Applied Math 261b—Scientific Computing
Final Exam
Department of Applied Mathematics
The University of Western Ontario

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Saturday, April 24, 1999
9:00am–12:00 noon

Put your name and student number on your exam booklet(s). All questions are of equal weight. A marking scheme for each sub-question is indicated in square brackets. Textbooks, notes, calculators, and palmtop computers are allowed. Do questions 1, 2, 3, and ONE of 4 or 5.

PLACE YOUR ANSWERS IN YOUR EXAM BOOKLET. THE EXTRA SPACE ON THIS EXAM SHEET MAY BE USED FOR SCRAP PAPER.

1. (The SVD $A = U\Sigma V^T$).

   (a) [15 marks] You have computed the SVD of the 4 by 4 Hilbert matrix H, with $h_{ij} = 1/(i+j-1)$, by using Matlab:

```
>> format long
>> a = hilb(4)

a =

   1.00000000000000   0.50000000000000   0.33333333333333   0.25000000000000
   0.50000000000000   0.33333333333333   0.25000000000000   0.20000000000000
   0.33333333333333   0.25000000000000   0.20000000000000   0.16666666666667
   0.25000000000000   0.20000000000000   0.16666666666667   0.14285714285714

>> [u s v] = svd( a )

u =

   0.79260829116376  -0.58207569949724   0.17918629053545  -0.02919332316479
   0.45192312090160   0.37050218506709  -0.74191779062845   0.32871205576319
   0.32241639858183   0.50957863450180   0.10022813694719  -0.79141114583313
   0.25216116968824   0.51404827222216   0.63828252819361   0.51455274999715


s =

   1.50021428005924                  0                  0                  0
                  0   0.16914122022145                  0                  0
                  0                  0   0.00673827360576                  0
                  0                  0                  0   0.00009670230402


v =

   0.79260829116376  -0.58207569949724   0.17918629053545  -0.02919332316479
   0.45192312090160   0.37050218506709  -0.74191779062845   0.32871205576319
   0.32241639858182   0.50957863450180   0.10022813694719  -0.79141114583313
   0.25216116968824   0.51404827222216   0.63828252819361   0.51455274999715
```

   Use this factorization to compute the solution to $Hx = [1, 0, 0, 0]'$. Show your work. Check your work by explicitly computing the residual. Remember that U and V are orthogonal.

   (b) [10 marks] What is the condition number of H, and what is the estimated error in x?

2. Compute the condition number of the *roots* of the following equations. Assume that we are solving for x, and we are interested in the changes in x caused by changes in the parameter(s) a.

   (a) [10 marks] $x^2 + 2x + a = 0$.

   (b) [5 marks] $x^5 + 2x + a = 0$.

   (c) [5 marks] $x^2 - 2x + 1 + a = 0$ near $a = 0$. What goes wrong with the idea of condition number analysis, here?

   (d) [5 marks] $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$. Here we want a generalization of the notion of "condition number" which reflects the fact that any or all of the coefficients may change at once. Remember the chain rule for multivariate differentials: if $y = f(x_1, x_2, \ldots, x_k)$ then $\Delta y = f_1 \Delta x_1 + f_2 \Delta x_2 + \cdots + f_k \Delta x_k$ where $f_k = \partial f / \partial x_k$.

3. (Least squares, and matrix recognition)

(a) [15 marks] Convert the following linear least squares problem into a matrix formulation (that is, give the 5×2 matrix A and the right-hand side b so that the problem can be written as minimize r where $r = b - Ax$.)

$$\text{minimize} \sum_{k=1}^{5} \left(y_k - x_1 t_k - x_2(2t_k^2 - 1)\right)^2 . \tag{1}$$

The data (t_k, y_k) is given below, for $1 \le k \le 5$.

t	y
-2	0
-1	0
0	1
1	0
2	4

(b) [10 marks] Explain in a short paragraph how one can use the the QR factorization or the SVD ($A = U\Sigma V^T$) to solve the least-squares problem $Ax = b$.

Do Either Question 4 or Question 5, but not Both.

4. (Newton-Cotes Integration Formulae)

   (a) [15 marks] Your "frammistat" sensor takes readings of the "frammistat" level in your experiment at the start of a time interval, then at 68% of the time interval, and finally again at the end of that interval. You wish to find a good formula for the integral of the frammistat level, and you settle on a closed Newton-cotes type formula. By making your formula exact for constant frammistat levels, i.e. $f(t) = 1$ on $0 \leq t \leq h$, and also exact for $f(t) = t$ and $f(t) = t^2$, set up the equations that determine the weights in your Newton-Cotes type formula for

   $$\int_0^h f(t)\,dt \approx w_0 f(0) + w_{0.68} f(0.68h) + w_1 f(h)\,.$$

   (b) [10 marks]. You get a new frammistat sensor, that can sample at any 3 points in the interval. What sample points should you choose for maximum accuracy in the computation of the average frammistat level (remember that the average is just $(\int_0^h f(t)\,dt)/h$)? Justify your choice. Would your choice change if you were doing experiments on a continuous sequence of time intervals (that is, a whole collection of intervals one after the other, at times 0, h, $2h$, $3h$, etc)?

Do Either Question 4 or Question 5, but not Both.

5. (Taylor series methods and Runge-Kutta methods for solving Initial Value Problems.)

(a) [15 marks] Carry out two steps of the 2nd order Taylor series method to solve $x' = \cos(\pi t x)$, starting with $x(0) = 1$ and a step size of $h = 0.01$. Show your work by filling in the following table. The final column is for the residual $\hat{x}' - \cos(\pi t \hat{x})$, where $\hat{x}$ is your piecewise polynomial computed by Taylor series on each interval.

t	x	x'	x''	residual
0	1			
0.01				
0.02				

(b) [10 marks] Is the following implicit Runge-Kutta method at least of 2nd order?

$$\begin{aligned} k_1 &= f(x_n + \frac{h}{2} k_1) \\ x_{n+1} &= x_n + h k_1 \end{aligned}$$

Hint: use the same kind of analysis as in equation 9.2.2 on page 320 of your text.