; (b) $\mu_2 = 0.628$, showing the stable focus $E_{3+}$ and an unstable limit cycle; (c) $\mu_2 = 0.62255$, showing the stable focus $E_{3+}$ and the unstable homoclinic loop and (d) $\mu_2 = 0.62$, showing the stable focus $E_{3+}$.

portraits for the four cases are shown in Figs. 13(a)–13(d), respectively, which correspond to the four phase portraits in the B–T bifurcation diagram (see Fig. 12) from the top to the bottom. However, it should be noted that due to the transformation (66), the simulated phase portraits in the original $X$- and $Y$-axes have the saddle point $E_{3-}$ on the left side of the focus $E_{3+}$ (see Fig. 13), while the B–T bifurcation diagram (see Fig. 12) has the saddle point on the right side of the focus point.

It should be pointed out that although the four phase portraits in Figs. 13(a)–13(d) are similar to those four phase portraits in Figs. 11(c)–11(f), they are quite different since the former can be only obtained near the B–T bifurcation point, while the later can be found near any Hopf bifurcation point.

6. Conclusion
In this paper, we have studied four predator–prey models and paid particular attention on the Allee effect. It has been shown that strong Allee effect has great influence on the dynamics of the system, in particular on stability and bifurcations. Compared to the systems without the Allee effect, when the density of prey population is low, the species having a strong Allee effect are vulnerable to extinction due to predation. In general, the Allee effect makes the
dynamics of the systems more complicated. Especially, for the $B_3$ model, the Allee effect not only completely changes the stability of the equilibrium at the origin, but also changes the supercritical Hopf bifurcation to subcritical Hopf bifurcation with very limited parameter values for the bifurcating unstable limit cycles. Also this model with the Allee effect yields Bogdanov–Takens bifurcation, inducing more complex bifurcation behaviors. This study shows that including the Allee effect in predator–prey systems is necessary in order to have a more realistic analysis. Future works will focus on more complex systems $A_{iii}, B_{iii}, C_{iv}$, and $C_{vii}$.

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References


