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Abstract:

A review is presented of the quantization procedures applicable to linear gravitational fields in Minkowski space–time and of various interaction processes involving gravitons. The discussion is mainly concerned with those processes that in the Feynman diagrammatic approach involve gravitons on external lines and are of particular astrophysical interest because of their contribution to background gravitational radiation in the universe. More specifically they are graviton production from particle–antiparticle annihilation, gravitational bremsstrahlung, scattering of gravitons and photoproduction. Among the topics discussed are also, the graviton–particle vertex and some of its applications, and the problem of coherent emission of gravitational radiation in the laboratory.

1. Introduction

Although an extremely satisfactory classical description of gravitation exists, its complete quantization has so far defied every attempt. Einstein's general theory of relativity does in fact present from the point of view of quantum field theory, a variety of mathematical difficulties. These arise chiefly from the non-linearity of the field equations, the presence of a complicated gauge group, constraints in the initial value formulation of the theory, and the loss of the Poincare group so important to particle physicists (Ashtekar and Geroch [1], DeWitt [22]). The probable discovery of sources of strong gravitational fields such as black holes may have made the problem of the complete quantization of the gravitational field even more acute, as astrophysical observations require quantitative predictions which are difficult to make on the basis of a still non existing theory.

Strong gravitational fields, no matter how important and interesting, do not however encompass the whole realm of quantum phenomena where gravitation plays a role. Weak gravitational sources are indeed very numerous in our universe and are most readily available to us in the laboratory. The linearized general theory of relativity is adequate to deal with these sources, can be quantized consistently in a number of ways and the interactions of the quanta of the field, the gravitons, with other particles can be studied.

Most of these interactions have an interest in themselves and serve to ascertain in a clearer way, the significance and role of gravitation in physics. The study of gravitation would certainly not be complete without their consideration. Some processes, however, have a direct astrophysical interest. Bremsstrahlung, for instance, and photoproduction do generate sizeable amounts of gravitational radiation (GR) which can be comparable in magnitude with those of classical processes. All processes, then contribute to the creation of a background of gravitational radiation in the universe. This vast and as yet untapped source of information has great cosmological and astrophysical interest. In fact, because of the extreme weakness of their interactions gravitons are almost unabsorbable and could therefore convey to us information about the various evolutionary stages of the universe. With only a few exceptions, dictated mainly by the nature of the particular problem at hand, the processes considered in this article are linear. Thus, problems such as the fall of matter down a black hole (Zerilli [93], Davis, Ruffini, Press and Price [16]), the radiation due to the collision of two non-rotating black holes (Hawking [40]), the stimulated or spontaneous emission from rotating black holes (Unruh [76], Hawking [41]) and several other interesting problems for which the linearized theory is not adequate, do not fall within the scope of this review.

Most of the linear processes that have so far been studied in the literature are here reviewed and presented in as self contained a way as possible. Much work has been produced in this field in the late sixties that still lies scattered over numerous journals. Hence the desirability of a review which may help to reassess the role of these processes in astrophysics. This role has, sometimes and in our opinion unwarrantedly, been played down in the literature.

Although no real attempt is made to estimate the overall background of GR in the universe, which would obviously be of considerable astrophysical significance, linear processes can be distinctly singled out that contribute to the increase or, conversely, the decrease of GR in the universe and to the life and evolution of astrophysical objects. It is found, in particular, that high coherence can compensate, within limits, for the weakness of certain sources. From this point of view the problem of highly organized matter has indeed been barely discussed in the literature.

2. Quantization of linear gravitational fields

Quantization theories for linear gravitational fields have been proposed in the past by Rosenfeld [70] and Pauli and Fierz [28]. More recently, the problem of quantizing the linearized version of the general theory of relativity has been tackled along different lines by several authors with similar results. Worth of particular attention among the quantization procedures is the functional integrals method developed by DeWitt [21] and Fadeev and Popov [26] which is here only mentioned. Although we essentially follow in this article the approach of Gupta [33], the approach of Weinberg is also reported in some detail. Also interesting is the path dependent formalism of Mandelstam [52] because of its connection with the theory of gauge fields. In Mandelstam's approach any reference to a coordinate system is avoided and unphysical variables like the metric tensor $g_{\mu\nu}$ need not be mentioned as curvature is directly introduced from the very beginning. The quantization of curvature implies that space–time can no longer be viewed as consisting of a four dimensional infinity of points. Hence the results of measurements in quantum theory are no longer path independent. Again the first order perturbation calculations give results equivalent to those of flat space theories.

The question of renormalization is not dealt with in this paper. It is known that in the case of pure gravitational fields all one-loop divergencies can be eliminated by a field renormalization ('t Hooft and Veltman [74], DeWitt [21]). This is no longer the case as soon as gravitons interact with other particles ('t Hooft and Veltman [74], Deser and van Nieuwenhuizen [17–20]), with the exception of external off-mass shell gravitons which when added to a flat space renormalizable process leave it finite (van Nieuwenhuizen [77]). It is implicitly hoped that the lower order terms of perturbation theory are valid for most processes considered.

2.1. The linear approximation

Einstein's equations can be written in the form

$$R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} = -\tfrac{1}{2}\kappa^2 T_{\mu\nu} \tag{2.1}$$

where $R_{\mu\nu}$ is the Ricci tensor, $g_{\mu\nu}$ the metric tensor, $R \equiv R^{\mu}_{\mu}$, $\kappa = \sqrt{16\pi G}$, G the gravitational constant and $T_{\mu\nu}$ the energy momentum tensor of the "matter field". Matter here includes everything except the gravitational field.

$\gamma^{\alpha\beta}$ is defined as the energy momentum tensor density for the matter field

$$\gamma^{\alpha\beta} = T^{\alpha\beta}\sqrt{-g}\,. \tag{2.2}$$

The covariant divergence of $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ vanishes. It therefore follows that the divergence of $T_{\mu\nu}$ too vanishes:

$$\partial \tau_\mu^\nu / \partial x^\nu - \tfrac{1}{2} \tau^{\alpha\beta} \partial g_{\alpha\beta} = 0 , \tag{2.3}$$

$$\frac{\partial}{\partial x^\nu} (\tau_\mu^\nu + t_\mu^\nu) = 0 . \tag{2.4}$$

t_μ^ν is the pseudo tensor of the gravitational field and

$$\partial t_\mu^\nu / \partial x^\nu = -\tfrac{1}{2} \tau^{\alpha\beta} \, \partial g_{\alpha\beta} / \partial x^\mu . \tag{2.5}$$

The linear approximation is defined by expanding the metric tensor and keeping only the first order terms

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} \tag{2.6}$$

where $\eta_{\mu\nu}$ is the Minkowski metric. It follows that

$$\Box^2 h_{\mu\nu} - \left(\frac{\partial^2 h_{\mu\lambda}}{\partial x_\nu \partial x_\lambda} + \frac{\partial^2 h_{\nu\lambda}}{\partial x_\mu \partial x_\lambda} \right) + \frac{\partial^2 h_\lambda^\lambda}{\partial x_\nu \partial x_\mu} + \delta_{\mu\nu} \left(\frac{\partial^2 h_{\lambda\rho}}{\partial x_\lambda \partial x_\rho} - \Box^2 h_\lambda^\lambda \right) = \kappa T_{\mu\nu} , \tag{2.7}$$

$$\frac{\partial t_{\mu\nu}}{\partial x_\nu} = \frac{\kappa}{2} \frac{\partial h_{\nu\lambda}}{\partial x_\mu} T^{\nu\lambda} . \tag{2.8}$$

By using the de Donder–Lanczos condition

$$\partial h_{\mu\nu} / \partial x_\nu - \tfrac{1}{2} \partial h_\lambda^\lambda / \partial x_\mu = 0 \tag{2.9}$$

one obtains from (2.7)

$$\Box^2 h_{\mu\nu} - \tfrac{1}{2} \delta_{\mu\nu} \Box^2 h_\lambda^\lambda = \kappa T_{\mu\nu} . \tag{2.10}$$

By defining

$$h_{\mu\nu} \doteq \gamma_{\mu\nu} - \tfrac{1}{2} \delta_{\mu\nu} \gamma_\lambda^\lambda \tag{2.11}$$

and using (2.10) and (2.11) one gets

$$\Box^2 \gamma_{\mu\nu} = \kappa T_{\mu\nu} ; \qquad \partial \gamma_{\mu\nu} / \partial x_\nu = 0 \tag{2.12}$$

$$\frac{\partial t_{\mu\nu}}{\partial x_\nu} = \frac{\kappa}{2} \frac{\partial \gamma_{\nu\lambda}}{\partial x_\mu} T^{\nu\lambda} - \frac{\kappa}{4} \frac{\partial \gamma_\lambda^\lambda}{\partial x_\mu} T_\nu^\nu$$

$$t_{\mu\nu} = \tfrac{1}{2} \left\{ \frac{\partial \gamma_{\lambda\rho}}{\partial x_\mu} \frac{\partial \gamma^{\lambda\rho}}{\partial x_\nu} - \frac{1}{2} \frac{\partial \gamma_\lambda^\lambda}{\partial x_\mu} \frac{\partial \gamma_\rho^\rho}{\partial x_\nu} - \tfrac{1}{2} \delta_{\mu\nu} \left(\frac{\partial \gamma_{\lambda\rho}}{\partial x_\sigma} \frac{\partial \gamma^{\lambda\rho}}{\partial x_\sigma} - \frac{1}{2} \frac{\partial \gamma_\lambda^\lambda}{\partial x_\sigma} \frac{\partial \gamma_\rho^\rho}{\partial x_\sigma} \right) \right\} . \tag{2.13}$$

Hence the Hamiltonian density can be written as:

$$\mathcal{H} = \frac{1}{2} \left[\frac{\partial \gamma_{\lambda\rho}}{\partial t} \frac{\partial \gamma^{\lambda\rho}}{\partial t} - \frac{1}{2} \frac{\partial \gamma_\lambda^\lambda}{\partial t} \frac{\partial \gamma_\rho^\rho}{\partial t} + \frac{1}{2} \frac{\partial \gamma_{\lambda\rho}}{\partial x_\sigma} \frac{\partial \gamma^{\lambda\rho}}{\partial x_\sigma} - \frac{1}{4} \frac{\partial \gamma_\lambda^\lambda}{\partial x_\sigma} \frac{\partial \gamma_\rho^\rho}{\partial x_\sigma} \right] . \tag{2.14}$$

It also follows from (2.12) that

$$\Box^2\, \partial\gamma_{\mu\nu}/\partial x_\nu = \kappa\, \partial T_{\mu\nu}/\partial x_\nu\ ; \qquad \Box^2\, \partial\gamma_{\mu\nu}/\partial x_\nu = 0\ . \tag{2.15}$$

The two equations agree only in the first approximation. However, for the exact non linear gravitational field the supplementary conditions are exactly compatible with the field equations.

2.2. The quantization scheme of Gupta

The Lagrangian density is constructed as a function of the independent variables $\gamma_{\mu\nu}$ and γ

$$\mathcal{L}_G = -\frac{1}{4}\left(\frac{\partial\gamma_{\mu\nu}}{\partial x_\lambda}\frac{\partial\gamma^{\mu\nu}}{\partial x_\lambda} - \frac{1}{2}\frac{\partial\gamma}{\partial x_\lambda}\frac{\partial\gamma}{\partial x_\lambda}\right)\ . \tag{2.16}$$

Thus the field equations are:

$$\Box^2\gamma_{\mu\nu} = 0\ ; \qquad \Box^2\gamma = 0 \tag{2.17}$$

and the Hamiltonian density becomes

$$\mathcal{H} = \frac{1}{2}\left[\frac{\partial\gamma_{\mu\nu}}{\partial t}\frac{\partial\gamma^{\mu\nu}}{\partial t} - \frac{1}{2}\frac{\partial\gamma}{\partial t}\frac{\partial\gamma}{\partial t} + \frac{1}{2}\frac{\partial\gamma_{\mu\nu}}{\partial x_\lambda}\frac{\partial\gamma^{\mu\nu}}{\partial x_\lambda} - \frac{1}{4}\frac{\partial\gamma}{\partial x_\lambda}\frac{\partial\gamma}{\partial x_\lambda}\right]\ . \tag{2.18}$$

Equation (2.18) will be reconciled with (2.14) by choosing the supplementary conditions in such a way that the expectation values of γ and γ^μ_μ coincide. In terms of γ's, we have

$$\Pi_{11} = \frac{\partial L}{\partial(\partial\gamma_{11}/\partial t)} = \frac{1}{2}\frac{\partial\gamma_{11}}{\partial t} \tag{2.19}$$

and the commutation relations are:

$$[\gamma_{11}(\boldsymbol{r}, t), \Pi_{11}(\boldsymbol{r}', t)] = \mathrm{i}\delta(\boldsymbol{r}' - \boldsymbol{r}) \tag{2.20}$$

$$\left[\gamma_{11}(\boldsymbol{r}, t)\, ,\ \frac{\partial\gamma_{11}}{\partial t}(\boldsymbol{r}', t)\right] = 2\mathrm{i}\delta(\boldsymbol{r}' - \boldsymbol{r}) \tag{2.21}$$

$$[\gamma_{11}(x), \gamma_{11}(x')] = 2\mathrm{i}D(x - x')\ , \tag{2.22}$$

where

$$D(x - x') \equiv -\frac{2\mathrm{i}}{(2\pi)^4}\ \mathcal{L}t_{\rho\to 0} \int \mathrm{d}k'\, \frac{\exp\{-\mathrm{i}k'(x - x')\}}{k'^2 + \rho^2} \tag{2.23}$$

is the singular function for the gravitational field. Since $\gamma_{12} = \gamma_{21}$ and $\Pi_{12} = \partial L/\partial(\partial\gamma_{12}/\partial t)$, the following commutation relation also holds:

$$[\gamma_{12}(x), \gamma_{12}(x')] = \mathrm{i}D(x - x') \tag{2.24}$$

and, more in general

$$[\gamma_{\mu\nu}(x), \gamma_{\lambda\rho}(x')] = \mathrm{i}(\delta_{\mu\lambda}\delta_{\nu\rho} + \delta_{\mu\rho}\delta_{\nu\lambda})\, D(x - x') \tag{2.25}$$

$$[\gamma(x), \gamma(x')] = -4\mathrm{i}D(x - x')\,. \tag{2.26}$$

$\gamma_{\mu\nu}$ and γ are then expanded into plane waves in the form

$$\gamma_{\mu\nu} = \sum_k \frac{1}{2\sqrt{k}} \{a_{\mu\nu}(k) \exp\{\mathrm{i}(k\cdot r - \omega t)\} + a^+_{\mu\nu}(k) \exp\{-\mathrm{i}(k\cdot r - \omega t)\}\} \tag{2.27}$$

$$\gamma = \sum_k \frac{2}{2\sqrt{k}} \{a(k) \exp\{\mathrm{i}(k\cdot r - \omega t)\} + a^+(k) \exp\{-\mathrm{i}(k\cdot r - \omega t)\}\}. \tag{2.28}$$

Substitution of the above two equations into (2.24) and (2.25) gives:

$$[a_{\mu\nu}(k), a^+_{\lambda\rho}(k)] = \delta_{\mu\nu}\delta_{\nu\rho} + \delta_{\mu\rho}\delta_{\nu\lambda} \tag{2.29}$$

$$[a(k), a^+(k)] = -1\,. \tag{2.30}$$

Use of the last six equations in the Hamiltonian density and omission of the zero point energy give the Hamiltonian of the gravitational field:

$$\int \mathcal{H}\, \mathrm{d}v = \sum_k k\{\tfrac{1}{2} a^+_{\mu\nu}(k)\, a^{\mu\nu}(k) - a^+(k)\, a(k)\}\,. \tag{2.31}$$

The following four operators are defined for convenience:

$$\begin{aligned} a'_{11}(k) &= \tfrac{1}{2}(a_{11}(k) - a_{22}(k))\,; \qquad a'_{22}(k) = \tfrac{1}{2}(a_{11}(k) + a_{22}(k)) \\ a'_{33}(k) &= \frac{1}{\sqrt{2}}\, a_{33}(k)\,; \qquad a'_{00}(k) = \frac{1}{\sqrt{2}}\, a_{00}(k)\,. \end{aligned} \tag{2.32}$$

They obey the commutation rules

$$\begin{aligned} &[a'_{11}(k), a'^+_{11}(k)] = 1\,; \qquad [a'_{22}(k), a'^+_{22}(k)] = 1\,, \\ &[a'_{33}(k), a'^+_{33}(k)] = 1\,; \qquad [a'_{00}(k), a'^+_{00}(k)] = 1\,. \end{aligned} \tag{2.33}$$

The Hamiltonian can then be written as:

$$\begin{aligned} \int \mathcal{H} \mathrm{d}v = \sum_k k\, \{ & a'^+_{11} a'^{11} + a'^+_{22} a^{22\prime} + a'^+_{33} a'^{33} + a'^+_{00} a'^{00} - a^+ a \\ & + a^+_{12} a^{12} + a^+_{23} a^{23} + a^+_{31} a^{31} - a^+_{10} a^{10} - a^+_{20} a^{20} - a^+_{30} a^{30}\}\,. \end{aligned} \tag{2.34}$$

There are therefore eleven types of gravitons corresponding to eleven independent components of $\gamma_{\mu\nu}$ and γ. Since the commutators involving $a_{10}(k)$ and $a(k)$ have a negative sign it is convenient to use an indefinite metric in dealing with the components γ_{10} and γ.

2.3. The supplementary conditions

Because of the introduction of an indefinite metric, any state Ψ is normalised as

$$\Psi^+\Psi = (-1)^{n_{10}+n_{20}+n_{30}+n} \tag{2.35}$$

where n_{10}, n_{20}, n_{30} and n are the occupation numbers of the a_{10}-, a_{20}-, a_{30}- and a- gravitons.

The supplementary conditions

$$\frac{\partial\gamma^+_{\mu\nu}}{\partial x_\mu}\Psi = 0\ ; \qquad (\gamma^{+\mu}_\mu - \gamma^+)\,\Psi = 0 \tag{2.36}$$

where $\gamma^+_{\mu\nu}$ and γ^+ are the positive frequency parts of $\gamma_{\mu\nu}$ and γ, now determine the allowable wave functions ψ that describe the state of the gravitational field and ensure in particular that

$$\langle\partial\gamma_{\mu\nu}/\partial x_\mu\rangle = 0\ ; \qquad \langle\gamma^\mu_\mu\rangle = \langle\gamma\rangle\,. \tag{2.37}$$

Substitution of the Fourier expansions of $\gamma_{\mu\nu}$ in (2.36) and the choice of k along the x_3 axis gives:

$$(a_{3i}(k) - a_{0i}(k))\,\Psi = 0\ ; \qquad i = 1, 2 \tag{2.38}$$

$$(a_{33}(k) - a_{00}(k))\,\Psi = 0\ ; \qquad (\sqrt{2}a'_{33}(k) - a_{03}(k))\,\Psi = 0 \tag{2.39}$$

$$(a_{30}(k) - a_{00}(k))\,\Psi = 0\ ; \qquad (a_{30}(k) - \sqrt{2}a'_{00}(k))\,\Psi = 0\,. \tag{2.40}$$

These equations eliminate states in which a_{3i} and a_{0i} gravitons are present in the absence of interaction. Also from equations (2.36) and the Fourier expansions of $\gamma_{\mu\nu}$ and γ, one obtains

$$(a_{\mu\mu}(k) - 2a(k))\,\Psi = 0\,, \qquad (a'_{22}(k) - a(k))\,\Psi = 0 \tag{2.41}$$

which imply the absence of a'_{22}- and a-gravitons in a pure gravitational field. Hence, the supplementary conditions eliminate 9 types of gravitons for a pure gravitational field:

$$n_{23} = n_{31} = n_{01} = n_{02} = n_{03} = n_{22} = n_{33} = n_{00} = n = 0 \tag{2.42}$$

while the occurrence of negative probabilities in real states as predicted by eq. (2.36) is prevented.

Only gravitons of the types a_{12} and a_{11} can exist and the vacuum state may be defined as that containing no a_{12} and a_{11} gravitons. Since all the components of $\gamma^+_{\mu\nu}$ and γ^+ contain absorption operators, the vacuum state Ψ_0 satisfies the equations:

$$\gamma^+_{\mu\nu}\Psi_0 = 0\ ; \qquad \gamma^+\Psi_0 = 0\,. \tag{2.43}$$

The only non vanishing component of the spin operator is then

$$\rho'_{12} = \sum_k 2[a^+_+(k)a_+(k) - a^+_-(k)a_-(k)] = 2\sum_k (n_+(k) - n_-(k)) \tag{2.44}$$

where n_+ and n_- are the numbers of gravitons corresponding to the operators $a_+(k)$ and $a_-(k)$. Equation (2.44) shows that gravitons are particles of spin 2 with two independent spin states corresponding to spin axis parallel or antiparallel to the direction of motion.

2.4. The gravitational field in interaction

The interaction of the gravitational field with matter is represented by the Lagrangian density:

$$\mathcal{L} = -\frac{1}{4}\left(\frac{\partial\gamma_{\mu\nu}}{\partial x_\lambda}\frac{\partial\gamma^{\mu\nu}}{\partial x_\lambda} - \frac{1}{2}\frac{\partial\gamma}{\partial x_\lambda}\frac{\partial\gamma}{\partial x_\lambda}\right) - \frac{\kappa}{2}(\gamma_{\mu\nu} - \tfrac{1}{2}\delta_{\mu\nu}\gamma)T^{\mu\nu} . \tag{2.45}$$

The above equation gives rise to the field equations:

$$\Box^2\gamma_{\mu\nu} = \kappa T_{\mu\nu} ; \qquad \Box^2\gamma^\mu_\mu = \kappa T^\mu_\mu . \tag{2.46}$$

The supplementary conditions are modified in the Heisenberg representation to:

$$[\partial\gamma^+_{\mu\nu}/\partial x_\mu]\,\Psi = 0 \tag{2.47}$$

$$(\gamma^\mu_\mu - \gamma)^+\,\Psi = 0 \tag{2.48}$$

where $[\quad]^+$ denotes the positive frequency part. In the absence of interaction they are indentical to the previous conditions given in (2.37). The terms $\partial\gamma_{\mu\nu}/\partial x_\mu$ and $(\gamma^\mu_\mu - \gamma)$ can be split into positive and negative frequency parts in the presence of interaction in the Heisenberg representation. Also the following equations hold:

$$\Box^2\,\partial\gamma_{\mu\nu}/\partial x_\mu = 0 \tag{2.49}$$

$$\Box^2(\gamma^\mu_\mu - \gamma) = 0 . \tag{2.50}$$

Passing from the Heisenberg representation to the interaction representation, the following equations are obtained:

$$[\partial\gamma^+_{\mu\nu}/\partial x_\mu + \kappa \int D^+(x' - x)\,T_{0\nu}(x')\,\mathrm{d}v]\,\Psi(t) = 0 \tag{2.51}$$

$$(\gamma^{\mu+}_\mu - \gamma^+)\Psi = 0 . \tag{2.52}$$

$D^+(x - x')$ is the positive frequency part of $D(x - x')$. Those gravitons which could not exist in a pure gravitational field can now appear in virtual states in the presence of interaction due to the $T_{0\nu}$ term. The whole theory is Lorentz invariant.

In particular, the field equations for the electromagnetic field are given by:

$$\partial\mathcal{F}^{\mu\nu}/\partial x^\nu = -\rho^\mu \tag{2.53}$$

$$\partial\mathcal{A}^\mu/\partial x^\mu = 0 , \qquad \mathcal{F}^{\mu\nu} = \sqrt{-g}\,F^{\mu\nu} \tag{2.54}$$

$$F^{\mu\nu} = \partial A^\nu/\partial x^\mu - \partial A^\mu/\partial x^\nu = g^{\mu\lambda}g^{\nu\rho}F_{\lambda\rho} \tag{2.55}$$

$$\mathcal{A}^\mu = \sqrt{-g}\,A^\mu ; \qquad \rho^\mu = \sqrt{-g}\,J^\mu . \tag{2.56}$$

The total Lagrangian density of the gravitational and electromagnetic fields may then be split into two parts

$$\mathcal{L}_{\mathrm{tot}} = \mathcal{L} + \mathcal{L}_{\mathrm{R}} \tag{2.57}$$

and

$$\mathcal{L}_R = -\tfrac{1}{4}\sqrt{-g}\,\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu} = -\tfrac{1}{4}\sqrt{-g}\,g^{\mu\lambda}g^{\nu\rho}\,\mathcal{F}_{\lambda\rho}\,\mathcal{F}_{\mu\nu} \tag{2.58}$$

which, by using the Lorentz condition $\partial A^\alpha/\partial x^\alpha = 0$, becomes

$$\mathcal{L}_R = -\tfrac{1}{4}\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu} - \tfrac{1}{2}\kappa\gamma_{\alpha\beta}(F^{\alpha\mu}F^{\beta}_{\mu} - \tfrac{1}{4}\delta^{\alpha\beta}F^{\mu\nu}F_{\mu\nu}) + O(\kappa^2)\,. \tag{2.59}$$

The first order interaction term is given by

$$-\tfrac{1}{2}\kappa\gamma_{\mu\nu}T^{\mu\nu} \tag{2.60}$$

where $T^{\mu\nu}$ is the usual flat space energy-momentum tensor of the electromagnetic field.

2.5. Weinberg's theory of massless particles

Linear gravitational fields can also be quantized by means of Weinberg's "physical particles only" approach [85–87]. In this approach particles are defined as the irreducible representations of the inhomogeneous Lorentz group and play a more fundamental role than fields since particle operators enter the interaction in the form of free fields which transform according to various representations of the homogeneous Lorentz group and commute with each other at spacelike separations. The S matrix is Lorentz invariant and the Feynman rules for gravitons (and photons) can be obtained. In order to account for the inverse square law forces, potentials are introduced which a priori may not be invariant. Thereby, the Lorentz invariance of the S matrix becomes a difficult requirement to satisfy. The salient feature of the formalism is that for massless particle of spin $j = 2$, the only Lorentz invariant theory is just that of Einstein. With an exactly conserved current, in the weak field limit, the field equations reduce to those derived by Gupta. Weinberg's approach is here presented with a view to illustrate how the gulf between the theory of massless particles and the rest of particle physics could possibly be bridged.

2.5.1. The fields

Weinberg shows that in the perturbation theory for massive particles of any spin, the amplitudes for emission or absorption of a pàrticle with spin j vanishes necessarily as $p^{j-1/2}$ for momentum $p \to 0$ when $m \to 0$.

The necessity to avoid this difficulty which would make the existence of inverse square laws impossible leads to the introduction of potentials whose jth derivatives give the fields. For simplicity, the fields for particles of spin j are chosen to be transformed under the $(2j + 1)$ dimensional representations $(j\ 0)$ and $(0\ j)$ of the homogeneous Lorentz group. They are indicated by $\phi_\sigma(x)$ and $\chi_\sigma(x)$ respectively and can be written as tensors $F_-(x)$ and $F_+(x)$ of rank $2j$ as follows:

$$F_{\pm}^{[\mu_1\nu_1][\mu_2\nu_2]\dots[\mu_j,\nu_j]} = (2\pi)^{-3/2}\,\mathrm{i}^{+j}\int\frac{\mathrm{d}^3p}{(2|p|)^{1/2}}\,[p^{\mu_1}e_{\pm}^{\nu_1}(\boldsymbol{p}) - p^{\nu_1}e_{\pm}^{\mu_1}(\boldsymbol{p})]$$

$$\dots[p^{\mu_j}e_{\pm}^{\nu_j}(\boldsymbol{p}) - p^{\nu_j}e_{\pm}^{\mu_j}(\boldsymbol{p})]\,[a(\boldsymbol{p},\pm j)\,\mathrm{e}^{\mathrm{i}p\cdot x} + b^*(\boldsymbol{p},\mp j)\,\mathrm{e}^{-\mathrm{i}p\cdot x}]\,. \tag{2.61}$$

b^* is the antiparticle creation operator, a is the particle annihilation operator, b is the antiparticle annihilation operator, a^* is the particle creation operator.

Therefore, $F_-(x)$ and $F_+(x)$ are just linear combinations of $\phi(x)$ and $\chi(x)$ and vice versa. The polarization vectors $e^{\mu}_{\pm}(\boldsymbol{p})$ in (2.61) are defined as

$$e^{\mu}_{\pm}(\boldsymbol{p}) = R^{\mu}_{\nu}(\hat{p})e^{\nu}_{\pm} \tag{2.62}$$

$$e^{1}_{\pm} = 1/\sqrt{2}\,; \qquad e^{2}_{\pm} = \pm \mathrm{i}/\sqrt{2}\,; \qquad e^{3}_{\pm} = e^{0}_{\pm} = 0\,. \tag{2.63}$$

$R^{\mu}_{\nu}(p)$ is a pure rotation that carries the z-axis into the direction of p.

i) $F_{\pm}(x)$ each have not more than $(2j+1)$ linearly independent components.

ii) $F_{\pm}(x)$ actually are tensors, that is, they transform as:

$$U(\Lambda)\,F_{\pm}^{[\mu_1\nu_1]\ldots[\mu_j,\nu_j]}\,U^{-1}(\Lambda) = \Lambda^{\mu_1}_{\rho_1}\,\Lambda^{\nu_1}_{\eta_1}\ldots\Lambda^{\mu_j}_{\rho_j}\Lambda^{\nu_j}_{\eta_j}\,F_{\pm}^{[\rho_1\eta_1]\ldots[\rho_j,\eta_j]}(\Lambda x)\,. \tag{2.64}$$

Λ^{μ}_{ρ} is a proper orthochronous Lorentz transformation. The corresponding unitary operator is $U(\Lambda)$ while $U^{-1}(\Lambda)$ is its inverse. Also

$$\Lambda^{\mu}_{\rho}\Lambda^{\sigma}_{\mu} = \delta^{\sigma}_{\rho}\,. \tag{2.65}$$

The $U(\Lambda)$ forms an infinite dimensional (and reducible) unitary, representation of the Inhomogeneous Lorentz Group:

$$U(\bar{\Lambda},\bar{a})\;U(\Lambda,a) = U(\bar{\Lambda}\Lambda,\,\bar{\Lambda}a+\bar{a})$$

$$U(\bar{\Lambda})\;U(\Lambda) = U(\bar{\Lambda}\Lambda)\,. \tag{2.66}$$

Therefore, from (2.64), $F_{\pm}$ transforms according to some reducible or irreducible representation of the homogeneous Lorentz group. The most general representation that can be constructed from $a(p,+j)\exp(\mathrm{i}p\cdot x) + b^{*}(p,-j)\exp(-\mathrm{i}p\cdot x)$ is a sum of representations of type

$$F_{+}:(0,j)\,(\tfrac{1}{2},j+\tfrac{1}{2})\,(1,j+1)\ldots \tag{2.67}$$

while the most general representation that can be constructed from $a(p,-j)\exp(\mathrm{i}p\cdot x) + b^{*}(p,+j)\exp(\mathrm{i}p\cdot x)$ is a sum of

$$F_{-}:(j,0)\,(j+\tfrac{1}{2},\tfrac{1}{2})\,(j+1,1)\ldots \tag{2.68}$$

as has been shown by Weinberg.

The only way such sums can have dimensionality $\leqslant 2j+1$ is for them to consist just of the first terms $(0,j)$ and $(j,0)$. The representation determines the fields uniquely, so F_{+} must be χ and F_{-} must be ϕ.

The following properties of $F_{\pm}$ are mentioned to number the components of $F_{\pm}$:

1) The $F_{\pm}$'s are symmetric under interchange of any two index pairs,
2) Antisymmetric under interchange of indices within a pair,
3) They are either self-dual or anti-self-dual within each index pair,
4) The complete trace on any two index pairs vanishes.

The conditions of antisymmetry and self duality or antiself-duality lower the number of independent components for each index pair from 16 to 6 to 3. The conditions 1 to 3 would give $F_{\pm}(x)$ the same number of components as for a symmetric tensor in 3 dimensions, i.e.,

$$N_j = \binom{j+2}{2} = \frac{(j+1)(j+2)}{2}\,. \tag{2.69}$$

Condition 4 imposes N_{j-2} constraints. Therefore, the number of independent components left is not greater than $2j+1$,

$$N_j - N_{j-2} \leqslant 2j+1\ . \tag{2.70}$$

The proof that the $F_\pm(x)$ are tensors, is contained in Weinberg's article [87]. The fields $F_\pm(x)$ obey the free field equations

$$\Box^2 F_\pm^{[\mu_1\nu_1]\cdots}(x) = 0 \tag{2.71}$$

$$\partial_{\mu_1} F_\pm^{[\mu_1\nu_1]\cdots}(x) = 0\ . \tag{2.72}$$

For $j=1$, (2.61) gives two of Maxwell's equations. The five independent components of $F_\pm^{[\mu\nu][\lambda\eta]}$ are identified with the left or right handed parts of the source-free Riemann–Christoffel tensor.

The potentials can now be introduced and are found to be suitable to formulate a theory of photons and gravitons. The potentials are not tensors and their transformation properties are shown in the next subsection. The S matrix follows as a natural consequence of the introduction of the potentials and the current operator in the interaction representation as shown in subsection 2.5.3.

The transformation of the S matrix follows from the Lorentz transformation rule for one particle states and the imposition of two conditions on the current operator in the Heisenberg representation.

2.5.2. The potentials

The tensor fields $F_\pm(x)$ are the jth derivatives of the potentials as can be seen from (2.61),

$$A_\pm^{\mu_1\cdots\mu_j}(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{\mathrm{d}^3 p}{(2|p|)^{1/2}}\ e_\pm^{\mu_1}(\boldsymbol{p}) \ldots e_\pm^{\mu_j}(\boldsymbol{p})[a(\boldsymbol{p},\pm j)\mathrm{e}^{\mathrm{i}p\cdot x} + (-1)^j b^*(\boldsymbol{p},\mp j)\,\mathrm{e}^{-\mathrm{i}p\cdot x}] \tag{2.73}$$

$$F_\pm^{[\mu,\nu]}(x) = \partial^\mu A_\pm^\nu(x) - \partial^\nu A_\pm^\mu(x) \tag{2.74}$$

$$F_\pm^{[\mu\nu][\lambda\eta]}(x) = \partial^\mu\partial^\lambda A_\pm^{\nu\eta}(x) - \partial^\mu\partial^\eta A_\pm^{\nu\lambda}(x) - \partial^\nu\partial^\eta A_\pm^{\mu\lambda}(x) + \partial^\nu\partial^\eta A_\pm^{\mu\lambda}(x) + \partial^\nu\partial^\eta A_\pm^{\mu\lambda}(x)\ . \tag{2.75}$$

Since $A_\pm$ are free of the objectionable factors p^j, they can be used to construct a theory of photons and gravitons. The $A_\pm(x)$ cannot be tensors, because the traceless part of a symmetric tensor of rank j transforms according to the $(j/2, j/2)$ representation of the homogeneous Lorentz group and this is not one of the representations (2.67) and (2.68) allowed for massless particles of helicity $\lambda = \pm j$. Also, $A_\pm$ vanishes if any of its indices is $=0$, a property hardly admissible for a tensor. The potentials therefore transform in the following way:

$$U(\Lambda)A_\pm^{\mu_1\cdots\mu_j}(x)\,U^{-1}(\Lambda) = \Lambda_{\nu_1}^{\mu_1}\ldots A_{\nu_j}^{\mu_j}(2\pi)^{-3/2}\int\frac{\mathrm{d}^3p}{(2|p|)^{1/2}}\ [e_\pm^{\nu_1}(\boldsymbol{p}) - p^{\nu_1}f_\pm(\boldsymbol{p},\Lambda)]\,M \tag{2.76}$$

$$M \equiv [e_\pm^{\nu_j}(\boldsymbol{p}) - p^{\nu_j}f_\pm(p,\Lambda)]\ [a(\boldsymbol{p},\pm j)\exp(\mathrm{i}p\cdot\Lambda x) + (-1)^j\, b^*(\boldsymbol{p},\mp j)\exp(-\mathrm{i}p\cdot\Lambda x)]\ . \tag{2.77}$$

Equation (2.76) can be expressed in a more compact form as:

$$U(\Lambda)A_\mu^{\mu_1\cdots\mu_j}(x)U^{-1}(\Lambda) = \Lambda_{\nu_1}^{\mu_1}\ldots\Lambda_{\nu_j}^{\mu_j}A_\pm^{\nu_1\cdots\nu_j}(\Lambda x) + \sum_{r=1}^{j}\partial^{\mu_r}\phi_\pm^{\mu_1\cdots\mu_{r-1}\mu_{r+1}\cdots\mu_j}(x_j\Lambda)\ . \tag{2.78}$$

The second term on the right-hand side of the above equation denotes the gradient terms and the $A^{\pm}$ are tensors except for gradient terms. Since the photon and graviton are their own antiparticles, it is convenient to restrict any further discussion to purely neutral particles and define phases such that

$$b(\boldsymbol{p}, \pm j) = (-1)^j\, a(\boldsymbol{p}, \pm j)\,. \tag{2.79}$$

Then

$$A_{\pm}^{\mu_1 \dots \mu_j \dagger}(x) = A_{\mp}^{\mu_1 \dots \mu_j}(x)\,. \tag{2.80}$$

Therefore, Hermitian fields can be defined such that

$$A^{\mu_1 \dots \mu_j}(x) = A_{+}^{\mu_1 \dots \mu_j}(x) + A_{-}^{\mu_1 \dots \mu_j}(x)$$

$$= (2\pi)^{-3/2} \int \frac{\mathrm{d}^3 p}{(2|\boldsymbol{p}|)^{1/2}} \sum e_{\pm}^{\mu_1}(\boldsymbol{p}) \dots e_{\pm}^{\mu_j}(\boldsymbol{p})\, [a(\boldsymbol{p}, \pm j)\, \mathrm{e}^{\mathrm{i}p\cdot x} + a^*(\boldsymbol{p}, \mp j)\, \mathrm{e}^{-\mathrm{i}p\cdot x}] \tag{2.81}$$

$$B^{\mu_1 \dots \mu_j}(x) = -\mathrm{i}A_{+}^{\mu_1 \dots \mu_j}(x) + \mathrm{i}A_{-}^{\mu_1 \dots \mu_j}(x)$$

$$= -\frac{\mathrm{i}}{(2\pi)^{3/2}} \int \frac{\mathrm{d}^3 p}{(2|\boldsymbol{p}|)^{1/2}} \sum_{\pm} (\pm)\, e_{\pm}^{\mu_1}(\boldsymbol{p}) \dots e_{\pm}^{\mu_j}(\boldsymbol{p})[a(\boldsymbol{p}, \pm j)\mathrm{e}^{\mathrm{i}p\cdot x} + a^*(\boldsymbol{p}, \mp j)\mathrm{e}^{-\mathrm{i}p\cdot x}] \tag{2.82}$$

with A carrying intrinsic parity $(-1)^j$, while B has intrinsic parity $-(-1)^j$. A^{μ} and $A^{\mu\nu}$ can therefore couple to normal tensors like the electric current and energy momentum tensor, while the B's have to couple to tensors of abnormal parity, like the current of magnetic monopoles.

2.5.3. Lorentz invariance and current conservation

The space like components of the current operator in the interaction representation are defined by

$$g_{i_1, i_2 \dots i_j}(x) = -\delta H'(x^0)/\delta A^{i_1 \dots i_j}(x)\,. \tag{2.83}$$

From the potentials (2.73) it can be seen that the S matrix for emission of a particle with $m = 0$, helicity $\pm j$, and momentum p in a transition $\alpha \to \beta$ is

$$S_{\beta\alpha}(\boldsymbol{p}, \pm j) = \frac{i}{(2\pi)^{3/2}}\, \frac{1}{(2|\boldsymbol{p}|)^{1/2}}\, e_{\mp}^{i_1}(\boldsymbol{p}) \dots e_{\mp}^{i_j}(\boldsymbol{p}) \int \mathrm{d}^4 x\, \mathrm{e}^{-\mathrm{i}p\cdot x} \sum_{n=0}^{\infty} \frac{(-\mathrm{i})^n}{n!}$$

$$\times \int_{-\infty}^{\infty} \mathrm{d}t_1 \dots \mathrm{d}t_n \langle \mathrm{b}|\, T\{H'(t_1) \dots H'(t_n)\, g_{i_1 \dots i_j}(x)\}\,|\mathrm{a}\rangle\,. \tag{2.84}$$

The above S matrix can be expressed in terms of exact energy eigen states and Heisenberg representation operators as

$$S_{\beta\alpha}(\boldsymbol{p}, \pm j) = \frac{\mathrm{i}}{(2\pi)^{3/2}}\, \frac{1}{(2|\boldsymbol{p}|)^{1/2}}\, e_{\mp}^{i_1}(\boldsymbol{p}) \dots e_{\mp}^{i_j}(\boldsymbol{p})\, \langle \mathrm{b\ out}|\, g_{i_1 \dots i_j}^{\mathrm{H}}(p)\,|\mathrm{a\ in}\rangle \tag{2.85}$$

$$g^{(\mathrm{H})}_{i_1\ldots i_j}(p) = \int \mathrm{d}^4x\ \mathrm{e}^{-\mathrm{i}p\cdot x}\langle \mathrm{b\ out}|g^{\mathrm{H}}_{i_1\ldots i_j}(x)|\mathrm{a\ in}\rangle \tag{2.86}$$

$$g^{\mathrm{H}}(x) = \mathrm{e}^{\mathrm{i}Ht}g(0)\,\mathrm{e}^{-\mathrm{i}Ht} = \mathrm{e}^{\mathrm{i}Ht}\,\mathrm{e}^{-\mathrm{i}H_0t}\,g(t)\,\mathrm{e}^{\mathrm{i}H_0t}\,\mathrm{e}^{-\mathrm{i}Ht}\,. \tag{2.87}$$

From the Lorentz transformation rule for one particle states the S matrix transforms as

$$S_{\beta\alpha}(\boldsymbol{p}, \pm j) = \left(\frac{\Lambda \boldsymbol{p}}{|\boldsymbol{p}|}\right)^{1/2} S_{\Lambda\beta\Lambda\alpha}(\Lambda\boldsymbol{p}, \pm j)\exp\{\mp \mathrm{i}j^{\Theta}(\boldsymbol{p}, \Lambda)\}\,. \tag{2.88}$$

This is possible if and only if

(i) $g^{(\mathrm{H})}_{i_1\ldots i_j}(p)$ is the spacelike part of a symmetric tensor $g^{(\mathrm{H})}_{\mu_1\ldots\mu_j}(p)$ with

$$U(\Lambda)g^{(\mathrm{H})}_{\mu_1\ldots\mu_j}(p)U^{-1}(\Lambda) = \Lambda^{\nu_j}_{\mu_j}\,g^{(\mathrm{H})}_{\nu_1\ldots\nu_j}(\Lambda p)\,. \tag{2.89}$$

(ii) The tensor is conserved

$$p^{\mu_1}g^{(\mathrm{H})}_{\mu_1\ldots\mu_j}(p) = 0\,. \tag{2.90}$$

The propagator for the $j = 2$ particle is given by

$$\langle T\{A^{\mu\nu}(x), A^{\lambda\eta}(y)\}\rangle_0 = (2\pi)^{-3/2}\int \frac{\mathrm{d}^3p}{(2|p|)^{1/2}}\,\Pi^{\mu\nu\lambda\eta}(\boldsymbol{p})\{\theta(x-y)\,\mathrm{e}^{\mathrm{i}p\cdot(x-y)} + \theta(y-x)\,\mathrm{e}^{\mathrm{i}p\cdot(y-x)}\} \tag{2.91}$$

$$\Pi^{\mu\nu\lambda\eta}(\boldsymbol{p}) = \sum_{\pm} e^{\mu}_{\pm}(\boldsymbol{p})\,e^{\nu}_{\pm}(\boldsymbol{p})\,e^{\lambda}_{\mp}(\boldsymbol{p})\,e^{\eta}_{\mp}(\boldsymbol{p}) \tag{2.92}$$

In momentum space, this gives

$$\Delta^{\mu\nu\lambda\eta}(q) = \mathrm{i}\int \mathrm{d}^4x\,\exp\{-\mathrm{i}q\cdot(x-y)\}\,\langle T\{A^{\mu\nu}(x), A^{\lambda\eta}(y)\}\rangle_0 = \Pi^{\mu\nu\lambda\eta}(q)/(q^2 - \mathrm{i}\epsilon)\,. \tag{2.93}$$

Letting $\boldsymbol{p} = \boldsymbol{k} = (001)$, using (2.63) and evaluating the polarization sum shows that the only non-vanishing components of (2.92) are

$$\Pi^{1111}(\boldsymbol{k}) = \Pi^{2222}(\boldsymbol{k}) = \Pi^{1212}(\boldsymbol{k}) = \Pi^{2112}(\boldsymbol{k}) = \Pi^{1221}(\boldsymbol{k}) = \Pi^{2121}(\boldsymbol{k}) = -\Pi^{1122}(\boldsymbol{k}) = -\Pi^{2211}(\boldsymbol{k}) = \tfrac{1}{2}\,. \tag{2.94}$$

There is a simple relation between $\Pi^{\mu\nu\lambda\eta}(k)$ and $\Pi^{\mu\nu}(k)$. $\Pi^{\mu\nu}(k)$ is given by

$$\Pi^{\mu\nu}(\boldsymbol{p}) = \sum_{\pm} e^{\mu}_{\pm}(\boldsymbol{p})\,e^{\nu}_{\mp}(\boldsymbol{p})\,. \tag{2.95}$$

Application of the rotation $R(\hat{q})$ enables to represent $\Pi^{\mu\nu\lambda\eta}(q)$ as:

$$\Pi^{\mu\nu\lambda\eta}(\boldsymbol{q}) = \tfrac{1}{2}\,[\Pi^{\mu\lambda}(\boldsymbol{q})\Pi^{\nu\eta}(\boldsymbol{q})^{\nu\eta}(\boldsymbol{q}) + \Pi^{\mu\eta}(\boldsymbol{q})\Pi^{\nu\lambda}(\boldsymbol{q}) - \Pi^{\mu\nu}(\boldsymbol{q})\Pi^{\lambda\eta}(\boldsymbol{q})] \tag{2.96}$$

where

$$\Pi^{\mu\nu}(\boldsymbol{q}) = \eta^{\mu\nu} + \frac{(\eta^{\mu}q^{\nu} + \eta^{\nu}q^{\mu})}{|q|^2}\,q^0 + \frac{q^2\eta^{\mu}\eta^{\nu}}{|\boldsymbol{q}|^2} \tag{2.97}$$

$$\eta^{\mu} = [0001]\,. \tag{2.98}$$

The propagator (2.93) can be split into three parts

$$\Delta_{\mathrm{C}}^{\mu\nu\lambda\eta}(q) = \Delta_{\mathrm{Cov}}^{\mu\nu\lambda\eta}(q) + \Delta_{\mathrm{Grad}}^{\mu\nu\lambda\eta}(q) + \Delta_{\mathrm{Loc}}^{\mu\nu\lambda\eta}(q) \tag{2.99}$$

$$\Delta_{\mathrm{Cov}}^{\mu\nu\lambda\eta}(q) = -\tfrac{1}{2}(\eta^{\mu\lambda}\eta^{\nu\eta} + \eta^{\mu\eta}\eta^{\nu\lambda} - \eta^{\mu\lambda}\eta^{\lambda\eta})/(q^2 - \mathrm{i}\epsilon) \tag{2.100}$$

while Δ_{Grad} contains all terms carrying a factor q^μ, q^ν, q^λ or q^η and Δ_{Loc} is the remaining part

$$\Delta_{\mathrm{Loc}}^{\mu\nu\lambda\eta}(q) = [(\eta^{\mu\lambda}\eta^{\nu}\eta^{\eta} + \eta^{\mu\eta}\eta^{\nu}\eta^{\lambda} + \eta^{\nu\lambda}\eta^{\mu}\eta^{\eta} + \eta^{\nu\eta}\eta^{\mu}\eta^{\lambda} - \eta^{\mu\nu}\eta^{\lambda}\eta^{\eta} - \eta^{\lambda\eta}\eta^{\mu}\eta^{\nu})/2|\boldsymbol{q}|^2]$$
$$+ \eta^{\mu}\eta^{\nu}\eta^{\lambda}\eta^{\eta}/2|\boldsymbol{q}|^4 \,. \tag{2.101}$$

The Δ_{Loc} term does not have a pole at $q = 0$, so its Fourier transform is temporally local, and its effect can be cancelled by adding a "Newtonian" term to the interaction.

The current $g_{\mu\nu}(x)$ has to be conserved to eliminate the $1/q^2$ pole in Δ_{Grad}. The only conserved symmetric tensor is the energy momentum tensor $\theta_{\mu\nu}$, so $A_{\mu\nu}$ has to be coupled to $\theta_{\mu\nu}$. For exact conservation $\theta_{\mu\nu}$ must involve the potential $A_{\mu\nu}$. Therefore the proof of Lorentz invariance becomes extremely difficult for the case of $j = 2$ (the graviton) but for $j = 1$ it is simple but longwinded.

The effective graviton propagator is

$$\Delta_{\mathrm{Cov}}^{\mu\nu\lambda\eta}(q) = -\tfrac{1}{2}\mathrm{i}[\eta^{\mu\lambda}\eta^{\nu\eta} + \eta^{\mu\eta}\eta^{\nu\lambda} - \eta^{\mu\nu}\eta^{\lambda\eta}]/(q^2 - \mathrm{i}\epsilon)\,. \tag{2.102}$$

2.5.4. Derivation of Einstein's equations

For the massless particle with $j = 2$ the Lorentz invariant theory leads to the derivation of Einstein's equations. The interaction representation potential in this case is traceless. It then follows that the traceless part of the Heisenberg representation potential is given by:

$$A_{\mathrm{H}}^{ij}(\boldsymbol{x}, t) - \tfrac{1}{3}\delta^{ij}A_{\mathrm{H}k}^{k}(\boldsymbol{x}, t) = U(t)A^{ij}(\boldsymbol{x}, t)U^{-1}(t)\,. \tag{2.103}$$

Analogously to the case of the scalar potential for the electromagnetic field, six components of the Heisenberg representation potential are introduced and lead to the equations

$$\nabla^2 A_{\mathrm{H}}^{i0}(x) = -g_{\mathrm{H}}^{i0}(x) \tag{2.104}$$

$$\nabla^2 A_{\mathrm{H}i}^{i}(x) = -\tfrac{3}{2}g_{\mathrm{H}}^{00}(x) \tag{2.105}$$

$$\nabla^2 A_{\mathrm{H}}^{00}(x) - \tfrac{1}{3}A_{\mathrm{H}k}^{k}(x) = -\tfrac{1}{2}g_{\mathrm{H}i}^{i}(x) - \tfrac{1}{2}g_{\mathrm{H}}^{00}(x) \tag{2.106}$$

$$\partial_i A_{\mathrm{H}}^{i0}(x) + \tfrac{2}{3}\partial_0 A_{\mathrm{H}k}^{k}(x) = 0\,. \tag{2.107}$$

The S matrix for a transition α–β due to an infinitesimal c-number $\delta g_{\mu\nu}$ can then be expressed as:

$$\delta S_{\beta\alpha} = -\mathrm{i}\int \mathrm{d}^4x \langle\beta\ \mathrm{out}|A_{\mathrm{H}}^{\mu\nu}(x)|\alpha\ \mathrm{in}\rangle\, \delta g_{\mu\nu}(x)\,. \tag{2.108}$$

The field equations satisfied by (2.103) are:

$$\Box^2[A_{\mathrm{H}}^{ij}(\boldsymbol{x}, t) - \tfrac{1}{3}\delta^{ij}A_{\mathrm{H}k}^{k}(\boldsymbol{x}, t)] = -\int \mathrm{d}^3y\, D^{ijkl}(\boldsymbol{x} - \boldsymbol{y})g_{\mathrm{H}}^{kl}(\boldsymbol{y}, t) \tag{2.109}$$

where

$$D^{ijkl}(\boldsymbol{r}) = (2\pi)^{-3} \int \mathrm{d}^3 p\, \Pi^{ijkl}(\boldsymbol{p}) \exp(\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}) = \tfrac{1}{2}\,[(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk} - \delta^{ij}\delta^{kl})\,\delta^3(\boldsymbol{x}-\boldsymbol{y}) + \tfrac{1}{2}(\partial^i\partial^k\delta^{jl} + \partial^j\partial^k\delta^{il} + \partial^i\partial^l\delta^{jk} + \partial^j\partial^l\delta^{ik} - \partial^i\partial^j\delta^{kl} - \partial^k\partial^l\delta^{ij})\,D(x-y) + \tfrac{1}{2}\partial^i\partial^j\partial^k\partial^l\,\mathcal{E}(\boldsymbol{x}-\boldsymbol{y})] \tag{2.110}$$

$$\mathcal{E}(\boldsymbol{x}-\boldsymbol{y}) = (2\pi)^{-3}\int \frac{\mathrm{d}^3 p}{|p|^4} \exp\{\mathrm{i}\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})\} = \mathcal{E}(0) - \frac{|\boldsymbol{x}-\boldsymbol{y}|}{8\pi}\,. \tag{2.111}$$

By using the current conservation condition one obtains

$$\Box^2 A^{ij}_{\mathrm{H}}(x) - \partial^0\partial^j A^{i0}_{\mathrm{H}}(x) - \partial_0\partial^i A^{j0}_{\mathrm{H}}(x) - \partial^i\partial^j[A^{00}_{\mathrm{H}}(x) - \tfrac{1}{3}A^{k}_{\mathrm{H}k}(x)] = -g^{ij}_{\mathrm{H}}(x) + \tfrac{1}{2}\delta^{ij}g^{\mu}_{\mathrm{H}\mu}(x)\,. \tag{2.112}$$

The traceless part of A^{ij}_{H} is divergenceless as is $A^{ij}(x)$,

$$\partial_i A^{ij}_{\mathrm{H}}(x) - \tfrac{1}{3}\partial_j A^{k}_{\mathrm{H}k}(x) = 0\,. \tag{2.113}$$

These equations can also be expressed as

$$R^{\mu\nu}_{\mathrm{H}}(x) = -g^{\mu\nu}_{\mathrm{H}}(x) + \tfrac{1}{2}g^{\mu\nu}g^{\lambda}_{\mathrm{H}\lambda}(x)$$

or equivalently as

$$R^{\mu\nu}_{\mathrm{H}}(x) - \tfrac{1}{2}g^{\mu\nu}R^{\lambda}_{\mathrm{H}\lambda}(x) = -g^{\mu\nu}_{\mathrm{H}}(x) \tag{2.114}$$

where $R^{\mu\nu}_{\mathrm{H}}$ is given by

$$R^{\mu\nu}_{\mathrm{H}}(x) = \Box^2 A^{\mu\nu}_{\mathrm{H}}(x) - \partial^\mu\partial_\lambda A^{\lambda\nu}_{\mathrm{H}}(x) - \partial^\nu\partial_\lambda A^{\lambda\mu}_{\mathrm{H}}(x) + \partial^\mu\partial^\nu A^{\lambda}_{\mathrm{H}\lambda}(x)\,. \tag{2.115}$$

Equations (2.110) and (2.111) must also be retained as gauge conditions on the potential. They arise because of the special way in which $A^{ij}(x)$ have been constructed. Equations (2.113) and (2.114) are recognized as Einstein's equations in the weak field approximation provided that $g^{\mu\nu}$ is taken as

$$g^{\mu\nu}(x) = \eta^{\mu\nu} + A^{\mu\nu}_{\mathrm{H}}(x)$$

and the energy momentum tensor $\theta^{\mu\nu}$ is for matter alone, which is however inconsistent with the requirement that the current $g^{\mu\nu}_{\mathrm{H}}$ be conserved. The matter tensor $\theta^{\mu\nu}$ has zero covariant divergence, but its ordinary divergence is $\neq 0$ if gravitational interactions must be accounted for. Gupta has shown that (2.113) with an exactly conserved right-hand side is exactly equivalent to Einstein's equations.

The Feynman rules for photons and gravitons can thus be formulated and are summed up in section 3.

3. Rules to calculate diagrams

The Feynman rules for photons, gravitons, electrons, and neutrinos etc. can now be formulated and are summed up in section 3. It is worthwhile to mention that the rules discussed are not a complete set of Feynman rules and cannot be used to calculate higher order terms involving closed graviton loops.

The wave function of an incident electron of momentum p_i and spin S_i is given by the solution of the Dirac equation,

$$(p - m)\Psi_i = 0 \tag{3.1}$$

$$\Psi_i(x) = \sqrt{\frac{m}{E_i V}}\, U(p_i, S_i)\exp(-ip_i \cdot x)\,. \tag{3.2}$$

For a neutrino of momentum p_i and spin S_i

$$\Psi_i(x) = \sqrt{\frac{1}{V}}\, U(p_i, S_i)\exp(-ip_i \cdot x)\,. \tag{3.3}$$

The wave function is normalized to unit probability in a box of volume V. In a similar way, the wave function in the final state is given by:

$$\bar{\Psi}_f(x) = \sqrt{\frac{m}{E_f V}}\, \bar{U}(p_f, S_f)\exp(ip_f \cdot x) \tag{3.4}$$

for the electron, and

$$\bar{\Psi}_f(x) = \sqrt{\frac{1}{V}}\, \bar{U}(p_f, S_f)\exp(ip_f \cdot x) \tag{3.5}$$

for the neutrino.

The wave functions in momentum space are obtained by use of the Fourier transform. $\sqrt{m/E_{i,f}V}\, U(p_{i,f}, S_{i,f})$ describes an electron in momentum space in either the initial or the final state and $\sqrt{(1/V)}\, U(p_{i,f}, S_{i,f})$ describes a neutrino in either the initial or the final state.

The four vector potential of a "photon" with momentum p_μ and polarization ϵ^μ is written as a plane wave

$$A^\mu(x, p) = \frac{\epsilon^\mu}{\sqrt{2pV}}\,(e^{ip\cdot x} + e^{-ip\cdot x}) \tag{3.6}$$

$$p_\mu p^\mu = 0\,; \tag{3.7}$$

ϵ^μ is the unit polarization vector and satisfies the transversality condition,

$$\epsilon_\mu p^\mu = 0\,. \tag{3.8}$$

The wave function of an incoming photon is given in momentum space by:

$$\Psi^{\mathrm{in}}_{\mathrm{ph}} = (2\pi)^{-3/2}\,\frac{1}{\sqrt{2|\boldsymbol{p}_i|}}\,\epsilon^\mu_\pm(\boldsymbol{p}_i)\,. \tag{3.9}$$

Similarly the wave function of an outgoing photon is given by:

$$\Psi^{\mathrm{out}}_{\mathrm{ph}} = (2\pi)^{-3/2}\,\frac{1}{\sqrt{2|\boldsymbol{p}_f|}}\,\epsilon^\mu_\pm(\boldsymbol{p}_f)\,. \tag{3.10}$$

The photon propagator is given by the equation:

$$\Box D_F(x - y) = \delta^4(x - y) \tag{3.11}$$

$$D_F(x-y) = \int \frac{d^4 q}{(2\pi)^4}\, e^{-iq\cdot(x-y)}\, D_F(q^2) \tag{3.12}$$

$$D_F(K^2) = \frac{-i}{(2\pi)^4}\, \eta^{\mu\nu}/(k^2 - i\epsilon)\,. \tag{3.13}$$

This last equation describes a virtual photon propagator ($\mu \to \nu$) for an internal line of momentum K.

The form of the electron–electron–photon vertex is dictated by the requirement that A^μ is coupled to a conserved covariant quantity.

The Lagrangian for the electron–electron–photon interaction is given by:

$$\mathcal{L}_I = -e\bar{\Psi}_F \gamma_\mu \Psi_i A^\mu\,. \tag{3.14}$$

In momentum space the electron–electron–photon vertex is given by:

$$-(2\pi)^4\, e\gamma_\mu \delta^4\,. \tag{3.15}$$

The potential for an external particle of momentum $\boldsymbol{p}$ and helicity ± 2 is given by:

$$A_{\pm}^{\mu_1,\mu_2} = (2\pi)^{-3/2} \int \frac{d^3 p}{\sqrt{2|p|}}\, \epsilon_{\pm}^{\mu}(\boldsymbol{p})\epsilon_{\pm}^{\mu_2}(\boldsymbol{p})[a(\boldsymbol{p}, \pm 2)\, e^{ip\cdot x} + b^*(\boldsymbol{p}, \mp 2)\, e^{-ip\cdot x}]\,. \tag{3.16}$$

The graviton is its own antiparticle like the photon,

$$b(\boldsymbol{p}, \pm 2) = a(\boldsymbol{p}, \pm 2)\,, \qquad b^*(\boldsymbol{p}, \pm 2) = a^*(\boldsymbol{p}, \pm 2)\,. \tag{3.17}$$

In momentum space the wave function of an incoming graviton is:

$$\Psi_{gr}^{in} = \frac{1}{(2\pi)^{3/2}}\, \frac{1}{\sqrt{2|p|}}\, \epsilon_{\pm}^{\mu}(\boldsymbol{p})\,\epsilon_{\pm}^{\nu}(\boldsymbol{p})\,. \tag{3.18}$$

The wave function of an outgoing graviton is:

$$\Psi_{gr}^{out} = \frac{1}{(2\pi)^{3/2}}\, \frac{1}{\sqrt{2|p|}}\, \epsilon_{\pm}^{\mu *}(\boldsymbol{p})\,\epsilon_{\pm}^{\nu *}(\boldsymbol{p})\,. \tag{3.19}$$

The symmetrical tensor $\theta^{\mu\nu}$ is given by:

$$\theta^{\mu\nu} = -\tfrac{1}{2}(\bar{\Psi}\gamma^\mu \partial^\nu \Psi + \bar{\Psi}\gamma^\nu \partial^\mu \Psi)\,. \tag{3.20}$$

The electron–electron–graviton vertex is given by:

$$-\tfrac{1}{2}\kappa(\gamma_\mu p_\nu^{e} + \gamma_\nu p_\mu^{e})\,; \qquad (\kappa = \sqrt{16\pi G})\,. \tag{3.21}$$

The neutrino–neutrino–graviton vertex can be expressed as:

$$-\tfrac{1}{2}\kappa(1 + i\gamma_5)(\gamma_\mu p_\nu + \gamma_\nu p_\mu)\,. \tag{3.22}$$

The scalar–scalar–graviton vertex is given by:

$$-\tfrac{1}{2}\kappa(p_{1\mu}p_{2\nu} + p_{1\nu}p_{2\nu} - m^2\eta_{\mu\nu}) \tag{3.23}$$

where p_1 and p_2 are the initial and final four momenta of the scalar particle and m is the mass of the scalar particle.

The photon–photon–graviton vertex is given by:

$$-\tfrac{1}{2}\{(\epsilon_{1\mu}p_{1\alpha} - \epsilon_{1\alpha}p_{1\mu})(\epsilon_{2\nu}p_2^{\alpha} - \epsilon_2^{\alpha}p_{2\nu}) - \tfrac{1}{4}\delta_{\mu\nu}(\epsilon_{1\alpha}p_{1\beta} - \epsilon_{1\beta}\beta_{1\alpha})(\epsilon_2^{\alpha}p_2^{\beta} - \epsilon_2^{\beta}p_2^{\alpha})\} \tag{3.24}$$

where p_1 and p_2 are the initial and final four momenta and ϵ_1 and ϵ_2 the respective polarizations. It is obtained from the interaction Lagrangian $-\frac{1}{2}\kappa h_{\mu\nu}T^{\mu\nu}$ for a particle and can be used to describe effects like the redshift, the deflection of light by the sun and also for interactions of fermions, scalar particles etc. with gravitons in the weak field approximation. $T^{\mu\nu}$ is the energy momentum tensor of the particle considered. Also the energy momentum tensor for scalar matter is given by the expression:

$$T_{\mu\nu} = \phi_{1\nu}\phi_{1\mu} - \tfrac{1}{2}\eta_{\mu\nu}\phi_{1\sigma}\phi_{1\sigma} - \tfrac{1}{2}m^2\eta_{\mu\nu}\phi^2 .$$

Table 3.1
Feynman rules for gravitons, photons, electrons, neutrinos, and scalar particles

Wave functions	
External particle of momentum $\boldsymbol{p}$	Wave function
An incoming photon	$(2\pi)^{-3/2}\dfrac{1}{(2\lvert\boldsymbol{p}\rvert)^{1/2}}\epsilon_{\pm}^{\mu}(\boldsymbol{p})$
An outgoing photon	$(2\pi)^{-3/2}\dfrac{1}{(2\lvert\boldsymbol{p}\rvert)^{1/2}}\epsilon_{\pm}^{*\mu}(\boldsymbol{p})$
An incoming graviton	$(2\pi)^{-3/2}\dfrac{1}{(2\lvert\boldsymbol{p}\rvert)^{1/2}}\epsilon_{\pm}^{\mu}(\boldsymbol{p})\,\epsilon_{\pm}^{\nu}(\boldsymbol{p})$
An outgoing graviton	$(2\pi)^{-3/2}\dfrac{1}{(2\lvert\boldsymbol{p}\rvert)^{1/2}}\epsilon_{\pm}^{\mu}(\boldsymbol{p})\,\epsilon_{\pm}^{\nu}(\boldsymbol{p})$
An incoming electron	$(2\pi)^{-3/2}\sqrt{\dfrac{m}{E_{\rm i}V}}\,U(\boldsymbol{p}_{\rm i}, S_{\rm i})$
An incoming neutrino	$(2\pi)^{-3/2}\sqrt{\dfrac{1}{V}}\,U(\boldsymbol{p}_{\rm i}, S_{\rm i})$
An outgoing electron	$(2\pi)^{-3/2}\sqrt{\dfrac{m}{E_{\rm f}V}}\,\bar{U}(\boldsymbol{p}_{\rm f}, S_{\rm f})$
An outgoing neutrino	$(2\pi)^{-3/2}\sqrt{\dfrac{1}{V}}\,\bar{U}(p_{\rm f}, S_{\rm f})$
An incoming scalar particle	$(2\pi)^{-3/2}\sqrt{\dfrac{m}{E_{\rm f}V}}\,U(\boldsymbol{p}_{\rm i}, S_{\rm i})$
An outgoing scalar particle	$(2\pi)^{-3/2}\sqrt{\dfrac{m}{E_{\rm f}V}}\,\bar{U}(\boldsymbol{p}_{\rm f}, S_{\rm f})$

Graviton vertices	
Particles	Vertex
Photon–photon–graviton	$-\frac{1}{2}\sqrt{16\pi G}\,\{(\epsilon_{1\mu}p_{1\alpha}-\epsilon_{1\alpha}p_{1\mu}) \times (\epsilon_{2\nu}p_2^{\alpha}-\epsilon_2^{\alpha}p_{2\nu}) -\frac{1}{4}\delta_{\mu\nu}(\epsilon_{1\alpha}p_{1\beta}-\epsilon_{1\beta}p_{1\alpha}) \times (\epsilon_2^{\alpha}p_2^{\beta}-\epsilon_2^{\beta}p_2^{\alpha})\}$
Electron–electron–graviton	$-\frac{1}{2}\sqrt{16\pi G}(\gamma_{\mu}p_{\nu}^{\rm e}+\gamma_{\nu}p_{\mu}^{\rm e})\,\delta^4$
Electron–electron–photon	$-e(2\pi)^4\gamma_{\mu}\delta^4$
Scalar–scalar–graviton	$-\frac{1}{2}(p_{1\mu}p_{2\nu}+p_{1\nu}p_{2\mu}-m^2\eta_{\mu\nu})\sqrt{16\pi G}$
Neutrino–neutrino–graviton	$-\frac{1}{2}\sqrt{16\pi G}(1+{\rm i}\gamma_5)(\gamma_{\mu}p_{\nu}+\gamma_{\nu}p_{\mu})\,\delta^4$

Propagators	
Photon propagator for an internal line carrying momentum $q(\mu \to \nu)$	$-{\rm i}(2\pi)^{-4}\,\eta^{\mu\nu}/(q^2-{\rm i}\epsilon)$
Scalar propagator	$-{\rm i}(2\pi)^{-4}/(q^2-{\rm i}\epsilon)$
Graviton propagator $(\mu\nu \to \lambda\eta)$	$\dfrac{-{\rm i}(2\pi)^{-4}}{2(q^2-{\rm i}\epsilon)}\,[\eta^{\mu\lambda}\eta^{\nu\eta}+\eta^{\mu\eta}\eta^{\nu\lambda}-\eta^{\mu\nu}\eta^{\lambda\eta}]$

Table 3.2.
Non-linear graviton vertices.
Although these vertices are beyond the scope of this review, they are introduced for convenience as they are used in a few instances in the course of the review.

Particles	Vertex
3 gravitons	$-2\sqrt{16\pi G}\,\{p_{1\nu_3} p_{3\nu_2}\eta_{\mu_1\nu_2}\eta_{\nu_1\mu_3} - \frac{1}{2}(p_2\cdot p_3)\eta_{\mu_1\nu_3}\eta_{\nu_1\mu_2}\eta_{\nu_2\mu_3} + \frac{1}{2}p_{3\mu_2}p_{2\nu_3}\eta_{\mu_1\nu_2}\eta_{\nu_1\mu_3}$
	$+\frac{1}{2}(p_2\cdot p_3)\,\eta_{\mu_1\nu_2}\eta_{\nu_1\mu_2}\eta_{\mu_3\nu_3} + \frac{1}{4}p_{1\mu_3}p_{2\nu_3}\eta_{\mu_1\nu_2}\eta_{\nu_1\mu_2} + \frac{1}{2}p_{1\nu_2}p_{1\mu_3}\eta_{\mu_1\nu_1}\eta_{\mu_2\nu_3}$
	$-\frac{1}{4}p_{2\mu_3}p_{3\nu_2}\eta_{\mu_1\nu_1}\eta_{\mu_2\nu_3} + \frac{1}{8}p_2\cdot p_3\,\eta_{\mu_1\nu_1}\eta_{\mu_2\nu_3}\eta_{\nu_2\mu_3} - \frac{1}{8}p_2\cdot p_3\,\eta_{\mu_1\nu_1}\eta_{\mu_2\nu_2}\eta_{\mu_3\nu_3}$
	$-\frac{1}{4}p_{2\nu_3}p_{2\mu_3}\eta_{\mu_1\nu_1}\eta_{\mu_2\nu_2} + \frac{1}{2}p_{2\nu_3}p_{2\mu_3}\eta_{\mu_1\nu_2}\eta_{\mu_2\nu_1}\}$
Two graviton-two scalar	$-\frac{1}{4}\mathrm{i}\kappa^2\,\{(\eta_{\mu\nu}\eta_{\lambda\kappa} - \eta_{\mu\lambda}\eta_{\nu\kappa} - \eta_{\mu\kappa}\eta_{\nu\lambda})p_1\cdot p_2 + \eta_{\nu\lambda}(p_{1\mu}p_{2\kappa} + p_{1\kappa}p_{2\mu}) + \eta_{\mu\kappa}(p_{1\lambda}p_{2\nu} + p_{1\nu}p_{2\lambda})$
	$+\eta_{\mu\lambda}(p_{1\nu}p_{2\kappa} + p_{1\kappa}p_{2\nu}) + \eta_{\nu\kappa}(p_{1\lambda}p_{2\mu} + p_{1\mu}p_{2\lambda})$
	$-\eta_{\mu\nu}(p_{1\lambda}p_{2\kappa} + p_{1\kappa}p_{2\lambda}) - \eta_{\lambda\kappa}(p_{1\mu}p_{2\nu} + p_{1\nu}p_{2\mu})\}$

4. Graviton production from particle–antiparticle annihilation

The production of gravitons from particle-antiparticle annihilation has been considered by Ivanenko and Sokolov [42, 43] and Ivanenko and Brodsky [45]. More in general the possibility of mutual transformations of matter and gravitons has been discussed by Ivanenko [44]. Ivanenko and Sokolov then calculated the cross sections of the transmutations of scalar particles into gravitons. For a scalar field result coincides up to a constant factor with the result obtained by Vladimirov [79] for the non-relativistic approximation to the electron–position annihilation into gravitons. Vladimirov's work is discussed below.

4.1. Electron–position annihilation

Vladimirov [79] has studied the following processes:

a) $e^+ + e^- \to g + g$
b) $e^+ + e^- \to g + \gamma$

which are represented by the diagrams (a–e) and (f–k) of fig. 4.1 where e^+ and e^- are the positron and electron, g is the graviton and γ is the photon. These reactions have a higher probability of occurence. In fact, particularly favourable conditions for these reactions exist in supernovae, where the core can reach 5×10^9 K and pair densities are consequently very high. The interaction Hamiltonian used by Vladimirov in a previous paper [80] makes use of

$$H = \tfrac{1}{4}\mathrm{i}(\sqrt{\kappa^2})\,(h_{\mu\nu} - \tfrac{1}{4}\delta_{\mu\nu}h)\left(\bar{\Psi}\gamma_\mu \frac{\partial\Psi}{\partial x_\nu} - \frac{\partial\bar{\Psi}}{\partial x^\nu}\gamma_\mu\Psi\right)$$

but yields however incomplete results. For this reason Vladimirov has used in his more recent

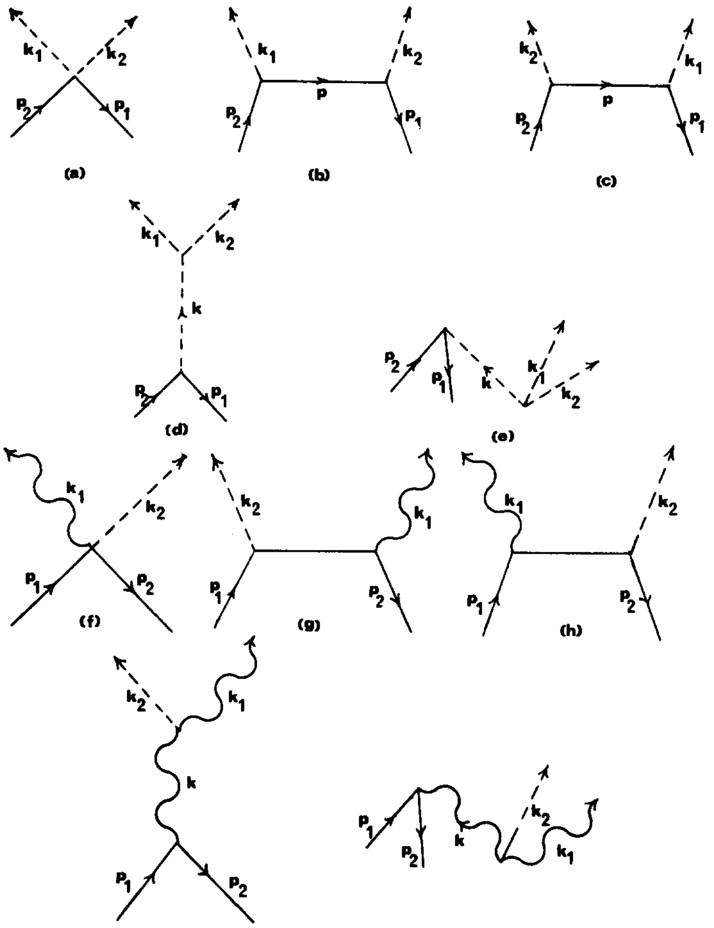

Fig. 4.1. (a)–(e) Broken lines represent gravitons: The processes $e^+ + e^- \to g + g$. (f)–(k) Wave line represents photon: The processes $e^+ + e^- \to g + \gamma$.

paper the tetrad formalism of Fock and Ivanenko. The tetrad formalism is in fact required to incorporate spinors in curved space–time. The influence of a gravitational field on a fermion field is therefore described by means of a tetrad field and all its equations are derivable from a variational principle. If h^μ_a, $h_{a\mu}$ denote the contravariant, covariant components of a tetrad numerated by an index a running from 1 to 4, with h^μ_4 being timelike and the other 3 tetrad vectors spacelike, then

$$h^{a\mu} = \eta^{ab} h^\mu_b , \qquad h^\mu_a = \eta_{ab} h^{b\mu} , \tag{4.1}$$

where $\eta_{ab} = \eta^{ab}$ is the Minkowski metric. The connection between the tetrad field and the metric field $g_{\mu\nu}(x)$ is given by

$$h_{\mu a} h^a_\nu = g_{\mu\nu} \qquad \text{or} \qquad h^\mu_a h^{a\nu} = g^{\mu\nu} . \tag{4.2}$$

Then the field equations can be derived by means of the variational principle $\delta \int (L + L^m)\, d^4x = 0$

where L is the gravitational Lagrangian and L^{m} is the matter Lagrangian which is a function of the tetrad field variables and the matter field variables and their first order derivatives. Also the Lagrangian is invariant under constant Lorentz rotations of the tetrads,

$$h^{\mathrm{a}'\mu}(x) = L_{\mathrm{a}}^{\mathrm{a}'}(x)\, h^{\mathrm{a}\mu}(x) \tag{4.3}$$

where $L_{\mathrm{a}}^{\mathrm{a}'}$ is the Lorentz rotation.

For a complete description of the interaction of the gravitational and spinor fields, it is necessary to take the Lagrangian density of the sum of the spinor field in curved space and the gravitational field:

$$\mathcal{L}_{\mathrm{int}} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_{\mathrm{grav}}\ ; \tag{4.4}$$

$\mathcal{L}_1$ is the first order term in the expansion of the equation:

$$\mathcal{L} = -\tfrac{1}{2}\sqrt{-g}\left[-\mathrm{i}\bar{\Psi}\gamma_{\mathrm{a}}h^{\mathrm{a}\mu}\frac{\partial\Psi}{\partial x^{\mu}} + \mathrm{i}\frac{\partial\bar{\Psi}}{\partial x^{\mu}}\gamma_{\mathrm{a}}h^{\mathrm{a}\mu}\Psi + 2m\bar{\Psi}\Psi + \frac{\mathrm{i}}{4}(\bar{\Psi}\gamma_{\mathrm{c}}\gamma_{\mathrm{a}}\gamma_{\mathrm{b}} - \gamma_{\mathrm{b}}\gamma_{\mathrm{a}}\gamma_{\mathrm{c}})\bar{\Psi}h^{\mathrm{c}\mu}h^{\mathrm{a}\nu}h^{\mathrm{b}\sigma}\Delta_{\mu,\nu\sigma}\right] \tag{4.5}$$

and is of the form

$$\mathcal{L}_1 = \tfrac{1}{4}\sqrt{\kappa^2}(h_{\mu\nu} + \tfrac{1}{2}\eta_{\mu\nu}h)\left(-\mathrm{i}\bar{\Psi}\gamma_{\mu}\frac{\partial\Psi}{\partial x^{\nu}} + \mathrm{i}\frac{\partial\bar{\Psi}}{\partial x_{\nu}}\gamma_{\mu}\Psi\right). \tag{4.6}$$

The density $\mathcal{L}_2$, the expansion of (4.5) to order κ, has the form

$$\mathcal{L}_2 = \frac{\kappa^2}{16}(h_{\mu\mathrm{a}}h_{\nu\mathrm{a}} + \eta_{\mu\nu}h_{\mathrm{ab}}h_{\mathrm{ab}})\left(-\mathrm{i}\bar{\Psi}\gamma_{\mu}\frac{\partial\Psi}{\partial x_{\nu}} + \mathrm{i}\frac{\partial\bar{\Psi}}{\partial x_{\nu}}\gamma_{\mu}\Psi\right) + \frac{\kappa^2}{4}mh_{\mu\nu}h^{\mu\nu}\Psi$$
$$-\mathrm{i}\frac{\kappa^2}{16}\bar{\Psi}\left(\gamma_{\mu}h_{\nu\rho}\frac{\partial h_{\nu\rho}}{\partial x_{\mu}} - \gamma_{\nu}h_{\mu\rho}\frac{\partial h_{\nu\rho}}{\partial x_{\mu}} - \gamma_{\nu}\gamma_{\sigma}\gamma_{\mu}h_{\sigma\rho}\frac{\partial h_{\nu\rho}}{\partial x_{\mu}}\right)\Psi + \kappa f(h). \tag{4.7}$$

The second line of (4.7) results from the introduction of the tetrad. $\mathcal{L}_{\mathrm{grav}}$ is the first order expansion of the Lagrangian density of the gravitational field. Therefore:

$$\mathcal{L}_{\mathrm{grav}} = -\sqrt{\frac{\kappa^2}{4}}\,h_{\mu\nu}\frac{\partial h_{\mathrm{ab}}}{\partial x_{\mu}}\frac{\partial h_{\mathrm{ab}}}{\partial x_{\nu}} + \sqrt{\frac{\kappa^2}{8}}\,h_{\mu\nu}\frac{\partial h}{\partial x_{\mu}}\frac{\partial h}{\partial x_{\nu}} - \sqrt{\frac{\kappa^2}{2}}\,h_{\mu\nu}\frac{\partial h_{\mu\mathrm{a}}}{\partial x_{\mathrm{b}}}\frac{\partial h_{\nu\mathrm{b}}}{\partial x_{\mathrm{a}}}$$
$$-\sqrt{\frac{\kappa^2}{4}}\,h_{\mu\nu}\frac{\partial h}{\partial x_{\mathrm{a}}}\frac{\partial h_{\mu\nu}}{\partial x_{\mathrm{a}}} + \sqrt{\frac{\kappa^2}{2}}\,h_{\mu\nu}\frac{\partial h_{\mu\mathrm{b}}}{\partial x_{\mathrm{a}}}\frac{\partial h_{\nu\mathrm{b}}}{\partial x_{\mathrm{a}}}. \tag{4.8}$$

A lengthy but straighforward calculation yields the differential cross section:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\kappa^4}{128}(4\pi)^2 pk_0\left[p^2m^2\cos^2\theta + p^4\sin^2\theta + 2p^4\sin^2\theta\cos^2\theta + \tfrac{1}{2}(m^2 - p^2\sin^2\theta)^2\right.$$
$$\left. + \frac{3p^2m^4 + m^2p^2(m^2 - p^2\sin^2\theta)}{m^2 + p^2\sin^2\theta} - \frac{p^8\sin^2\theta}{(m^2 + p^2\sin^2\theta)^2}\right] \tag{4.9}$$

where p is the momentum of the fermion and θ the angle between the direction of motion of the initial particles and the momentum of the graviton.

In the classical case, when $k_0^2 - m^2 \gg p^2$, the above formula takes the form:

$$\frac{d\sigma_{cl}}{d\Omega} = \frac{m^2\kappa^4}{64(8\pi)^2}\,\frac{1}{V}\,. \tag{4.10}$$

The radiation is equally probable in all directions.

In the ultrarelativistic case, in which $p^2 \sim k_0^2 \gg m^2$,

$$\frac{d\sigma_{ur}}{d\Omega} = \frac{\kappa^4 k_0^2}{128(8\pi)^2}\,(3\sin^2 2\theta + 2\sin^4\theta)\,. \tag{4.11}$$

From (4.11) one can see that there is no radiation of gravitons in the directions of propagation of the initial particles.

From (4.9) it is also possible to obtain a similar formula for the annihilation of two four component neutrinos ($m = 0$) into two gravitons (Wheeler and Brill [90]).

The total interaction Lagrangian of spinor, gravitational and electromagnetic fields is given by:

$$\begin{aligned}\mathcal{L}_{int} &= e\bar\Psi\gamma_\mu\Psi A^\mu - \tfrac14 e\sqrt{\kappa^2}\,\bar\Psi\gamma_\mu\Psi A^\mu h - \tfrac12 e\sqrt{\kappa^2}\,\bar\Psi\gamma_a\Psi A_\mu h^{a\mu} + \tfrac12\sqrt{\kappa^2} f_{\mu\rho}F^{\mu\nu}h^\rho_\nu \\ &\quad - \tfrac18\sqrt{\kappa^2}F_{\mu\nu}F^{\mu\nu}h + \tfrac12\sqrt{\kappa^2}\,m\bar\Psi\Psi h + \tfrac14\sqrt{\kappa^2}(h_{\mu\nu} + \tfrac12 g_{\mu\nu}h)\left(-i\bar\Psi\gamma^\mu\frac{\partial\bar\Psi}{\partial x^\nu}\gamma^\mu\Psi\right) + \sqrt{\kappa^2}f(3h) + O(\kappa^2)\,.\end{aligned} \tag{4.12}$$

$\sqrt{\kappa^2}f(3h)$ are terms containing the product of three graviton functions. A transformation of the pair into a photon and a graviton is possible for identical polarizations and the processes are represented by the diagrams f–k of fig. 4.1. If the calculation is in the center of mass system, the contributions of the diagrams f, i and k mutually cancel.

The differential cross section is given by

$$d\sigma = \frac{e^2\kappa^2}{4(8\pi)^2}\,\frac{p^3\sin^2\theta}{k_0(k_0^2 - p^2\cos^2\theta)}\left[1 + \sin^2\theta + \frac{\sin^2\theta(k_0^2 - 2p^2\sin^2\theta)}{k_0^2 - p^2\cos^2\theta}\right]d\Omega\,. \tag{4.13}$$

In the classical case $p^2 \ll k_0^2 = m^2$

$$\frac{d\sigma_{cl}}{d\Omega} = \frac{e^2\kappa^2}{4(8\pi)^2}\left(\frac{V}{1}\right)^3\sin^2\theta\,(1 + 2\sin^2\theta)\,. \tag{4.14}$$

Because of the presence of the factor $\sin^2\theta$ in (4.14) the production of gravitons and photons has a maximum at $\theta = \frac12\pi$. There is no radiation in the directions of the momenta of the initial particles. In the ultra relativistic case $p^2 \sim k_0^2 \gg m^2$ the cross section becomes

$$\frac{d\sigma_{ur}}{d\Omega} = \frac{e^2\kappa^2}{4(8\pi)^2}\,(1 + \cos^2\theta) \tag{4.15}$$

and no longer depends on the energy of the particles. The radiation of gravitons becomes strongest in the directions of the momenta of the original particles.

A comparison of the total cross sections for processes a) (non linear) and b) can be made in the classical as well as the ultra relativistic cases. In the non relativistic approximation the total cross section of two graviton annihilation has the form

$$\sigma_{cl} = \frac{r_g^2}{2048\pi V}\,, \qquad r_g = \kappa^2 m\,. \tag{4.16}$$

The total cross section for photon–graviton annihilation is

$$\sigma'_{\rm cl} = \frac{13 r_{\rm e} r_{\rm g}}{480\pi} \left(\frac{V}{1}\right)^3 , \qquad r_{\rm e} = \frac{e^2}{m} . \tag{4.17}$$

In the ultra relativistic approximation the total cross section for two graviton annihilation is

$$\sigma_{\rm ur} = \frac{\kappa^4 k_0^2}{768\pi} = \frac{r_{\rm g}^2}{768\pi} \frac{k_0^2}{m^2} \tag{4.18}$$

and increases with k_0^2. For $k_{0\,\rm cr} \sim \sqrt{r_{\rm e}/r_{\rm g}}\, mc^2$, the cross section of the two graviton annihilation becomes of the order of the cross section of two photon annihilation. For electrons, $k_{0\,\rm cr} = 10^{21}\, m$, the total cross section of the photon–graviton annihilation is

$$\sigma'_{\rm ur} = r_{\rm e} r_{\rm g}/48\pi = 1.26 \times 10^{-68}\, cm^2 \tag{4.19}$$

and does not depend on the energy or the mass of the colliding particles. At the same energies the cross section of two photon annihilation becomes of the order of the cross section of photon–graviton annihilation,

$$k'_{0\,\rm cr} \sim k_{0\,\rm cr} \sim \sqrt{r_{\rm e}/r_{\rm g}} . \tag{4.20}$$

In the ultra relativistic case, the cross section, unlike the electrodynamical case increases with energy and for $k_0 \sim 10^{21}\, m$ certainly exceeds the limits of the field approximation and becomes comparable with the cross section for the annihilation into photons.

In spite of the extremely small values obtained for the gravitational transmutations, such processes may be of importance on a cosmological scale as stressed by Wheeler [89].

5. Gravitational bremsstrahlung

Bremsstrahlung is a well defined process only within certain limits; the simultaneous emission of very soft gravitons for instance, too soft to be observable within the accuracy of the energy determination of the incident and outgoing particle, can never be excluded. Such radiation is always present, even in the usual elastic scattering. Since it is impossible to make a clean physical distinction between bremsstrahlung and radiationless scattering when the emitted graviton is extremely soft, the considerations are restricted to the emission of not too soft gravitons. The external field could be a Coulomb field. The contributions of Halpern and Laurent [36], Weinberg [88], Carmeli [15], Barker et al. [3] and Boccaletti [12] are considered in detail. The contribution to graviton emission from bremsstrahlung processes is certainly not insignificant as shown later in the calculations.

5.1. Weinberg's formula

Weinberg has obtained, as a consequence of a treatment in which the infrared divergence of quantum gravidynamics are removed with the same methods used in quantum electrodynamics a formula which gives the emission rate and spectrum of soft gravitons in an arbitrary collision process. The derived formula for gravitational bremsstrahlung is used to estimate the gravitational radiation emitted during thermal collisions in the sun which, within the solar system, is found to be a stronger source of gravitational radiation than classical sources such as planetary motion.

A short account of the treatment of the infrared divergences arising from very soft real and virtual photons and gravitons is given. In a combined theory of electromagnetism and gravitation, the infrared photons and gravitons simply supply independent correction factors to transition rates.

The attachment of a soft photon line with momentum q to an outgoing charged particle line in a Feynman diagram needs the introduction of an extra charged particle propagator with momentum $q + p$ and one extra vertex for the transition $p + q \to p$. If the soft photon line is attached to an incoming charged particle line, the extra propagator is for momentum $p - q$ and the transition is $p \to p - q$. The extra factor introduced in the matrix element in the limit $q \to 0$ is:

$$\sum_n e_n \eta_n p_n^\mu / (p_n \cdot q - \mathrm{i}\eta_n \epsilon)\,; \qquad \eta = \pm 1 \tag{5.1}$$

where the sum runs over all external lines in the original diagram. If a soft graviton line of momentum q is attached, in the limit $q \to 0$ the extra factor is:

$$\sqrt{16\pi G}\ \sum_n \eta_n p_n^\mu p_n^\nu / (p_n \cdot q - \mathrm{i}\eta_n \epsilon) \tag{5.2}$$

and the sum again runs over all external lines in the original diagram. The effect of attaching N soft graviton lines to an arbitrary Feynman diagram is just to multiply the matrix element by N factors of eq. (5.2). Because of this factorization it is possible to perform a sum over an unlimited number of very complicated Feynman diagrams.

The virtual infrared divergences are now discussed and the equations derived above are used to obtain expressions for the infrared virtual graviton corrections. It is shown further that the divergences cancel. In order to obtain these results, it is perhaps easier to deal before with soft photons and derive at the end the corresponding results for gravitons. The effect of adding N virtual infrared photon lines to a diagram that does not already involve any infrared lines is to multiply the matrix by N pairs of factors:

$$\sum_n e_n \eta_n p_n^\mu / (p_n \cdot q - \mathrm{i}\eta_n \epsilon)\,.$$

Each pair is connected by a photon propagator $\{-\mathrm{i}/(2\pi)^4\}\ \eta_{\mu\nu}/(q^2 - \mathrm{i}\epsilon)$ while a summation over the polarization indices and an integration over q are carried out. In addition one must divide by $2^N N!$ because the external line poles factor only if we sum over all places to which the two ends of each virtual infrared photon line are attached and this includes spurious sums over the N permutations of the lines and over the two directions each line might be thought to flow. The result is then:

$$\frac{1}{N!}\left[\frac{1}{2}\int_\lambda^\Lambda \mathrm{d}^4 q\, A(q)\right]^N \tag{5.3}$$

$$A(q) = \frac{-\mathrm{i}}{(2\pi)^4 (q^2 - \mathrm{i}\epsilon)} \sum_{nm} \frac{e_n e_m \eta_n \eta_m p_n \cdot p_m}{(p_n \cdot q - \mathrm{i}\eta_n \epsilon)(-p_m \cdot q - \mathrm{i}\eta_m \epsilon)}\,. \tag{5.4}$$

Λ is a cut off energy, a cut off $|q| \gg \lambda$ is imposed to display the logarithmic divergences as powers of $\ln \lambda$, where $\lambda \ll \Lambda$. This cut off only affects the infrared lines because it is only these that give infrared divergences for $\lambda = 0$.

The negative sign of $p_{n'} q$ in the second denominator means that if q is the momentum emitted by line n, q must be absorbed by line m. Summation over N leads to the conclusion that S matrix for an arbitrary process may be expressed as:

$$S_{\beta\alpha} = S^0_{\beta\alpha} \exp\left(\tfrac{1}{2} \int_\lambda^\Lambda \mathrm{d}^4 q\, A(q)\right) . \tag{5.5}$$

$S^0_{\beta\alpha}$ is the S matrix without virtual photons of infrared type. The rate for $\alpha \to \beta$ is given by:

$$\Gamma_{\beta\alpha} = \Gamma^0_{\beta\alpha} \exp\left(\mathrm{Re} \int_\lambda^\Lambda \mathrm{d}^4 q\, A(q)\right) \tag{5.6}$$

$$\mathrm{Re} \int_\lambda^\Lambda \mathrm{d}^4 q A(q) = \left[-\frac{1}{2 \cdot (2\pi)^3} \int_\lambda^\Lambda \mathrm{d}^4 q\, \delta(q^2)\right] \cdot \sum_{nm} \frac{e_n e_m \eta_n \eta_m p_n \cdot p_m}{(p_n \cdot q)(p_m \cdot q)} = -A \ln \frac{\Lambda}{\lambda} \tag{5.7}$$

where A is the positive dimensionless constant:

$$A = \int \mathrm{d}^2\Omega\, A(\hat{q})$$

$$A(\hat{q}) = \frac{1}{2(2\pi)^3} \sum_{nm} \frac{e_n e_m \eta_n \eta_m (p_n \cdot p_m)}{(E_n - p_n \cdot \hat{q})(E_m - p_m \cdot \hat{q})} . \tag{5.8}$$

Integration gives:

$$A = -\frac{1}{8\pi^2} \sum_{nm} \eta_n \eta_m e_n e_m \frac{1}{\beta_{nmt}} \ln \frac{1 + \beta_{nm}}{1 - \beta_{nm}} . \tag{5.9}$$

β_{nm} is the relative velocity of particles n and m in the rest frame of either

$$\beta_{nm} = (1 - m_n^2 m_m^2 / (p_n - p_m)^2)^{1/2} . \tag{5.10}$$

Therefore:

$$\Gamma_{\beta\alpha} = \Gamma^0_{\beta\alpha} (\lambda/\Lambda)^A . \tag{5.11}$$

The same procedure applies to gravitons, with the substitution $A \to B$ where

$$B = \frac{G}{2\pi} \sum_{nm} \eta_n \eta_m m_n m_m \frac{1 + \beta_{nm}^2}{\beta_{nm} (1 - \beta_{nm}^2)^{1/2}} \ln \frac{1 + \beta_{nm}}{1 - \beta_{nm}} . \tag{5.12}$$

Hence:

$$B(\hat{q}) = \frac{16\pi G}{2(2\pi)^3} \sum_{nm} \eta_n \eta_m \frac{\{(p_n \cdot p_m)^2 - \tfrac{1}{2} m_n^2 m_m^2\}}{(E_n - p_n \cdot \hat{q})(E_m - p_m \cdot \hat{q})} \tag{5.13}$$

and one obtains

$$\Gamma_{\beta\alpha} = \Gamma^0_{\beta\alpha}(\lambda/\Lambda)^B \ . \tag{5.14}$$

Since $B > 0$, the above equation shows that all processes have zero rate in the limit $\lambda \to 0$ just as all charged particles have zero rate for $\lambda \to 0$ in electrodynamics. The paradox is resolved by taking into account the infrared divergences due to emission of real soft gravitons.

The S matrix elements for emitting N real soft gravitons in a process $\alpha \to \beta$ is obtained by multiplying the S matrix for $\alpha \to \beta$ by N factors of the type (5.2) and then contracting each of these factors with the appropriate graviton "wave function":

$$(2\pi)^{-3/2}(2|q|)^{-1/2}\ \epsilon^*_\mu(q, \pm 1)\,\epsilon^*_\nu(q, \pm 1)$$

where q is the graviton momentum, $h = \pm 2$ is its helicity, ϵ_μ is the polarization vector. Therefore the graviton emission matrix element is:

$$S^{\rm gr}_{\beta\alpha}(1, 2, \ldots N) = S_{\beta\alpha} \prod_{r=1}^{N} (2\pi)^{-3/2} \frac{1}{2|q_r|^{1/2}} \sum_n \eta_n \frac{(p_n \cdot \epsilon^*(q_r, \frac{1}{2}h_r))^2}{p_n \cdot q_r} \ . \tag{5.15}$$

By squaring eq. (5.15), summing over helicities and dividing by $N!$ because gravitons are bosons one obtains the rate for emission of N soft gravitons with momenta near $q_1, q_2, q_3, \ldots q_N$

$$\Gamma^{\rm gr}_{\beta\alpha}(q_1, q_2, \ldots q_N)\,{\rm d}^3 q_1 \ldots {\rm d}^3 q_N = \frac{1}{N!}\,\Gamma_{\beta\alpha} \prod_{r=1}^{N} B(q_r)\,{\rm d}^3 q_r \ , \tag{5.16}$$

where $\Gamma_{\beta\alpha} = |S_{\beta\alpha}|^2$,

$$B(q) = (2\pi)^{-3/2} \frac{1}{2|q|}\ 16\pi G \sum_{nm} \eta_n \eta_m p^\mu_n p^\nu_n p^\rho_m\, p^\sigma_m \Pi_{\mu\nu\rho\sigma}(q) \tag{5.17}$$

and

$$\Pi_{\mu\nu\rho\sigma}(q) = \sum_{\pm} \epsilon_\mu(q, \pm)\,\epsilon_\nu(q, \pm)\,\epsilon^*_\rho(q, \pm)\,\epsilon^*_\sigma(q, \pm 1) \ . \tag{5.18}$$

Also

$$\Pi_{\mu\nu\rho\sigma}(q) = \tfrac{1}{2}\{\Pi_{\mu\rho}(q)\Pi_{\nu\sigma}(q) + \Pi_{\mu\sigma}(q)\Pi_{\nu\rho}(q) - \Pi_{\mu\nu}(q)\Pi_{\rho\sigma}(q)\} \tag{5.19}$$

where $\Pi_{\mu\nu}(q) = \eta_{\mu\nu}$.

Therefore

$$B(q) = (2\pi)^{-3/2} \frac{16\pi G}{2|q|} \sum_{nm} \eta_n \eta_m \frac{\{(p_n \cdot p_m)^2 - \frac{1}{2}m^2_n m^2_m\}}{(p_n \cdot q)(p_m \cdot q)} \tag{5.20}$$

$$B(q) = B(\hat{q})/|q|^3 \ . \tag{5.21}$$

The rates for emission of N gravitons with energies near $\omega_1, \ldots \omega_N$ is given by integration of (5.16) over solid angles,

$$\Gamma^{gr}_{\beta\alpha}(w_1 \ldots w_N)\,dw_1 \ldots dw_N = \frac{B^N}{N!}\Gamma_{\beta\alpha}\frac{dw_1}{w_1}\frac{dw_2}{w_2}\ldots\frac{dw_N}{w_N}\,. \tag{5.22}$$

The preceding equation shows that the graviton emission rate contains logarithmic infrared divergences on integration. Use of the well known representation of the step function gives:

$$\Gamma^{gr}_{\beta\alpha}(\leqslant E) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin E\sigma}{\sigma}\exp\left\{B\int_{\lambda}^{E}\frac{dw}{w}e^{iw\sigma}\right\}d\sigma \tag{5.23}$$

and for $\lambda \to 0$

$$\Gamma^{gr}_{\beta\alpha}(\leqslant E) = (E/\lambda)^B\, b(B)\Gamma_{\beta\alpha} \tag{5.24}$$

where

$$b(B) = \frac{1}{\pi}\int_{-\infty}^{\infty}d\sigma\frac{\sin\sigma}{\sigma}\exp\left\{x\int_{0}^{1}\frac{dw}{w}(e^{iw\sigma}-1)\right\}. \tag{5.25}$$

By inserting (5.14), which displays the virtual infrared divergences, into (5.24)

$$\Gamma^{gr}_{\beta\alpha}(\leqslant E) = (E/\Lambda)^B\; b(B)\Gamma^0_{\beta\alpha}\,. \tag{5.26}$$

It is seen that all dependence on the infrared cut off λ disappears. $\Gamma^0_{\beta\alpha}$ is the rate for the process $\alpha \to \beta$ without soft graviton emission, and without inclusion of virtual infrared gravitons. To lowest order in G the above equation gives the power spectrum of soft gravitons accompanying a reaction $\alpha \to \beta$ as:

$$E\,d\Gamma_{\beta\alpha}(\leqslant E) = B\,\Gamma^0_{\beta\alpha}\,dE\,. \tag{5.27}$$

The rate of emission of energy in soft gravitational radiation during collisions is:

$$P(\leqslant\Lambda) = \int_{0}^{\Lambda}E\,d\Gamma(\leqslant E)\,. \tag{5.28}$$

"Soft" here means that the emitted energy E is $<\Lambda$. By using (5.26) one obtains:

$$P(\leqslant\Lambda) = \frac{B}{1+B}\,b(B)\,\Lambda\Gamma_0 \approx B\Lambda\Gamma_0\,. \tag{5.29}$$

If the particles involved in the collision are non relativistic then (5.10) may be expanded in terms of V_n and V_m with $V = p/E$

$$\beta^2_{nm} = V^2_n + V^2_m - 2V_n\cdot V_m - V^2_n V^2_m - 3(V_n\cdot V_m)^2 + 2(V^2_n + V^2_m)(V_n\cdot V_m)\,. \tag{5.30}$$

Use of (5.30) in (5.12) gives:

$$\frac{1+\underline{\underline{\beta^2_{nm}}}}{2\beta^2_{n'm}(1-\beta^2)^{1/2}}\ln\left(\frac{1+\beta_{nm}}{1-\beta_{nm}}\right)^{1/2} = 1 + \tfrac{11}{6}(V_n - V_m)^2 + \tfrac{63}{40}(V^2_n + V^2_m)^2 + \ldots\,. \tag{5.31}$$

Because of energy momentum conservation the following equations hold

$$\sum_n \eta_n m_n (1 + \tfrac{1}{2} V_n^2 + \tfrac{3}{8} V_n^4 + \ldots) = 0\,, \qquad \sum_n \eta_n m_n V_n (1 + \tfrac{1}{2} V_n^2 + \ldots) = 0\,. \tag{5.32}$$

Use of (5.31) to (5.32) in (5.13) gives in nonvanishing order in V

$$B = \frac{G}{\pi} (\tfrac{16}{5}\, Q_{ij} Q_{ij} + \tfrac{94}{15}\, Q_{ii}^2) \tag{5.33}$$

$$Q_{ij} = \tfrac{1}{2} \sum_n \eta_n m_n V_{ni} V_{nj}\,. \tag{5.34}$$

In non relativistic elastic two body scattering

$$Q_{ij} Q_{ij} = \tfrac{1}{2} \mu^2 V^4 \sin^2\theta_c\,, \qquad Q_{ii} = 0 \tag{5.35}$$

where μ is the reduced mass, $V = |V_1 - V_2|$ is the relative velocity and θ_c is the scattering angle in the center of mass system. Therefore:

$$B = \frac{8G}{5\pi} \mu^2 V^4 \sin^2\theta_c\,. \tag{5.36}$$

The rate for such collisions per cm^3/sec is $V n_1 n_2\, d\sigma/d\Omega$ where n_1 and n_2 are the number densities of particles 1 and 2. Hence the total power emitted in soft gravitational radiation attributable to 1–2 collisions is:

$$P(\leqslant \Lambda) = \frac{8G}{5\pi} \mu^2 V^5 n_1 n_2 V_s \Lambda \int \frac{d\sigma}{d\Omega} \sin^2\theta_c\, d\Omega\,; \tag{5.37}$$

V_s is the volume of the source. "Soft" radiation can be defined by taking the cutoff $\Lambda \approx \frac{1}{4}\mu V^2$. This formula can be used to estimate the thermal gravitational radiation from the sun. The most frequent collisions are the Coulomb collisions between electrons and protons or electrons. Therefore:

$$\mu = m_e\,, \qquad V = \sqrt{\frac{3kT}{m_e}}\,, \qquad n_1 = n_e\,, \qquad n_2 = n_e + n_p = 2n_e\,, \tag{5.38}$$

$$\int \frac{d\sigma}{d\Omega} \sin^2\theta_c\, d\Omega = \frac{8\pi e^2}{(3kT)^2} \ln \Lambda_D\,. \tag{5.39}$$

Λ_D is the ratio of the Debye shielding radius to the average impact parameter. The solar gravitational radiation is given by:

$$P_\odot = \frac{32G}{5} (3kT)^{3/2} \frac{1}{\sqrt{m_e}} n_e^2 V_\odot \frac{e^4}{\hbar c^5} \ln \Lambda_D\,. \tag{5.40}$$

In the sun's core $T_c \approx 10^7$ K, $n_e = 3 \times 10^{25}\ cm^{-3}$, $V_\odot = 2 \times 10^{31}\ cm^{-3}$

$$\ln \Lambda_D = 4\,. \tag{5.41}$$

Therefore the power is calculated to be

$$P_{\odot} = 6 \times 10^{14}\ \text{erg/sec}\,. \tag{5.42}$$

As a comparison, classical quadrupole radiation for the Jupiter–sun system yields only 7.6×10^{11} erg/sec at extremely low frequencies.

The thermal gravitational radiation from the sun appears to be the dominant source of gravitational radiation in the solar system. A binary star like Sirius A and B radiates more classically, the power is 8×10^{14} erg/sec, and also more thermally. Thermal collisions may possibly provide the most important source of gravitational radiation in the universe.

The power spectrum formula of eq. (5.17) is both classical and quantum mechanical. Also, since everything in the universe is almost transparent to gravitons, eq. (5.37) may be used directly to compute the thermal gravitational radiation from any hot body.

5.2. Gravitational bremsstrahlung from the sun

Carmeli [15] has calculated the solar gravitational radiation power by determining the spectral resolution of low frequency gravitational radiation by a system of colliding particles. The spectral resolution is obtained by application of Fourier analysis to the Landau and Lifshitz's formula [51] for the gravitational radiation intensity. In general, in the spectral distribution of the radiation accompanying a collision, the main part of the intensity is contained in the frequencies $\omega \sim 1/\tau$, where τ is the order of magnitude of the duration of the collision. For this interval of frequencies, however, one cannot obtain a general formula for the distribution. The "tail" of the distribution at low frequencies, satisfying the condition $\omega\tau \ll 1$, is, however, easier to handle. This is in fact the case discussed by Weinberg, as seen in section 5.1. This part of the calculation by Carmeli is thus a classical version of Weinberg's treatment and the result for the spectral power of the gravitational radiation is in accord with the result of Weinberg. By using Carmeli's method, however, one can also estimate the total gravitational radiation of the scattering process for all possible frequencies, and without resolution into Fourier integrals. In fact, if ΔE denotes the gravitational radiation accompanying a collision of two charged particles, then ΔE is obtained from the radiation formula by integration over the time interval $(-\infty, \infty)$ of the collision at all possible frequencies

$$\Delta E = \int_{-\infty}^{\infty} I\, \mathrm{d}t = \int_{-\infty}^{\infty} \frac{G}{45}\left(\frac{\mathrm{d}^3 D_{kl}}{\mathrm{d}t^3}\right)^2 \tag{5.43}$$

where D_{ik} is the mass quadrupole moment

$$D_{ik} = \int \rho(3x^i x^k - \delta^{ik} x_s x_s)\, \mathrm{d}^3 x\,. \tag{5.44}$$

The rate for such collisions/cm^3 sec is $v_1 n_1 n_2\, \mathrm{d}\sigma/\mathrm{d}\Omega$. Hence the total power radiated for any two particles 1 and 2 is

$$P = \int V_s \Delta E v n_1 n_2 \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\, \mathrm{d}\Omega$$

or (Spitzer [73])

$$P = V_s v n_1 n_2 \int_0^\infty \Delta E \; 2\pi\rho \, d\rho \, . \tag{5.45}$$

The integral in (5.45) is the effective radiation and its value for quadrupole gravitational radiation can be calculated by following the method of Landau and Lifshitz [51]. The result is:

$$\int_0^\infty d\rho \int_{-\infty}^\infty dt \, I \, 2\pi\rho = \frac{32\pi G}{9} \mu e^2 V^3 \, . \tag{5.46}$$

The total power is therefore:

$$P = \frac{32\pi G}{9} \mu^1 e^2 v^4 n_1 n_2 V_s \, . \tag{5.47}$$

Use of the equations (5.38) due to non relativistic collisions in the sun gives the total radiation as

$$P = \frac{64\pi G}{9c^5} \frac{1}{m} e^2 n_e^2 (3kT)^2 V_s \, . \tag{5.48}$$

Using the same data taken by Weinberg for the sun's core:

$$T \approx 10^7 \text{ K}, \qquad n_e \approx 3 \times 10^{25} \text{ cm}^{-3} \, ; \qquad V_s \approx 2 \times 10^{31} \text{ cm}^3 \, . \tag{5.49}$$

The solar gravitational radiation power is found to be

$$P \approx 5 \times 10^{15} \text{ erg/sec.} \tag{5.50}$$

This value is about 10 times larger than the one obtained from eq. (5.37). It also does not require any cut off as the integral (5.46) unlike (5.37) converges.

5.3. Bremsstrahlung in neutron stars

Boccaletti [12] has applied Weinberg's results to study the emission of high frequency gravitational radiation from neutron stars.

The data used now are

$V_s \sim 10^{19}$ cm^3 for the volume of the neutrino star ,

$\rho \sim 10^{14}$ g/cm^3 for the mass density ,

n = particle number = 0.6×10^{38} cm^{-3} , (5.51)

$\rho_F \sim 1.3 \times 10^{-4}$ cm^{-1} ,

$E_F = P_F^2/2m \sim 5 \times 10^{-5}$ erg ~ 30 MeV ; $\qquad \Lambda \sim \frac{1}{2} E_F$;

$V = 0.75 \times 10^{10}$ cm/sec is the center of mass velocity, m is the neutron mass and n_{eff} is the effective reduced particle number due to the degenerate gas while n is the original particle density

$$n_{\text{eff}} = n\, kT/E_{\text{F}} \,. \tag{5.52}$$

The phenomenological cross section σ is

$$\sigma \sim 10^{-25}\ \text{cm}^2 \,. \tag{5.53}$$

Therefore the power is calculated to be

$$P(\leqslant \Lambda) = 1.6 \times 10^{26}\ \text{erg/sec} \,. \tag{5.54}$$

This value of P is calculated for the tail of the spectrum and the average graviton frequency is

$$\nu \approx 0.4 \times 10^{22}\ \text{Hz} \,. \tag{5.55}$$

The total gravitational radiation of a neutron star, including all the frequencies of the spectrum and without any cut off is:

$$P \sim 10^{27}\ \text{erg/sec} \tag{5.56}$$

with the same ratio of total to tail power found by Carmeli for Coulomb scattering in the sun.

5.4. Graviton bremsstrahlung with gravitational scattering

The formulation of Gupta [33] can be used to treat graviton bremsstrahlung in a manner similar to photon bremsstrahlung. A discussion of the problem of infrared divergences in processes involving the emission of photons can be found for instance in Jauch and Rohrlich [47] or Yennie et al. [92]. From Weinberg's formulation [88] it can be accepted that the infrared divergence in graviton bremsstrahlung is cancelled by the diagram that contributes in radiationless scattering, though Weinberg has not discussed the self energy diagrams in radiationless scattering which are necessary to cancel the infrared divergent renormalization constant in the vertex diagram. The problem, though quite similar to that in quantum electrodynamics, is more cumbersome to resolve in the present case.

The graviton bremsstrahlung cross section for a spinless particle is derived by Barker et al. [3] by including linear as well as non linear gravitational interactions. They also show that the total probability for the emission of non physical gravitons in this process vanishes. The complete elimination of the infrared divergence in graviton bremsstrahlung is carried out by using the vertex and self energy diagrams suggested by the work of Dyson [25] and Ward [81].

The collision between two spinless particles of masses m and M is considered with the restrictions $M \gg m$ and $\frac{1}{2} M V_M^2 \ll M$. These conditions allow full consideration of the gravitational field which is not done if the scatterer is represented by means of an external field.

The initial and final propagation four vectors of the heavy particle are q and q', and that of the particle of mass m are p and p'. The propagation vector of the emitted graviton is k. The scattering operator for the diagrams of fig. 5.1 is given by:

$$S = -\mathrm{i}(2\pi)^4\, \delta(p - p' - k + q - q')\, \delta(p_0 - p_0' - k_0) \frac{\kappa^3 M}{4V^{5/2}} \frac{1}{\sqrt{2p_0 p_0' k_0}\,(k + p' - p)^2}$$
$$\times\, I_{ij}\, a_{ij}^*(k) a^*(q') a(q)\, a^*(p') a(p) \tag{5.57}$$

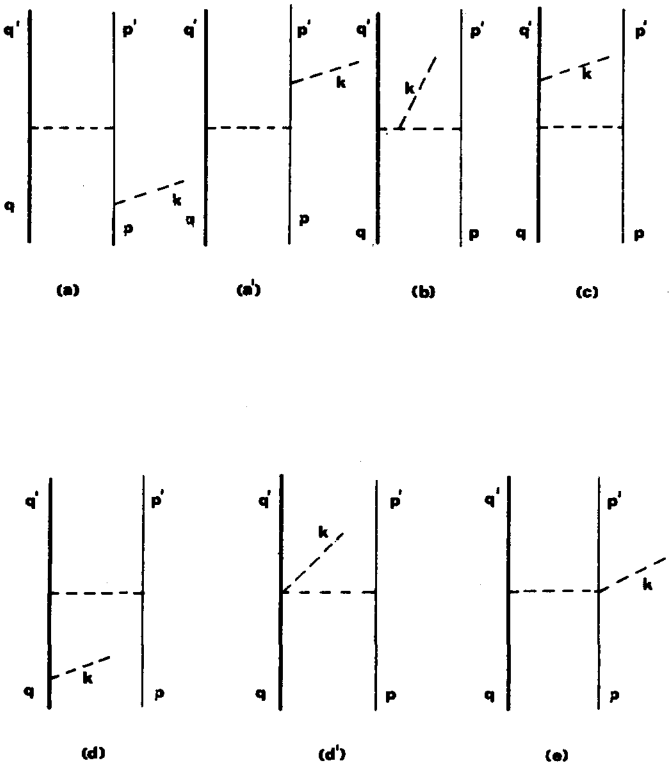

Fig. 5.1. The lowest order diagrams for graviton bremsstrahlung with gravitational scattering.

where

$$I_{ij} = A_{ij} + B_{ij} + C_{ij} \tag{5.58}$$

$$A_{ij} = (\tfrac{1}{2}\mu^2 - p_0'^2)\frac{p_i p_j}{p \cdot k} - (\tfrac{1}{2}\mu_0^2 - p_0^2)\frac{p_i' p_j'}{p' \cdot k}$$

$$B_{ij} = \frac{2(p_0^2 - \frac{1}{2}\mu^2)\, p_i' p_j' + 2(p_0'^2 - \frac{1}{2}\mu^2)\, p_i p_j - p_i' p_j [(4p_0' p_0 - \frac{1}{2}\mu^2) + (\boldsymbol{k} + \boldsymbol{p}' - \boldsymbol{p})^2]}{(p' - p)^2 - k_0^2} \tag{5.59}$$

$$C_{ij} = \frac{(\boldsymbol{k} + \boldsymbol{p}' - \boldsymbol{p})^2 p_i' p_j}{(\boldsymbol{p}' - \boldsymbol{p})^2 - k_6^2}, \qquad k^2 = 0\,, \qquad p^2 = p'^2 = -\mu^2\,. \tag{5.60}$$

A_{ij} represents the contribution of diagrams (a) and (a'), while B_{ij} and C_{ij} represent the contributions of diagrams (b) and (c). The remaining diagrams give vanishing contributions in view of the large mass of the scatterer.

After summation over the two polarization states of the graviton, the graviton bremsstrahlung cross section becomes

$$\mathrm{d}\sigma = \frac{\kappa^6 M^2}{32(2\pi)^4} \frac{\mathrm{d}k_0}{k_0} \frac{|p'|}{|p|} \frac{\sin\theta \sin\theta' \,\mathrm{d}\theta\, \mathrm{d}\theta'\, \mathrm{d}\phi'}{(\boldsymbol{k}+\boldsymbol{p}'-\boldsymbol{p})^4((\boldsymbol{p}'-\boldsymbol{p})^2 - k_0^2)^2} [J^2 p^4 \sin^4\theta + K^2 p'^4 \sin^4\theta'$$
$$+ L^2 p'^2 p^2 \sin^2\theta \sin^2\theta' + 2JKp'^2 p^2 \sin^2\theta \sin^2\theta' \cos 2\phi'$$
$$+ 2JL|p|^3|p'| \sin^3\theta \sin\theta' \cos\phi' + 2KL|p'|^3|p| \sin^3\theta' \sin\theta \cos\phi' \tag{5.61}$$

with

$$J = (\tfrac{1}{2}\mu^2 - p_0'^2)\, \frac{((\boldsymbol{p}'-\boldsymbol{p})^2 - k_0^2 - 2k_0(|p|\cos\theta - p_0))}{|p|\cos\theta - p_0}$$
$$K = (p_0^2 - \tfrac{1}{2}\mu^2)\, \frac{((\boldsymbol{p}'-\boldsymbol{p})^2 - k_0^2 + 2k_0(|p'|\cos\theta' - p_0'))}{|p'|\cos\theta' - p_0'} \tag{5.62}$$
$$L = 4(\tfrac{1}{2}\mu^2 - p_0' p_0)k_0 \,.$$

The angle between $\boldsymbol{p}$ and $\boldsymbol{k}$ is θ, the polar angles of p' are θ' and ϕ'. In the non relativistic approximation, (5.61) can be easily integrated over the angles. When $p^2 \ll \mu^2$

$$k_0 = p_0 - p_0' \approx (\boldsymbol{p}^2 - \boldsymbol{p}'^2)/2\mu$$

so that

$$J = \tfrac{1}{2}\mu[\boldsymbol{p}^2 - \boldsymbol{p}'^2 + (\boldsymbol{p}'-\boldsymbol{p})^2]\,; \qquad L = -\mu(\boldsymbol{p}^2 - \boldsymbol{p}'^2)\,; \qquad K = \tfrac{1}{2}\mu[\boldsymbol{p}^2 - \boldsymbol{p}'^2 - (\boldsymbol{p}'-\boldsymbol{p})^2]\,;$$
$$(\boldsymbol{k}+\boldsymbol{p}'-\boldsymbol{p})^2 = (\boldsymbol{p}'-\boldsymbol{p})^2\,; \qquad (\boldsymbol{p}'-\boldsymbol{p})^2 - k_0^2 = (\boldsymbol{p}'-\boldsymbol{p})^2\,. \tag{5.63}$$

Integration over the angles ϕ', θ' and θ gives (in C.G.S. units)

$$\mathrm{d}\sigma = \frac{\kappa^6 M^2 \mu^2 c^5 \hbar}{15(4\pi)^3} \frac{\mathrm{d}k_0}{k_0} \left(5\frac{|\boldsymbol{p}'|}{|\boldsymbol{p}|} + 3\frac{(\boldsymbol{p}^2 + \boldsymbol{p}'^2)}{2|\boldsymbol{p}|^2} \ln \frac{|\boldsymbol{p}| + |\boldsymbol{p}'|}{|\boldsymbol{p}| - |\boldsymbol{p}'|}\right)$$

or

$$\mathrm{d}\sigma = \frac{64}{15}\left(\frac{GMm}{\hbar c}\right)^2 \frac{G\hbar}{c^3} \frac{\mathrm{d}\mathcal{E}}{\mathcal{E}} \left(5\sqrt{2-\mathcal{E}} + \tfrac{3}{2}(2-\mathcal{E}) \ln \frac{1+\sqrt{1-\mathcal{E}}}{1-\sqrt{1-\mathcal{E}}}\right) \tag{5.64}$$

where $\mathcal{E}$ is the ratio of the graviton energy and the incident particle energy. The logarithmic term is entirely due to diagrams (a) and (a′) which involve linear gravitational interaction, and it dominates the bremsstrahlung cross section in the non relativistic approximation. However, the general result (5.61) shows that the contributions of diagrams (b) and (c), which involve non linear gravitational interactions, are quite important at relativistic energies.

5.5. Graviton bremsstrahlung with Coulomb scattering

The graviton emission during the Coulomb scattering of a spinless particle of charge e by a heavy particle of charge Ze can also be estimated along lines similar to those expounded in section 5.4.

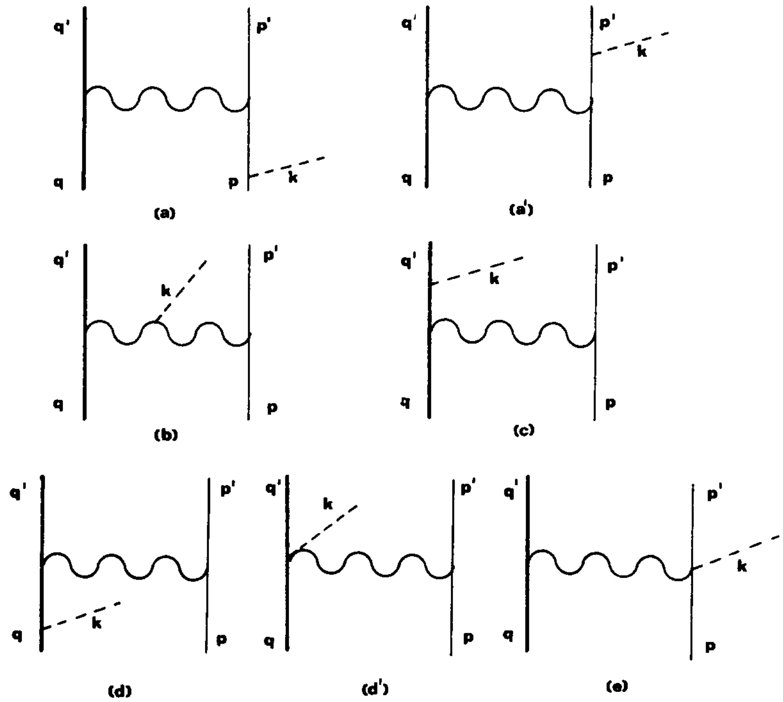

Fig. 5.2. Graviton bremsstrahlung with Coulomb scattering.

However, in addition to the particle–graviton coupling terms, one must consider the particle–photon coupling terms

$$\mathrm{i}eA_\mu[(\partial^\mu U^*)\,U - U^*(\partial^\mu U)] - e^2 A_\mu A^\mu U^* U\,, \tag{5.65}$$

the particle–photon–graviton coupling terms

$$\mathrm{i}e\kappa(h_{\mu\nu} - \tfrac{1}{2}\delta_{\mu\nu}h^\lambda_\lambda)A^\mu[(\partial^\nu U^*)\,U - U^*\partial^\nu U] + \mathrm{O}(e\kappa^2, e^2\kappa) \tag{5.66}$$

and the photon–graviton coupling terms

$$-\tfrac{1}{2}\kappa\, h_{\mu\nu}(F^\mu_\rho F^{\nu\rho} - \tfrac{1}{4}\delta^{\mu\nu}F_{\lambda\rho}F^{\lambda\rho}) + \mathrm{O}(\kappa^2)\,. \tag{5.67}$$

The lowest order diagrams for the process under consideration are shown in fig. 5.2 and can be treated in the same manner as the diagrams of fig. 5.1. The diagrams (a), (a′), and (b) only contribute to the scattering cross section which can be expressed as

$$\begin{aligned}\mathrm{d}\sigma = {} & \frac{Z^2 e^4 \kappa^2}{8(2\pi)^4}\,\frac{\mathrm{d}k_0}{k_0}\,\frac{|\boldsymbol{p}'|}{|\boldsymbol{p}|}\,\frac{\sin\theta\,\sin\theta'\,\mathrm{d}\theta\,\mathrm{d}\theta'\,\mathrm{d}\phi'}{(\boldsymbol{k}+\boldsymbol{p}'-\boldsymbol{p})^4\,((p'-p)^2-k_0^2)^2}\,[J'^2\boldsymbol{p}^4\sin^4\theta + K'^2\boldsymbol{p}'^4\sin^4\theta' \\ & + L'^2\boldsymbol{p}'^2\boldsymbol{p}^2\sin^2\theta\,\sin^2\theta' + 2J'K'\boldsymbol{p}'^2\boldsymbol{p}^2\sin^2\theta\,\sin^2\theta'\cos 2\phi' \\ & + 2J'L'|\boldsymbol{p}|^3|\boldsymbol{p}'|\sin^3\theta\,\sin\theta'\cos\phi' + 2L'K'\,|\boldsymbol{p}'|^3\,|\boldsymbol{p}|\sin^3\theta'\sin\theta\cos\phi']\end{aligned} \tag{5.68}$$

where

$$J' = p_0' [(p_0' - p)^2 - k_0^2 - 2k_0(|p| \cos\theta - p_0)]/(p\cos\theta - p_0)$$

$$K' = -p_0 \frac{[(p' - p)^2 - k_0^2 + 2k_0(|p'|\cos\theta' - p_0')]}{|p'|\cos\theta' - p_0'} \tag{5.69}$$

$$L' = 2(p_0 + p_0')\, k_0 \,.$$

Replacement of $(GMm/\hbar c)^2$ by $(Ze^2/4\pi\hbar c)^2$ in (5.64) gives the cross section in the non relativistic approximation.

6. Scattering of gravitons and gravitational scattering

This section is concerned with the scattering of gravitons by spinless particles, the gravitational scattering of several particles and the bending of light in a gravitational field.

6.1. The scattering of gravitons by spinless particles

Barker et al. [4] considered the scattering of gravitons by spinless particles and the annihilation of two spinless particles into two gravitons and have obtained the scattering and annihilation cross sections for various polarization states of the graviton. The property of invariance under gauge transformations is satisfied. The authors have only imposed the requirement of asymptotic gauge invariance and by doing so have obtained weaker gauge conditions than those used by Jackiw [46]. The processes are represented by fig. 6.1.

For neutral and charged particles the interaction Lagrangians used are respectively

$$\mathcal{L}_{\text{int}} = -\tfrac{1}{2}\kappa h_{\mu\nu}[\partial^\mu U_0 \partial^\nu U_0 - \tfrac{1}{2}\delta^{\mu\nu}\partial_\rho U_0 \partial^\rho U_0 - \tfrac{1}{2}\delta^{\mu\nu}\mu^2 U_0 U_0] - \tfrac{1}{16}\kappa^2\mu^2(h^\mu_\mu h^\nu_\nu - 2h_{\mu\nu}h^{\mu\nu})\, U_0 U_0 + \mathrm{O}(\kappa^3) \tag{6.1}$$

and

$$\mathcal{L}_{\text{int}} = -\kappa h_{\mu\nu}[\partial^\mu U^* \partial^\nu U - \tfrac{1}{2}\delta^{\mu\nu}\partial_\rho U - \tfrac{1}{2}\delta^{\mu\nu}\mu^2 U^* U] - \tfrac{1}{8}\kappa^2\mu^2(h^\mu_\mu h^\nu_\nu - 2h_{\mu\nu}h^{\mu\nu})\, U^* U + \mathrm{O}(\kappa^3) \tag{6.2}$$

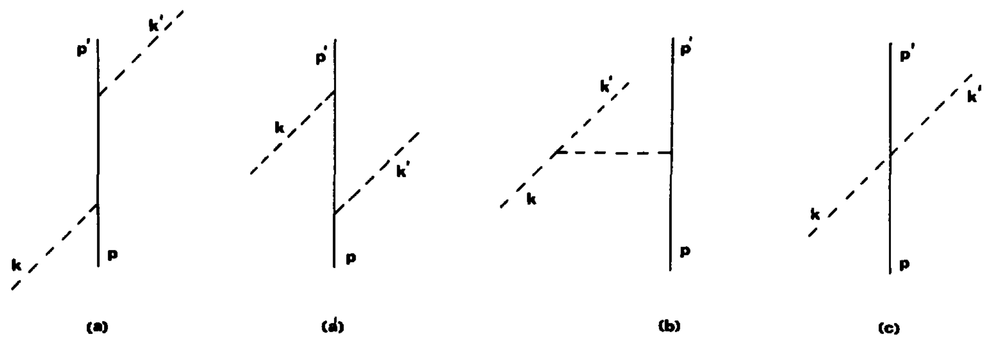

Fig. 6.1. Graviton scattering by a spinless particle.

while the graviton coupling terms are represented by

$$\mathcal{L}_{\text{int}} = -\tfrac{1}{2}\kappa(h_{\mu\nu} - \tfrac{1}{2}\delta_{\mu\nu}h^{\rho}_{\rho})\,[\tfrac{1}{2}\partial^{\mu}h_{\alpha\beta}\partial^{\nu}h^{\alpha\beta} - \partial^{\nu}h^{\mu}_{\alpha}\partial^{\alpha}h^{\lambda}_{\lambda} + \partial^{\beta}h^{\mu}_{\alpha}\partial^{\alpha}h^{\nu}_{\beta}$$

$$+\tfrac{1}{2}\partial^{\alpha}h^{\lambda}_{\lambda}\partial_{\alpha}h^{\mu\nu} - \partial_{\alpha}h^{\mu}_{\beta}\partial^{\alpha}h^{\nu\beta}] + \mathrm{O}(\kappa^{2})\,. \tag{6.3}$$

Expressions (6.1) to (6.3) are given as ordered products, in which U_0 and U represent the field operators for neutral and charged spinless particles, and $h_{\mu\nu}$ represents the field operator for gravitons.

The contribution of the observable gravitons to the scattering operator in the interaction picture is obtained by means of the Fourier decomposition,

$$h_{\mu\nu} = \frac{1}{\sqrt{V}}\sum_{k}\left(\frac{1}{2k_0}\right)^{1/2}[a_{\mu\nu}(k)\exp\{\mathrm{i}(k\cdot r - \omega t)\} + a^{*}_{\mu\nu}(k)\exp\{-\mathrm{i}(k\cdot r - \omega t)\}]$$

$$\omega t = k_0 x_0\,,\qquad k_0 = |k| \tag{6.4}$$

$$a_{\mu\nu}(k) = e_{\mu\nu,+}(k)\,a_{+}(k) + e_{\mu\nu,-}(k)a_{-}(k) \tag{6.5}$$

$$e_{\mu\nu,\pm}(k) = \frac{1}{\sqrt{2}}\,[(e^{1}_{\mu}(k)\,e^{1}_{\nu}(k) - e^{2}_{\nu}(k)e^{2}_{\nu}(k)) + \mathrm{i}(e^{1}_{\mu}(k)\,e^{2}_{\nu}(k) + e^{2}_{\mu}(k)\,e^{1}_{\nu}(k))] \tag{6.6}$$

$a_{+}(k)$ and $a_{-}(k)$ are annihilation operators for gravitons with their spin axes parallel and antiparallel to k, while $e^1(k)$ and $e^2(k)$ are unit vectors such that k, $e^1(k)$ and $e^2(k)$ are mutually perpendicular and $e^1_4 = e^2_4 = 0$.

Also $a\,(p)$ and $a^{*}\,(p')$ are the annihilation and creation operators for the spinless particle.

In the laboratory system,

$$\boldsymbol{p} = 0\,,\qquad p_0 = \mu\,,\qquad \boldsymbol{k} = \boldsymbol{k}' + \boldsymbol{p}'$$

$$\mu + k_0 = k'_0 + p'_0\,. \tag{6.7}$$

The polarization vectors associated with the gravitons are chosen such that

$$e^{1}(k') = e^{1}(k) = \frac{k'\times k}{|k'\times k|}\,,\qquad e^{2}(k') = \frac{k'\times e^{1}(k')}{|k'|}\,,\qquad e^{2}(k) = \frac{k\times e^{1}(k)}{|k|}\,. \tag{6.8}$$

Then the contributions of the diagrams (a) and (a') of fig. 6.1 vanish and the contributions of the remaining diagrams take a much simpler form. The scattering operator for the diagrams can be expressed as:

$$S = \frac{\mathrm{i}}{V^{2}}(2\pi)^{4}\,\delta(p + k - p' - k')\,\frac{1}{(\mu p'_0 k_0 k'_0)^{1/2}}\,\frac{k^{2}}{4}\,\frac{1}{2}\,\frac{\mu k_0 k'_0}{k_0 - k'_0}\,a^{*}_{ij}(k')\,a_{ij}(k)\,,$$

$$a^{*}(p')a(p) = \mathrm{i}V^{-2}(2\pi)^{4}\,\delta(p + k - p' - k')\,\frac{k^{2}}{16}\left(\frac{\mu k_0 k'_0}{p'_0}\right)^{1/2}\frac{1}{(k_0 - k'_0)^{1/2}}\left[\left(1 + \frac{k\cdot k'}{k_0 k'_0}\right)^{2}\right. \tag{6.9}$$

$$\left.\times\,(a^{*}_{+}(k')a_{+}(k) + a^{*}_{-}(k')a_{-}(k)) + \left(1 - \frac{k\cdot k'}{k_0 k'_0}\right)^{2}(a^{*}_{+}(k')a_{-}(k) + a^{*}_{-}(k')a_{+}(k))\right]a^{*}(p')a(p)\,.$$

Asymptotic gauge invariance requires that the scattering operator be invariant when subject to transformations of the form

$$a^*_{\mu\nu}(q) \to a^*_{\mu\nu}(q) - iq_\mu \lambda^*_\nu(q) - iq_\nu \lambda^*_\mu(q)$$

$$a_{\alpha\beta}(p) \to a_{\alpha\beta}(p) + ip_\alpha \lambda_\beta(p) + ip_\beta \lambda_\alpha(p) \tag{6.10}$$

a requirement manifestly satisfied.

The resulting scattering cross sections for various polarizations states of the gravitons are

$$\frac{d\sigma_{++}}{d\Omega} = \frac{d\sigma_{--}}{d\Omega} = G^2 m^2 \cot^4 \tfrac{1}{2}\theta \cos^4 \tfrac{1}{2}\theta/(1 + 2\epsilon \sin^2 \tfrac{1}{2}\theta)^2$$

$$\frac{d\sigma_{+-}}{d\Omega} = \frac{d\sigma_{-+}}{d\Omega} = G^2 m^2 \sin^4 \tfrac{1}{2}\theta/(1 + 2\epsilon \sin^2 \tfrac{1}{2}\theta)^2 \tag{6.11}$$

where ϵ is the incident graviton energy in units of m and θ is the graviton scattering angle. On averaging over the initial polarization states and summing over the final ones,

$$\frac{d\sigma}{d\Omega} = G^2 m^2 (\cot^4 \tfrac{1}{2}\theta \cos^4 \tfrac{1}{2}\theta + \sin^4 \tfrac{1}{2}\theta)/(1 + 2\epsilon \sin^2 \tfrac{1}{2}\theta)^2 \tag{6.12}$$

which, in the non relativistic approximation is

$$\frac{d\sigma}{d\Omega} = G^2 m^2 [\cot^4 \tfrac{1}{2}\theta \cos^4 \tfrac{1}{2}\theta + \sin^4 \tfrac{1}{2}\theta] \, . \tag{6.13}$$

The non relativistic cross section agrees with the results of Jackiw [46], Gross and Jackiw [32], DeWitt [21] but (6.12) disagrees with the result given by DeWitt [21].

6.2. Gravitational scattering of neutrinos

Boccaletti et al. [11] have considered the gravitational scattering of neutrinos in the one graviton exchange approximation. The problem is a priori interesting because neutrinos have zero mass and spin $\frac{1}{2}$ and must therefore exhibit the characteristic behaviour of fermions and at the same time bear some relationship to that of photons, because of the zero mass. A behaviour similar to that of photons should also be expected on the basis of the work of Papapetrou and Corinaldesi [63] who found that the spin of the particle has practically no influence on the value of the deflection of light in a Schwarzschild field. An interesting point however, has been raised by Kobzarev and Okun [48] who have observed that from the point of view of the gravitational interaction there is no reason why four component neutrinos should not exist. Boccaletti et al. have shown in their calculations that two and four component neutrinos are scattered in the same amount by bosons and in the small angle limit have the same cross sections as photons except in the case of neutrino–neutrino scattering which gives two slightly different cross sections for the two and four component cases.

6.2.1. Neutrinos in the gravitational field of a large mass

The behaviour of a neutrino beam in a gravitational field can be represented by the diagrams of fig. 6.2.

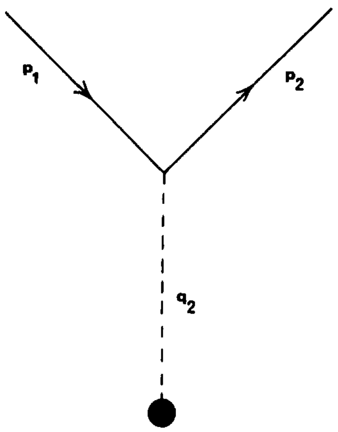

Fig. 6.2. Neutrino scattering in an external gravitational field.

The matrix element is given by

$$\langle p_2|M|p_1\rangle = \frac{\kappa^2 M_\odot}{8(2\pi)^2}\ \frac{1}{2}\int \sqrt{2}\,\delta^{\mu 0}\delta^{\nu 0}\bar{u}^{\mathrm{f}}_{\nu}\tfrac{1}{2}(1+\mathrm{i}\gamma_5)(\gamma_\nu p_\mu + \gamma_\mu p_\nu)\,u^{\mathrm{i}}_\nu \frac{1}{|q|^2}\,\delta^4(p_2+q-p_1)\,\mathrm{d}^3 q\,. \tag{6.14}$$

The differential cross section can be obtained from (6.14) in the usual way. The result is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{16}\,\frac{\kappa^4 M_\odot^2}{4(2\pi)^2}\,\frac{\cos^2(\theta/2)}{\sin^4(\theta/2)}\,. \tag{6.15}$$

In the limit of small angles one obtains

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\kappa^4 M_\odot^2}{4(2\pi)^2}\,\frac{1}{\theta^4}\,. \tag{6.16}$$

Therefore a neutrino beam is deflected, by a large mass in the same amount as a light beam. There is also no difference in this respect between a two component and a four component neutrino.

6.2.2. *Neutrino–scalar and neutrino–photon scattering*

The cross section of neutrino–scalar particle and neutrino–photon scattering can be also evaluated in the one graviton exchange approximation. The matrix element for neutrino–scalar particle scattering is represented by

$$\langle k_2 p_2|M|k_1 p_1\rangle = \frac{k^2}{2(2\pi)^2\, 8\sqrt{k_1^0 k_2^0}}\,\frac{1}{(p_1-p_2)^2}\,\bar{u}^{\mathrm{f}}_\nu\,\tfrac{1}{2}(1+\mathrm{i}\gamma_5)(\gamma^\nu p^\mu + \gamma^\mu p^\nu)\,u^{\mathrm{i}}_\nu(k_{1\mu}k_{2\nu} + k_{1\mu}k_{2\nu} - m^2\delta_{\mu\nu})\,. \tag{6.17}$$

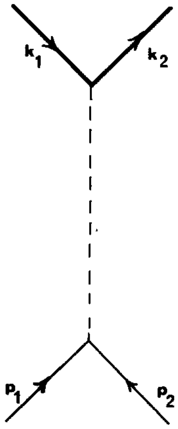

Fig. 6.3. Neutrino–scalar particle scattering.

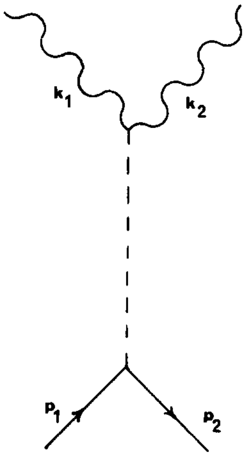

Fig. 6.4. Neutrino–photon scattering.

k_1 and k_2 are initial and final four momenta of the scalar particle and p_1 and p_2 the four momenta of the neutrino and $p = p_1 + p_2$ in fig. 6.3. In the small angle limit the differential cross section is given by

$$\frac{d\sigma}{d\Omega} = \frac{\kappa^4}{4(2\pi)^2}\,\frac{w^2}{\theta^4}\,, \tag{6.18}$$

where w is the total c.m. energy and θ the scattering angle in the c.m. system. Expression (6.18) is equal to the cross section for the problem in which the neutrino is replaced by a photon (Boccaletti et al. [9]). The diagram for neutrino–photon scattering is shown in fig. 6.4.

The matrix element is given by

$$\langle k_2 p_2 | M | k_1 p_1 \rangle = \frac{\kappa^2}{4(2\pi)^2\sqrt{k_1^0 k_2^0}}\,\bar{u}_\nu^{\mathrm{f}}\,\tfrac{1}{2}(1 + \mathrm{i}\gamma_5)(\gamma^\nu p^\mu + \gamma^\mu p^\nu)\,u_\nu^{\mathrm{i}}\,\frac{1}{(p_1 - p_2)^2}$$
$$\times\,[(e_{1\mu}k_{1\alpha} - e_{1\alpha}k_{1\mu})(e_{2\nu}k_2^\alpha - e_2^\alpha k_{2\nu}) - \tfrac{1}{4}\delta_{\mu\nu}(e_{1\alpha}k_{1\beta} - e_{1\beta}k_{1\alpha})(e_2^\alpha k_2^\beta - e_2^\beta k_2^\alpha)]\,. \tag{6.19}$$

For small angles, the differential cross section can be approximated by

$$\frac{d\sigma}{d\Omega} = \frac{4}{4}\,\frac{\kappa^4}{(2\pi)^2}\,\frac{k^2}{\theta^4} \tag{6.20}$$

which is equal to photon–photon scattering cross section

6.2.3. Neutrino–neutrino gravitational scattering

The matrix element is given by

$$\langle p_3 p_4 | M | p_1 p_2 \rangle = \frac{\kappa^2}{16(2\pi)^2}\,\frac{1}{2}\,\Big[\bar{u}(p_3)\,\tfrac{1}{2}(1 + \mathrm{i}\gamma_5)(\gamma^\nu p^\mu + \gamma^\mu p^\nu)\,u(p_1)\,\frac{1}{(p_1 - p_3)^2}\,\bar{u}(p_4)\tfrac{1}{2}(1 + \mathrm{i}\gamma_5)$$
$$\times\,(\gamma_\nu p'_\mu + \gamma_\mu p'_\nu)\,u(p_2) - \bar{u}(p_4)\tfrac{1}{2}(1 + \mathrm{i}\gamma_5)(\gamma^\nu q^\mu + \gamma^\mu q^\nu)\,u(p_1)\,\frac{1}{(p_1 - p_4)^2}\,\bar{u}(p_3)$$
$$\times\,\tfrac{1}{2}(1 + \mathrm{i}\gamma_5)(\gamma_\nu q'_\mu + \gamma_\mu q'_\nu)\,u(p_2)\Big] \tag{6.21}$$

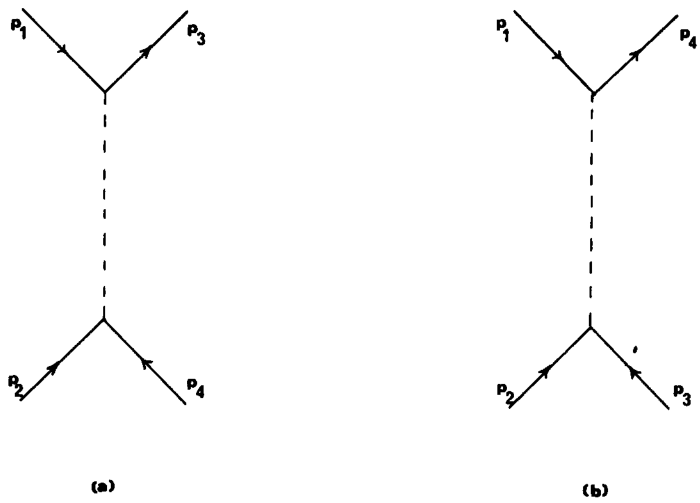

Fig. 6.5. Neutrino–neutrino gravitational scattering.

where $p = p_1 + p_3$, $p' = p_2 + p_4$, $q = p_1 + p_4$, $q' = p_2 + p_3$ in fig. 6.5.

In the c.m. system, in which k is the common modulus of the momenta and θ the scattering angle between p_1 and p_3, the differential cross section in the small angle limit is given by

$$\frac{d\sigma}{d\Omega} = \frac{5}{4} \frac{\kappa^4}{(2\pi)^2} \frac{k^2}{\theta^4} \tag{6.22}$$

if the contribution of the γ_5 terms are included.

If the contribution of the γ_5 terms are subtracted the resultant cross section is given by:

$$\frac{d\sigma}{d\Omega} = \frac{4}{4} \frac{\kappa^4}{(2\pi)^2} \frac{k^2}{\theta^4} . \tag{6.23}$$

From the above two equations it is seen that the gravitational scattering of two component neutrinos is slightly different from that of four component neutrinos. Moreover, (6.23) is exactly the same as the correspondent photon–photon cross section. In the case of a four component neutrino, the virtual gravitons could be converted into pairs of anomalous neutrinos and antineutrinos, that is left $\bar{\nu}$ and right ν. These pairs cannot be scattered absorbed or emitted through weak interactions and would only be deflected by gravitational fields and would themselves be sources of these fields as pointed out by Kobzarev and Okun [48].

6.3. Gravitational photon–photon and photon–scalar particle scattering

Boccaletti et al. [9] have calculated the cross sections for the photon–photon and photon–scalar particle scattering both classically and by using the linearized quantum theory of gravitation.

The correspondence principle would obviously connect these two methods. It is therefore confirmed by both calculations that the gravitational action of a beam of light is twice as great as the Newtonian one.

As pointed out by the authors, the comparison between classical and quantized theory is not a purely academic problem. The investigations into the peculiarities of gravitational interaction between photons or between photons and particles is interesting at least for two reasons:

i) On astrophysical grounds, because of the possible insight into astrophysics and perhaps into cosmology and the various applications and consequences on both domains.

ii) It may be possible in a not too distant future to produce more and more intense photon beams and therefore test some of the theoretical predictions.

The photon–photon and photon–scalar particle scattering are discussed in the lowest order in the perturbative approach, all diagrams of figs. 6.6 and 6.7 are with only an intermediate graviton.

The Lagrangian density for the interaction between the e.m. and gravitational fields is given by:

$$\mathcal{L}_{\mathrm{I}} = -\tfrac{1}{2}\kappa h_{\mu\nu} T_{\mu\nu} . \tag{6.24}$$

$h_{\mu\nu}$ represents the gravitational field, $T_{\mu\nu}$ is the energy momentum tensor of the c.m. field; the angular differential cross section for unpolarized photons is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{2\kappa^4}{(2\pi)^2}\,\frac{k^2}{\sin^2\theta}\cdot\frac{1}{4}\,[1+\cos^{16}\tfrac{1}{2}\theta+\sin^{16}\tfrac{1}{2}\theta] . \tag{6.25}$$

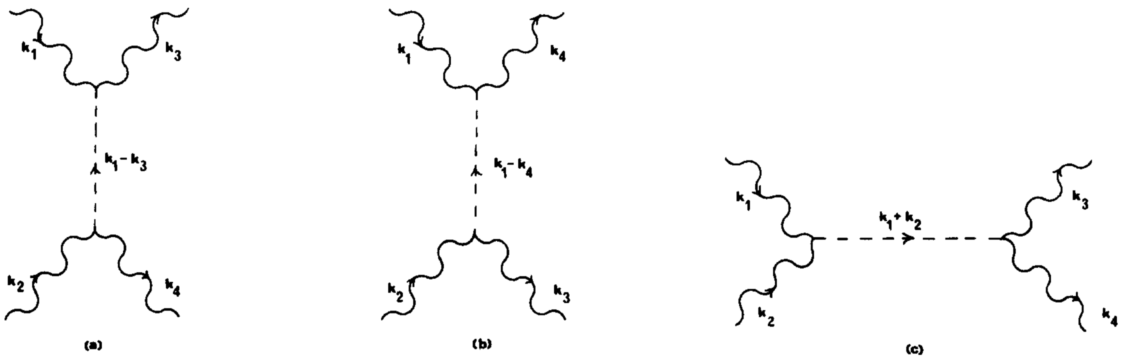

Fig. 6.6. Photon–photon scattering.

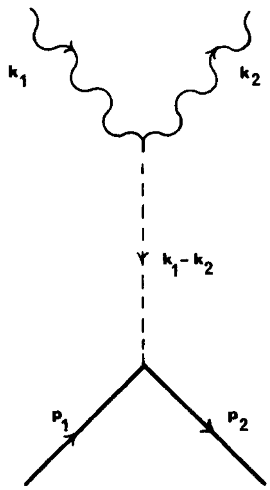

Fig. 6.7. Photon–scalar particle scattering.

The expression in square brackets is identical to the result obtained by Barker et al. [5, 6], though calculated by completely different methods. In the c.m. frame k is the photon momentum and θ is the scattering angle. For small angles

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{quantum}} \approx \frac{4\kappa^4}{(2\pi)^2}\frac{k^2}{4\theta^4} \approx 64G^2\frac{k^2}{\theta^4}. \tag{6.26}$$

The classical calculation yields 4 times the value of (6.26). This is equivalent to saying that the classical deflection angle is greater by a factor of 2. This factor may be accounted for by considering that in the classical case a fixed beam replaces the photon of the quantum case as a source of gravitational field. But in the quantum problem the photon too, is deflected. For symmetry reasons, a factor 2 for the deflection angle should therefore appear.

Photon–scalar particle scattering

K_1 is the four momentum of the initial photon and K_2 that of the final photon; p_1 and p_2 are the analogous quantities for the scalar particle.

The energy momentum tensor for scalar matter is

$$T_{\mu\nu} = \phi_{1\nu}\phi_{1\mu} - \tfrac{1}{2}\delta_{\mu\nu}\phi_{1\sigma}\phi_{1\sigma} + \tfrac{1}{2}m^2\phi^2\eta_{\mu\nu} \tag{6.27}$$

and the differential cross section is represented by

$$\frac{d\sigma}{d\Omega} = \frac{\kappa^4}{16(2\pi)^2}\,\frac{w^4}{4(w-k)^2}\left(\frac{1+\cos\theta}{1-\cos\theta}\right)^2 . \tag{6.28}$$

For small angles (6.28) becomes

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{quantum}} \sim \frac{\kappa^2}{4(2\pi)^2}\left(\frac{w^2}{w-k}\right)^2\frac{1}{\theta^4} \approx 16G^2\left(\frac{w^2}{w-k}\right)^2\frac{1}{\theta^4}\ \text{(c.g.s.)} \tag{6.29}$$

where w is the total energy in the center of mass system and k is the photon energy.

The classical differential cross section is given by

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{cl}} \sim 16G^2\frac{w^4}{k^2}\frac{1}{\theta^4} . \tag{6.30}$$

The classical formula differs from the quantum one by a factor of $k^2/(w-k)^2$. This means that the classical angle differs from the quantum one by a factor of $k/(w-k)$, that is by the ratio of the photon energy to the particle energy. The symmetry arguments previously advanced are no longer valid.

In the case when the ratio $k/(w-k) = \frac{1}{2}$, i.e., the photon–photon scattering, $w = 3k$.

It then follows that the difference between the classical and quantum formula is the same as in the case of photon–photon scattering.

A massive particle exerts a gravitational action exactly as that of a photon of the same energy. In different words, the gravitational deflection of a particle by a beam of light is double that as would be obtained by replacing the beam by an average density of matter equal to that of light. Barker et al. [4] have obtained a similar result by applying the quantum theory of gravitation to the case of photon–scalar particle scattering. Barker et al. give the gravitational potential and find that light is deflected by a heavy object by twice the amount predicted by the Newtonian theory.

6.4. Bending of light in an external gravitational field

Boccaletti et al. have calculated the effect of the bending of a light beam in a gravitational field and considered the interaction of a photon with the external gravitational field of the sun. This corresponds to one of the three crucial tests of Einstein's theory and is shown in fig. 6.8.

The interaction Lagrangian density is known from the work of Gupta and follows from (2.60).

The first order wave function in momentum space is therefore:

$$-\tfrac{1}{2}\kappa\delta^{\mu\lambda}(\tilde{\epsilon}^{\nu\rho}-\tfrac{1}{4}\delta^{\nu\rho}\tilde{\epsilon})(\epsilon^1_\lambda k^1_\rho-\epsilon^1_\rho k^1_\lambda)(\epsilon^2_\mu k^2_\nu-\epsilon^2_\nu k^2_\mu) \tag{6.31}$$

where ϵ^1, k^1, and ϵ^2, k^2 are the polarization vector and momentum of the two photons.

Fig. 6.8 is the analog of Rutherford scattering in quantum electrodynamics.

The matrix element M in the external field approximation (Jauch and Rohrlich [47]) is given by:

$$\langle k_2|M|k_1\rangle=\frac{\kappa^2 M_\odot}{(2\pi)^2(4k_1^0k_2^0)^{1/2}}\,\frac{1}{2}\int\delta^{\mu\lambda}(\sqrt{2}\delta^{\nu0}\delta^{\rho0}-\tfrac{1}{4}\sqrt{2}\delta^{\nu\rho})$$

$$\times(\epsilon^1_\lambda k^1_\rho-\epsilon^1_\rho k^1_\lambda)(\epsilon^2_\mu k^2_\nu-\epsilon^2_\nu k^2_\mu)\,\frac{1}{|q|^2}\,\delta^4(k_2+q-k_1)\,\mathrm{d}^3q\,. \tag{6.32}$$

$\tilde{h}^{00}=-\kappa M_\odot/4\pi r$ is the static external potential and $M_\odot$ is the solar mass.

$$\phi(q)=-\left(\frac{\kappa}{\sqrt{2}}M_\odot/(2\pi)^{3/2}\right)\cdot\frac{1}{|q|^2} \tag{6.33}$$

is the Fourier transform of $\tilde{h}^{00}$. $\boldsymbol{q}$ is the momentum of the virtual graviton. The graviton polarization tensor has in this case only the time components

$$\tilde{\epsilon}^{\nu\rho}=\sqrt{2}\delta^{\nu0}\delta^{\rho0}\,,\qquad\tilde{\epsilon}=\sqrt{2}\,. \tag{6.34}$$

The gauge is chosen, such that $\epsilon^0_1=\epsilon^0_2=0$. Hence

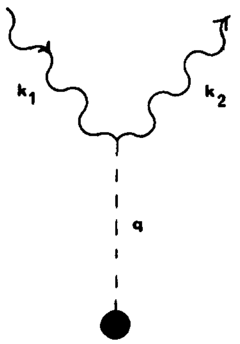

Fig. 6.8. Photon scattering in an external gravitational field.

$$\langle k_2 | M | k_1 \rangle = \frac{\sqrt{2}\kappa^2 M_\odot}{2(2\pi)^2 (4k_1^0 k_2^0)^{1/2}} \int \frac{1}{|\boldsymbol{q}|^2} [k_1^0 k_2^0 (\epsilon_1 \epsilon_2) - \tfrac{1}{2}(k_1 k_2)(\epsilon_1 \epsilon_2) + \tfrac{1}{2}(\epsilon_1 k_2)(\epsilon_2 k_1)]\, \delta^4(k_2 + q - k_1)\, \mathrm{d}^3 q \tag{6.35}$$

and

$$\mathrm{d}\sigma = \frac{\kappa^4 M_\odot^2}{(2\pi)^2 (2k_1^0 k_2^0)^4}\; \tfrac{1}{2} \sum_{\mathrm{pol}} [k_1^0 k_2^0 (\epsilon_1 \epsilon_2) - \tfrac{1}{2}(k_1 k_2)(\epsilon_1 \epsilon_2) + \tfrac{1}{2}(\epsilon_1 k_2)(\epsilon_2 k_1)]^2$$

$$\times \frac{\delta(k_2^0 - k_1^0)\, \mathrm{d}^3 k_2}{(k_1^{02} + k_2^{02} - 2k_1^0 k_2^0 \cos\theta)^2} = \frac{\kappa^4 M_\odot^2}{(4\pi)^2 16} \int \left(\frac{1 + \cos\theta}{1 - \cos\theta}\right)^2 \mathrm{d}\Omega = \frac{\kappa^4 M_\odot^2}{(4\pi)^2 16} \int \cos^4 \tfrac{1}{2}\theta\, \mathrm{d}\Omega\,. \tag{6.36}$$

For small angles, $\tan\theta \sim \theta$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \approx \frac{\kappa^4 M_\odot^2}{4(2\pi)^2} \frac{1}{\theta^4}\,. \tag{6.37}$$

According to Einstein's theory the angle by which a photon is deviated in the gravitational field

$$\theta = \frac{4GM_\odot}{r}\,, \tag{6.38}$$

where r is the least distance from the centre of the sun attained by the travelling photon.

Equation (6.38) is the same as the one resulting from Einstein's general relativity and is therefore a good check of the quantized theory.

7. Annihilation of spinless particles into gravitons

The formalism in the calculation of the scattering of gravitons by spinless particles has been extended to the annihilation of two neutral or oppositely charged spinless particles with propagation vectors p and p' into two gravitons with propagation vectors k and k' as shown in fig. 7.1. It takes into account the self-interaction of gravity as shown in section 4. In the c.m. system

$$\boldsymbol{p}' = -\boldsymbol{p}\,; \quad \boldsymbol{k}'_0 = -\boldsymbol{k}\,, \quad p'_0 = p_0 = k'_0 = k_0\,. \tag{7.1}$$

The polarization vectors associated with the two gravitons are chosen as:

$$e^1(k') = \frac{k' \times p'}{|k' \times p'|}\,; \quad e^1(k) = \frac{k \times p}{|k \times p|} = e^1(k')\,; \quad e^2(k') = \frac{k' \times e^1(k')}{|k'|}\,;$$

$$e^2(k) = \frac{k \times e^1(k)}{|k|} = -e^2(k')\,. \tag{7.2}$$

The scattering operator can be expressed as

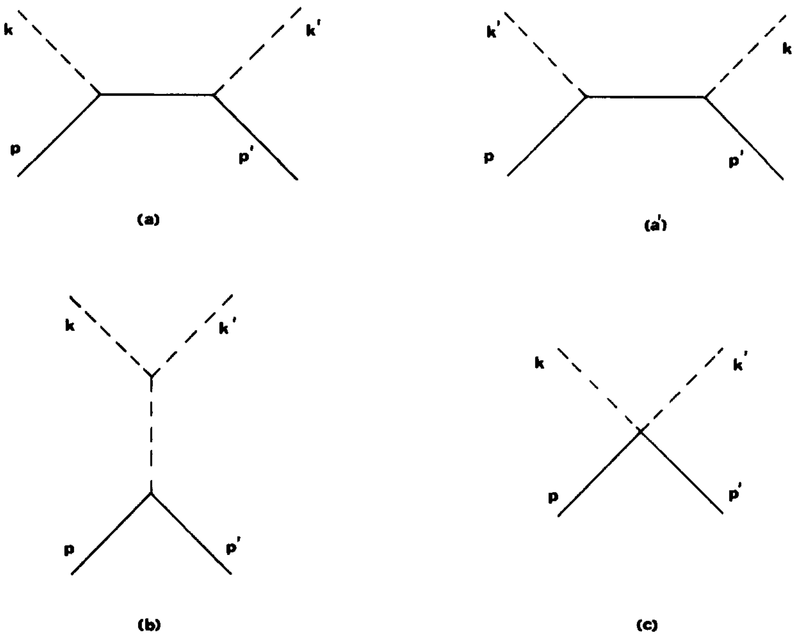

Fig. 7.1. Annihilation of two neutral or oppositely charged spinless particles into two gravitons.

$$S = -\mathrm{i}V^{-2}(2\pi)^4\,\delta(p+p'-k-k')a(p')a(p)[(a_+^*(k')a_+^*(k') + a_-^*(k')a_-(k))(F_a + F_a' + F_b + F_c) + (a_+^*(k')a_-^*(k) + a_-^*(k')a_+(k))(F_a + F_{a'})] \tag{7.3}$$

where

$$F_a = -\frac{\kappa^2}{128}\frac{(\mu^4 - (p-k)^2(p+k)^2)^2}{p_0^6(\mu^2 + (p-k)^2)}\,, \qquad F_{a'} = -\frac{\kappa^2}{128}\frac{(\mu^4 - (p+k)^2(p-k)^2)^2}{p_0^6(\mu^2 + (p+k)^2)}\,,$$

$$F_b = -\frac{\kappa^2}{32}\,p_0^4\,[\mu^2(\mu^2 - 4p_0^2) - (p-k)^2(p+k)^2]\,, \qquad F_c = -\frac{\kappa^2}{4}\frac{\mu^2}{p_0^2}\,. \tag{7.4}$$

The resulting cross sections for various polarization state of the two gravitons are

$$\frac{\mathrm{d}\sigma_{++}}{\mathrm{d}\Omega} = \frac{\mathrm{d}\sigma_{--}}{\mathrm{d}\Omega} = \frac{G^2E^2}{4\beta}\left[1 - \beta^2 - \beta^2\sin^2\theta + \frac{\beta^4\sin^4\theta}{1-\beta^2\cos^2\theta}\right]^2 \tag{7.5}$$

$$\frac{\mathrm{d}\sigma_{+-}}{\mathrm{d}\Omega} = \frac{\mathrm{d}\sigma_{-+}}{\mathrm{d}\Omega} = \frac{G^2E^2}{4\beta}\left[\frac{\beta^4\sin^4\theta}{1-\beta^2\cos^2\theta}\right]^2 \tag{7.6}$$

with

$$E = p_0\,, \qquad \beta = |p|/p_0 = v \tag{7.7}$$

where E is the particle energy, v its velocity, and θ is the angle between k and p. The total cross section can be obtained by integrating both $d\sigma_{++}$ and $d\sigma_{--}$ over θ from 0 to $\frac{1}{2}\pi$ and integrating either $d\sigma_{+-}$ or $d\sigma_{-+}$ over θ from 0 to π. Thus:

$$\sigma_{++} = \sigma_{--} = \frac{\pi G^2 E^2}{4\beta^2}\left[\beta - 3\beta^3 + 3\beta^5 - \beta^7 + (\tfrac{1}{2} - 2\beta^2 + 3\beta^4 - 2\beta^6 + \tfrac{1}{2}\beta^8)\ln\frac{1+\beta}{1-\beta}\right] \tag{7.8}$$

$$\sigma_{+-} = \sigma_{-+} = \frac{\pi G^2 E^2}{2\beta^2}\left[7\beta - \tfrac{53}{3}\beta^3 + \tfrac{191}{15}\beta^5 - \beta^7 + (10\beta^2 - \tfrac{7}{2} - 9\beta^4 + 2\beta^6 + \tfrac{1}{2}\beta^8)\ln\frac{1+\beta}{1-\beta}\right]. \tag{7.9}$$

The total cross section is given by

$$\sigma = \sigma_{++} + \sigma_{--} + \sigma_{+-}\ . \tag{7.10}$$

In the non relativistic approximation

$$\sigma = \pi G^2 m^2/\beta \tag{7.11}$$

while in the extreme relativistic approximation

$$\sigma \approx \tfrac{8}{15}\pi G^2 E^2\ . \tag{7.12}$$

The above results are qualitatively in agreement with the earlier estimates of Wheeler and Brill [90] and of Ivanenko and Sokolov [42] but the numerical factors in the above two equations disagree with the results of DeWitt [21].

8. The graviton–particle vertex, the proton–neutron mass difference and related problems

The analysis of the graviton–particle vertex in itself, that is quite separate from other parts of a Feynman diagram, can be used in a number of problems, including the calculations of the proton–neutron mass difference (Pagels [62]), and the binding energies of the deuteron and singlet deuteron (Hare and Papini [38, 39]). These applications are essentially based on the close analogy existing between electric charge and gravitational mass as quantities indicating the source strength of the electromagnetic and gravitational fields respectively. This may also be expressed by saying that as the electric and magnetic structures of a particle are obtained in the particle–photon vertex, so its mechanical properties are contained in the particle–graviton vertex. Dispersion relations may then be obtained for the form factors appearing in both vertices.

The nucleon-graviton vertex of fig. 8.1 that corresponds to the trace of the energy-momentum tensor can be represented in the form (Pagels [62])

$$\bar{u}(P)\Sigma(w^2) = \bar{u}(P)[G(w^2) + G'(w^2)\hat{l}\,] \tag{8.1}$$

where G and G' are nucleon mechanical form factors. It satisfies TCP-invariance. At threshold, where all lines are on the mass shell, one obtains

$$G(M^2) = M \tag{8.2}$$

which is the graviton–nucleon coupling constant. Following Bincer's method [7] one can then derive sidewise dispersion relation for the mechanical form factor. For proton and neutron they are

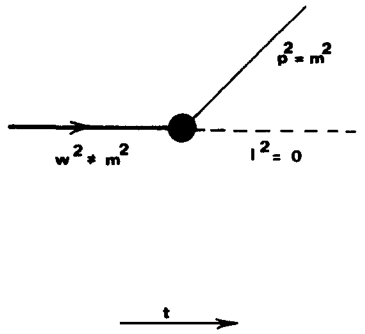

Fig. 8.1. The nucleon–graviton vertex.

$$G_{p,n}(w^2) = M_{p,n} + \frac{w^2 - M_{p,n}^2}{\pi} \int_{M_{p,n}^2}^{\infty} \frac{\mathrm{Im}\, G_{p,n}(w'^2)\, dw'^2}{(w'^2 - M_{p,n}^2)(w'^2 - w^2 - i\epsilon)} . \tag{8.3}$$

From these one can immediately derive the proton–neutron mass difference

$$\delta M = M_p - M_n = \frac{1}{\pi} \int_{M_p^2}^{\infty} \frac{\mathrm{Im}\, G_p(w^2)\, dw^2}{w^2 - M_p^2} - \frac{1}{\pi} \int_{M_n^2}^{\infty} \frac{\mathrm{Im}\, G_n(w^2) dw^2}{w^2 - M_n^2} . \tag{8.4}$$

The additional assumption of low energy dominance, already tested successfully in the calculation of anomalous magnetic moments (Drell and Pagels [23]; Parson [68]) restricts the integrations in (8.4) to $\lambda^2 M_p^2$ and $\lambda^2 M_n^2$ respectively with $\lambda^2 \approx 3$ (Drell and Pagels [23]). The contributions to Im $G_{p,n}$ can then be calculated by expanding the vertex as shown in fig. 8.2. The Nγ and Nπ con-

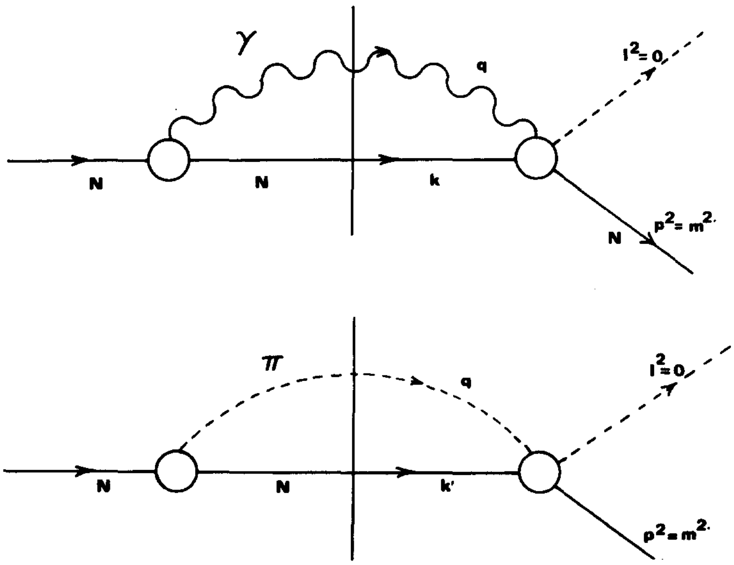

Fig. 8.2. Contributions to Im $G_{p,n}$.

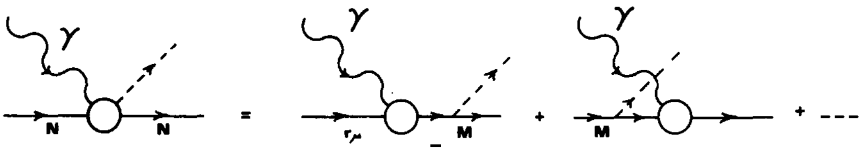

Fig. 8.3. Nγ contributions to Im $G(w^2)$.

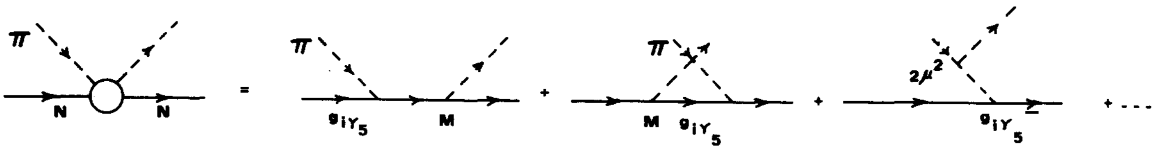

Fig. 8.4. Nπ contributions to Im $G(w^2)$.

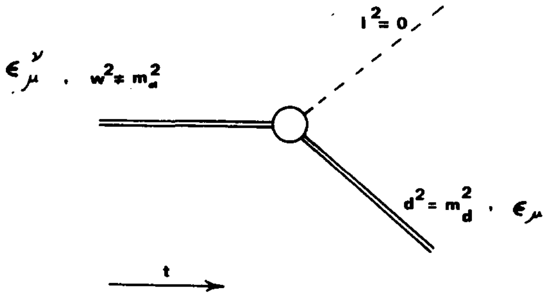

Fig. 8.5. The two nucleon intermediate state.

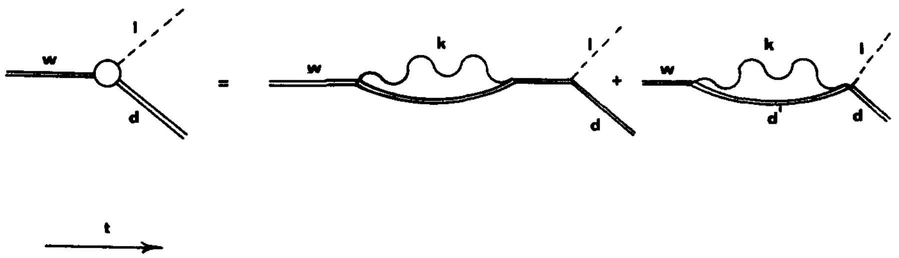

Fig. 8.6. The electromagnetic contribution.

tributions can be further expanded in figs. 8.3 and 8.4. Most of the diagrams in figs. 8.3 and 8.4 can be calculated in the pole approximation. This is sufficient to give the correct experimental value for δM. Completely analogous procedures are applied to the determination of the deuteron binding energy (Hare and Papini [39]). The vertex of interest is $\Sigma_{\mu\nu}(w^2)$ in this case and ϵ_μ and ϵ_μ^w are the polarization vectors for the off and on-shell deuterons respectively. Figs. 8.5, 8.6 and 8.7 represent the two nucleon intermediate state, the electromagnetic contribution and the break-up contribution respectively. The Lorentz condition $d \cdot \epsilon = 0$ and TCP invariance restrict the vertex to the expression

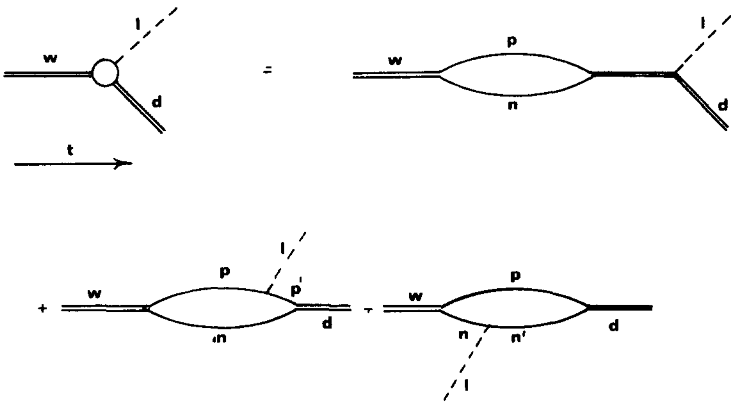

Fig. 8.7. The breakup contribution.

$$\epsilon_\mu \Sigma_{\mu\nu}(w^2) = F_\mathrm{d}(w^2)\epsilon_\nu + F'_\mathrm{d}(w^2)(\epsilon \cdot l) l_\nu \,. \tag{8.5}$$

One concludes, however in this case that

$$F_\mathrm{d}(M_\mathrm{d}^2) = 2M_\mathrm{d}^2 \,. \tag{8.6}$$

The subtracted dispersion relation now is

$$G(w^2) = M + \frac{w^2 - M^2}{\pi} \int_{M^2}^{\infty} \frac{\mathrm{Im}\, G(w'^2)\, \mathrm{d}w'^2}{(w'^2 - M^2)(w'^2 - w^2 - \mathrm{i}\epsilon)} \tag{8.7}$$

and the binding energy

$$B = M_\mathrm{n} + M_\mathrm{p} - M_\mathrm{d} \tag{8.8}$$

can be calculated as

$$B = \frac{1}{\pi} \int_{M_\mathrm{p}^2}^{\infty} \frac{\mathrm{Im}\, G_\mathrm{p}(w'^2)\, \mathrm{d}w'^2}{w'^2 - M_\mathrm{p}^2} + \frac{1}{\pi} \int_{M_\mathrm{n}^2}^{\infty} \frac{\mathrm{Im}\, G_\mathrm{n}(w'^2)\, \mathrm{d}w'^2}{w'^2 - M_\mathrm{n}^2} - \frac{1}{\pi} \int_{M_\mathrm{d}^2}^{\infty} \frac{\mathrm{Im}\, G_\mathrm{d}(w'^2)\mathrm{d}w'^2}{w'^2 - M_\mathrm{d}^2} \tag{8.9}$$

which can be extended to the more general case of a light nucleus (Hare and Papini [38]). Again low energy dominance restricts the integrations in (8.9) to $\lambda'^2 M_\mathrm{d}^2$ and $\lambda^2 M_\mathrm{N}^2$ respectively. λ'^2 is determined by the relation

$$\lambda'^2 M_\mathrm{d}^2 - M_\mathrm{d}^2 = \lambda^2 M_\mathrm{N}^2 - M_\mathrm{N}^2 \,. \tag{8.10}$$

Since λ^2 is already known, so is λ'^2. They yield the correct value of the binding energy. The same method applied to the singlet deuteron indicates that the system is unbound. The analysis of the particle–graviton vertex, and in the particular problems considered the assumption of threshold dominance appear therefore to be able to link and explain the neutron–proton mass difference, the deuteron binding energy and the non-existence of a bound singlet neutron–proton pair.

9. Photoproduction of gravitons

Photoproduction processes of the type $\gamma + e \to e + g$ have been first studied by Vladimirov [79]. Perhaps more interesting, at least from the astrophysical point of view, is the interaction of photons with a static electromagnetic field and the consequent emission of gravitons. Strong sources of electromagnetic radiation do in fact exist in the universe, among them are quasars, pulsars, galaxies and in our planetary system, the sun. The linear theory within certain limits is adequate to deal with them.

Weber and Hinds [83] have studied the photoproduction of gravitons starting from a Hamiltonian formulation. They have shown that the processes of the creation of gravitons by Coulomb scattering of photons and by scattering in a magnetostatic field occur and have calculated the cross sections for a large volume containing a uniform field. Such interactions might lead to observable effects over a long period of time.

Also Papini and Valluri [64–67] have considered the process of photoproduction of gravitons in static electromagnetic fields and have applied the results to the study of gravitational radiation from a variety of astrophysical objects. Radiation rates have been estimated in different frequency ranges of the electromagnetic spectrum. Photoproduction has been found to contribute a significant amount of gravitational radiation.

9.1. Graviton photoproduction in static external gravitational fields

Boccaletti et al. [8] have considered the emission of a graviton by a photon in a gravitational external field as represented by the two diagrams of fig. 9.1.

k_1 and k_2 are the four momenta of the initial and final photon respectively. p is the four

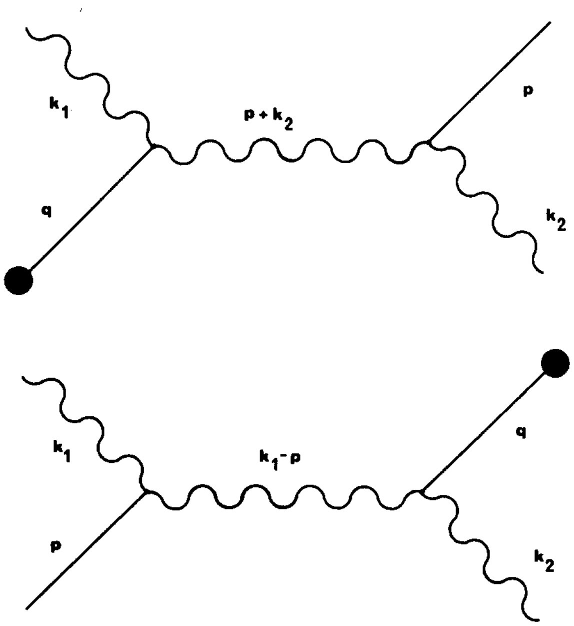

Fig. 9.1. Graviton photoproduction in a static external field.

momentum of the emitted graviton and q that of the virtual graviton. The matrix element corresponding to the two diagrams is given by:

$$\langle pk_2|M|k_1\rangle = \frac{\kappa^2}{(2\pi)^2(8\bar{k}_1^0 k_2^0 p^0)^{1/2}} \int \delta^{\mu\lambda}(\sqrt{2}\delta^{\nu 0}\delta^{\rho 0} - \frac{\sqrt{2}}{4}\delta^{\nu\rho})(\epsilon'_\lambda k'_\rho - \epsilon'_\rho k'_\lambda)\frac{1}{(p+k_2)^2}$$

$$\times [\delta_{\mu\sigma}(p+k_2)_\nu(p+k_2)_\tau - \delta_{\nu\sigma}(p+k_2)_\mu(p+k_2)_\tau - \delta_{\mu\tau}(p+k_2)_\nu(p+k_2)_\sigma + \delta_{\nu\tau}(p+k_2)_\mu(p+k_2)_\sigma]$$

$$\times (\epsilon_\eta^2 k_\xi^2 - \epsilon_\xi^2 k_\eta^2)\,\delta^{\eta\sigma}(\tilde{\epsilon}^{\tau\xi} - \tfrac{1}{4}\delta^{\tau\xi}\tilde{\epsilon}) + \delta^{\mu\lambda}(\sqrt{2}\delta^{\nu 0}\delta^{\rho 0} - \frac{\sqrt{2}}{4}\delta^{\nu\rho})(\epsilon_\lambda^2 k_\rho^2 - \epsilon_\rho^2 k_\lambda^2)\frac{1}{(k_1-p)^2}$$

$$\times [\delta_{\mu\sigma}(k_1-p)_\nu(k_1-p)_\tau - \delta_{\nu\sigma}(k_1-p)_\mu(k_1-p)_\tau - \delta_{\mu\tau}(k_1-p)_\nu(k_1-p)_\sigma + \delta_{\nu\tau}(k_1-p)_\mu(k_1-p)_\sigma]$$

$$\times (\epsilon'_\eta k'_\xi - \epsilon'_\xi k'_\eta)\,\delta^{\eta\sigma}(\tilde{\epsilon}^{\tau\xi} - \tfrac{1}{4}\delta^{\tau\xi}\tilde{\epsilon})\,\phi(q)\,\delta^4(k_2+p+q-k_1)\,\mathrm{d}^3q\,. \tag{9.1}$$

In the case of an extended source, like a galaxy, the Fourier integral is evaluated from a minimum r_0 that is the radius of the galaxy, and this imposes a constraint on the exchange momentum q

$$\mathrm{d}\sigma = (2\pi)^2 \int_{\mathrm{f}}\int_{\mathrm{i}} \delta(k_2^0 + p^0 - k_1^0)|\langle pk_2|M|k_1\rangle|^2$$

$$= \frac{k^6 m^2}{8(2\pi)^5}\int\left[\frac{\delta(k_2^0+p^0-k_1^0)}{|k_1-k_2-p|^4\; k_1^0 k_2^0 p^0}\,\tfrac{1}{2}\sum_{\mathrm{pol}}\Big(\sum_{\mathrm{grav}} 2\{\text{states of the graviton}\}^2\;\mathrm{d}^3k_2\;\mathrm{d}^3p\right]$$

$$= \frac{\kappa^6 m^2}{8(2\pi)^5}\int_0^{k_1^0}\mathrm{d}k_2^0\int_0^{\pi}\mathrm{d}\theta_2\int_0^{2\pi}\mathrm{d}\phi_2\int_0^{\pi}\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\phi\; Q\,\tfrac{1}{2}\sum_{\mathrm{pol}}\Big(\sum_{\mathrm{grav}} 2\{\cdots\}^2\Big). \tag{9.2}$$

Q is given by

$$Q = \frac{k_2^0}{4k_1^0}(k_1^0 - k_2^0)\sin\theta\,\sin\theta_2\,[k_1^{02} + k_2^{02} - k_1^0k_2^0(1-\cos\theta_2) + k_1^0(k_1^0-k_2^0)\cos\theta - k_2^0(k_1^0-k_2^0)$$

$$\times(\sin\theta_2\cos\phi_2\sin\theta\cos\phi + \sin\theta_2\sin\phi_2\sin\theta\sin\phi + \cos\theta_2\cos\theta)^{-2}]\,. \tag{9.3}$$

Σ_{pol} means the sum over the polarizations of the photon and the Σ_{grav} 2{states of graviton} indicates the square modulus of the expression between curly brackets in (9.2) summed over graviton polarizations. There are four possibilities for the photon and hence four amplitudes for ϵ_1 and ϵ_2 parallel and then perpendicular to the (k_1, k_2) plane. The explicit values of the polarization vectors are:

$$\epsilon_{1\parallel} = (\cos\phi_2, \sin\phi_2, 0)\,,\qquad \epsilon_{1\perp} = (\sin\phi_2, -\cos\phi_2, 0) \tag{9.4a}$$

$$\epsilon_{2\parallel} = (\cos\theta_2 \cos\phi_2, \cos\theta_2 \sin\phi_2, -\sin\theta_2)\,, \qquad \epsilon_{2\perp} = (\sin\phi_2, -\cos\phi_2, 0)\,. \tag{9.4b}$$

The constant in (9.4) has been taken as the average mass of a galaxy. The cross section in cm^2 is of the order of:

$$\sigma \approx 10^{24}/10^{25} \approx 10^{-1}\ \mathrm{cm}^2\,. \tag{9.5}$$

If the target is a galaxy, the cross section is very small and the contribution to the red shift of the light coming from the distant stars is negligible.

If the energy loss of a photon which emits a graviton is evaluated, with the use of the standard formula

$$\frac{-\mathrm{d}k_1}{\mathrm{d}x} = N \int_0^{k_1} p\,\frac{\partial\sigma}{\partial p}\,\mathrm{d}p\,, \tag{9.6}$$

N is the number of galaxies $/\mathrm{cm}^3$; considering a path of the order of the radius of the universe, Δk is found to be:

$$\Delta k_1 \sim 10^{-50}\ e\cdot v\,. \tag{9.7}$$

Therefore, it follows that the red shift of light cannot be explained as a "tired light" phenomenon.

9.2. The processes of the type $\gamma + \mathrm{e} \to \mathrm{e} + \mathrm{g}$

The calculations are based on the interaction Hamiltonian introduced by Gupta [33]. The

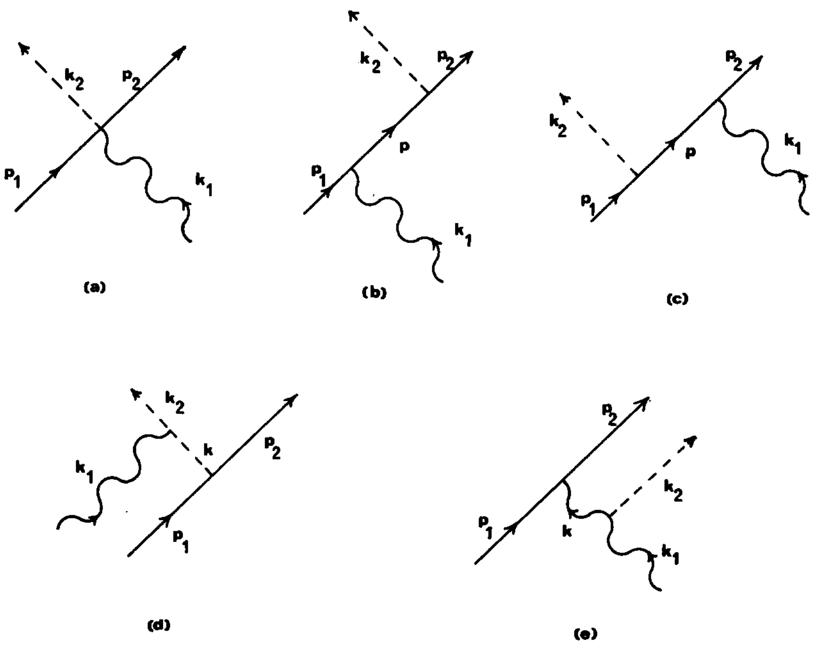

Fig. 9.2. The process $\gamma + \mathrm{e} \to \mathrm{e} + \mathrm{g}$.

diagrams for the process are given in fig. 9.2. The electron may be substituted by other fermions. Let E_1, be the initial energy of the fermion, k_{01} is the energy of the photon, k_{02} is the graviton energy.

When

$$E_1 \gg k_{01}\,, \qquad E_1 \gg k_{02} \tag{9.8}$$

the matrix element has the form:

$$M = \frac{ie\sqrt{k^2}}{(4\pi)^2\sqrt{k_{01}k_{02}}}\;\bar{u}_\nu^+(p_2)\,\frac{(p_1k_2)\hat{h}(p_1e)-(p_1k_1)\hat{e}}{(p_1k_1)(p_1k_2)}\;\bar{u}_{\nu'}(p_1)\;. \tag{9.9}$$

The diagrams b and c give the main contributions. The effective differential cross section of the process has the form:

$$d\sigma = (2\pi)^2\,\frac{k_{02}^2E_2}{E_2-p_1\cos\theta}\sum{}'|M(p_1,p_2)|^2\,d\Omega\,. \tag{9.10}$$

By squaring (9.9), averaging over initial and summing over the final spins, one obtains

$$\frac{d\sigma}{d\Omega} = \frac{e^2\kappa^2p^6\sin^4\theta\,\sin^2\phi}{8(4\pi)^2E_1(E_2-p_1\cos\theta)}\,\frac{k_{02}}{k_{01}}\left(\frac{p_1(k_2-k_1)}{(p_1k_1)(p_1k_2)}\right)^2 \tag{9.11}$$

where ϕ is the angle between $\boldsymbol{k}_1$ and $\boldsymbol{p}_1$ and θ is the angle between $\boldsymbol{k}_2$ and $\boldsymbol{p}_1$. In the relativistic case, $E \sim E_1 \sim E_2 \sim p_1 \sim p_2$ and by setting of $k_1 \sim k_2 \sim k$ one gets

$$\sigma \sim e^2\kappa^2E^2/k^2\,. \tag{9.12}$$

For k extremely small, eq. (9.12) diverges. This infrared divergence is of the same nature as those present in electrodynamics. In quantum electrodynamics the infrared catastrophe appears and hence the methods of perturbation theory would be inapplicable. If it is assumed that a significant number of gravitons exist in outer space, then the transformations of gravitons into photons must take place when fluxes of gravitons interact with matter.

In particular this effect via the factor $\sin^4\theta$ in (9.11) could enhance the electromagnetic radiation normally reaching earth from outer space if some objects moved at right angles to the radius vector from the earth to the object. The measurement of the effect and of the Doppler red shift would then make possible a complete estimate of velocity and direction of motion of astrophysical objects (Vladimirov [79]).

9.3. Photoproduction in static electromagnetic fields

The quantization of the linearised form of the gravitational field has in its applications to weak sources, yielded in general low rates for graviton emission. Strong extended sources of electromagnetic radiation in the universe like quasars, pulsars, etc. could improve considerably the graviton emission rate.

The first order diagram for the process is given in fig. 9.3. In fig. 9.3, l, q and k are the four

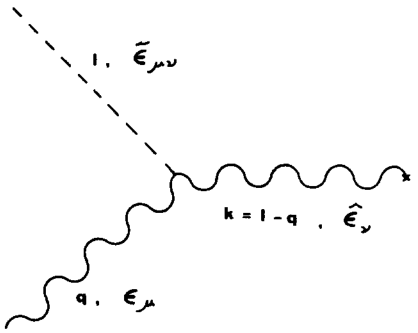

Fig. 9.3. The photon–photon–graviton vertex.

momenta of the graviton, the incoming photon and the photon of the static external field. The ϵ quantities are the corresponding polarization tensors. Therefore the interaction Lagrangian can be written as:

$$\mathcal{L}_{\mathrm{I}} = -\frac{\kappa}{2}\frac{\hat{\epsilon}^{\nu\mu}}{2}\,[(\epsilon\hat{\epsilon})(q_\mu l_\nu + q_\nu l_\mu) - (\epsilon\hat{\epsilon})\, 2q_\mu q_\nu - (q\hat{\epsilon})(\epsilon_\mu l_\nu + \epsilon_\nu l_\mu - \epsilon_\mu q_\nu - \epsilon_\nu q_\mu)$$

$$- (\epsilon l)(q_\mu \hat{\epsilon}_\nu + q_\nu \hat{\epsilon}_\mu) + (ql)(\epsilon_\mu \hat{\epsilon}_\nu + \epsilon_\nu \hat{\epsilon}_\mu) - \delta_{\mu\nu}\{(\epsilon\hat{\epsilon})(ql) - (q\hat{\epsilon})(\epsilon l)\}\,] \,. \qquad (9.13)$$

From it one derives the transition matrix

$$|\langle l|M|q\rangle|^2 = \left(\frac{1}{\sqrt{2l_0}\sqrt{2q_0}}\right)^2 \left(\frac{-\kappa}{4}\right)^2 \sum_{\mathrm{pol}} |\hat{\epsilon}_{\nu\mu}\,\{(q_\mu l_\nu + q_\nu l_\mu)(\epsilon\hat{\epsilon}) - (\epsilon_\mu l_\nu + \epsilon_\nu l_\mu)(q\hat{\epsilon})$$

$$+ (\epsilon_\mu \hat{\epsilon}_\nu + \epsilon_\nu \hat{\epsilon}_\mu)(ql) - (\hat{\epsilon}_\nu q_\mu + \hat{\epsilon}_\mu q_\nu)(\epsilon l) + (q_\mu \epsilon_\nu + q_\nu \epsilon_\mu)(l\hat{\epsilon})$$

$$- 2q_\mu q_\nu(\epsilon\hat{\epsilon}) - \delta_{\mu\nu}((ql)(\epsilon\hat{\epsilon}) - (\epsilon l)(q\hat{\epsilon}))\}\,\phi(K)|^2 \qquad (9.14)$$

where $\phi(K)$ is the potential due to the static electromagnetic field. The polarization sum gives 16 large sets of terms which results from the expansion. Using the expression

$$\Sigma\hat{\epsilon}_{\mu\nu}\hat{\epsilon}_{\alpha\beta} = -\tfrac{1}{2}(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\nu\alpha}\eta_{\mu\beta} - \eta_{\mu\nu}\eta_{\alpha\beta})$$

and considering the three cases

1) $\nu = \alpha\,, \quad \mu = \beta$
2) $\nu = \beta\,, \quad \mu = \alpha$
3) $\mu = \nu\,, \quad \alpha = \beta$

the summation over polarizations over all the 16 sets of terms is carried out. The differential cross section can then be written in the form:

$$\mathrm{d}\sigma/\mathrm{d}\Omega = (1/\sqrt{2l_0\, 2q_0})^2\left(\frac{-\kappa}{4}\right)^2 (2\pi)^2 \cdot 4 \int \mathrm{d}^3 l\, \delta(q_0 - l_0)\, \{3(ql)^2(\epsilon\hat{\epsilon})^2 - 4(\epsilon\epsilon)(ql)(q\hat{\epsilon})(l\hat{\epsilon})$$

$$- \delta(\epsilon l)(ql)(q\hat{\epsilon})(\epsilon\hat{\epsilon}) + 4(\epsilon l)^2(q\hat{\epsilon})^2 + 3(ql)^2(\epsilon\epsilon)(\hat{\epsilon}\hat{\epsilon})\}\, |\phi(K)|^2 \,. \qquad (9.15)$$

Since the vertex is covariant a convenient frame of reference can be chosen in order to simplify the calculations: q is chosen along the z axis. Then:

$$q = (q_0, 0, 0, \boldsymbol{q}) ; \quad q^2 = 0 = q_0^2 - \boldsymbol{q}^2 = l^2 = l_0^2 - \boldsymbol{l}^2 ; \quad q_0^2 = \boldsymbol{q}^2 ; \quad l_0^2 = \boldsymbol{l}^2 \tag{9.16}$$

$$l = (l_0, |\boldsymbol{l}| \sin\theta \cos\phi, |\boldsymbol{l}| \sin\theta \sin\phi, |\boldsymbol{l}| \cos\theta) \tag{9.17}$$

$$l - q = (0, |\boldsymbol{l}| \sin\theta \cos\phi, |\boldsymbol{l}| \sin\theta \sin\phi, |\boldsymbol{l}|(\cos\theta - 1)) \tag{9.18}$$

$$ql = q_0^2(1 - \cos\theta) . \tag{9.19}$$

Since $\epsilon_\mu q^\mu = 0$, two polarization vectors satisfying this condition are chosen such that:

(1) $\epsilon = (0, 1, 0, 0)$

(2) $\epsilon = (0, 0, 1, 0)$. (9.20)

The static Coulomb field is considered first. Then

$$\phi(K) = \frac{Ze}{|\boldsymbol{K}|^2} = \frac{Ze}{|\boldsymbol{l} - \boldsymbol{q}|^2} \tag{9.21}$$

and the time like component $\hat{\epsilon}$ does not vanish. Moreover

$$\epsilon\hat{\epsilon} = 0 ; \quad \hat{\epsilon}\hat{\epsilon} = -1 . \tag{9.22}$$

Thus the total cross section is given by

$$\sigma = \int \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\, \mathrm{d}\Omega = \frac{4}{(2l_0)(2q_0)} \left(\frac{-\kappa}{4}\right)^2 (2\pi)^2 \int l_0^2\, \mathrm{d}l_0\, \delta(q_0 - l_0)\, \mathrm{d}\Omega\, \{-3q_0^4(1 - \cos\theta)^2$$

$$- \tfrac{4}{2} q_0^4 \sin^2\theta (1 + \cos 2\phi) - 4q_0^4(\cos\theta - 1)\}\, (Ze)^2/(|\boldsymbol{l} - \boldsymbol{q}|^2)^2 \tag{9.23}$$

where θ and ϕ specify the orientation of the outgoing graviton or equivalently:

$$\sigma = 4\pi^4 GZ^2/e^2$$

$$\sigma = 4\pi^4 \frac{GZ^2/e^2}{c^4} \text{ (c.g.s. units)} . \tag{9.24}$$

The magnetic field is assumed to be that of a dipole. The potential due to the magnetic field can be calculated in momentum space from the expression for the potential in coordinate space by the use of the Fourier transformation.

Therefore (9.15) is evaluated and on simplification gives:

$$\tfrac{3}{4} q_0^4(1 - \cos\theta)^2\, (1 - 5\cos\theta) . \tag{9.25}$$

For the magnetic field of a dipole, the potential is given by

$$\phi_\mathrm{P} = -\frac{4\pi M}{\mathrm{i}} \frac{\cos\theta'}{|k|} \tag{9.26}$$

where M is the magnetic moment, θ' is the angle of inclination of the dipole with the direction of the incident photon. $\hat{\epsilon}$ can be chosen to have components

$$\hat{\epsilon} = (0, \sin \tfrac{1}{2}\theta \cos \phi, \sin \tfrac{1}{2}\theta \sin \phi, \cos \tfrac{1}{2}\theta)\,. \tag{9.27}$$

The total cross section is found to be given by

$$\sigma_{\mathrm{mag}} = \left(\frac{-\kappa}{4}\right)^2 (4\pi M \cos \theta')^2 \, (2\pi)^2 \, \tfrac{3}{4} \int \frac{q_0^4}{2q_0^2} \, \frac{(1 - \cos \theta)^2}{-1 - \cos \theta} \, (1 - 5 \cos \theta) \sin \theta \, \mathrm{d}\theta \, \mathrm{d}\phi$$

$$= 256\pi^6 G M^2 q_0^2 \cos^2 \theta' \qquad (q_0 = 2\pi/\lambda = 2\pi\nu/c) \tag{9.28}$$

$$\sigma_{\mathrm{mag}} = 1024\pi^8 \, \frac{G M^2 \nu^2}{c^6} \cos^2 \theta' \ \text{(c.g.s. units)}\,. \tag{9.29}$$

Eqs. (9.24) and (9.29) agree in order of magnitude with those previously calculated by Weber and Hinds [83] using the Hamiltonian formulation of the general theory of relativity.

9.3.1. Higher order corrections

In this section the second order radiative corrections are calculated. The four point interaction of the two fermions, photon and graviton is known from the work of Pagels [62].

Diagrams a and c of fig. 9.4 give the same contribution and are identical. Also diagrams b and d

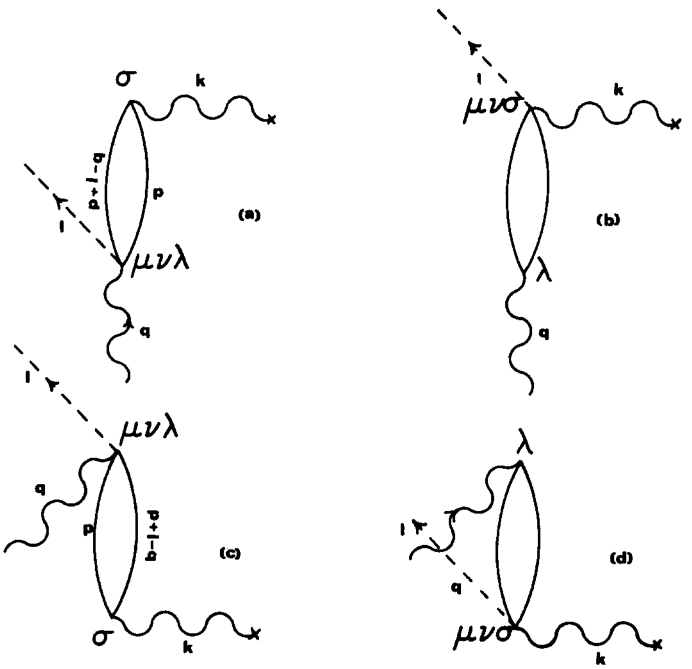

Fig. 9.4. The second order radiative corrections.

are identical. Therefore the polarization tensor $\Pi_{\mu\nu\lambda\sigma}(k, l, q)$ can be defined,

$$\Pi_{\mu\nu\lambda\sigma} = 2(T_{\mu\nu\lambda\sigma} + T_{\mu\nu\sigma\lambda}) . \tag{9.30}$$

Also $\langle f| T_{\mu\nu\lambda\sigma} |i\rangle$ and $\langle f| T_{\mu\nu\sigma\lambda} |i\rangle$ give the same contribution. The cross section is calculated initially for a screened static Coulomb field. The transition matrix element for diagram (a) is given by

$$\langle \epsilon_{\mu\nu} | T_{\mu\nu\lambda\sigma}(l, q, k, m) \frac{Ze}{|\boldsymbol{K}|^2} | \epsilon_\lambda \eta_\sigma \rangle =$$

$$(2\pi)^{8-N} e^2 \sqrt{16\pi G} \langle \epsilon_{\mu\nu} | \frac{\mathrm{Tr}}{\mathrm{i}\pi^2} \left\{ \int \mathrm{d}^4 p\, q_{\mu\nu} \gamma_\lambda \frac{p - m}{p^2 - m^2} \gamma_\sigma \frac{p + l - q - m}{(p + l - q)^2 - m^2} \right\} \frac{Ze}{|\boldsymbol{K}|^2} | \epsilon_\lambda \eta_\sigma \rangle . \tag{9.31}$$

After removal of the divergences by charge renormalisation one obtains

$$\sum_{\mathrm{f}} \sum_{\mathrm{i}} |\langle \epsilon_{\mu\nu} | T_{\mu\nu\lambda\sigma} | \epsilon_\lambda \eta_\sigma \rangle \phi(k)|^2 =$$

$$\frac{e^4_{\mathrm{Ren}}}{4} (128\pi G) Z^2 e^2 \left\{ 40 \frac{m^2}{q_0^2} \frac{1 - \cos\theta}{(1 - \cos\theta + \lambda')^2} - \frac{4(1 - \cos\theta)^2}{(1 - \cos\theta + \lambda')^2} - 100 \frac{m^4}{q_0^4} \frac{1}{(1 - \cos\theta + \lambda')^2} \right\} . \tag{9.32}$$

The total cross section can now be written as:

$$\sigma = \int \sum_{\mathrm{f}} \sum_{\mathrm{i}} |\langle f|M|i\rangle|^2\, \delta(q_0 - l_0)\, \mathrm{d}^3 l = (2\pi)^2 e^4_{\mathrm{Ren}} (16\pi G)(2Z^2 e^2)(2\pi)$$

$$\times \left\{ 40 \frac{m^2}{q_0^2} \left(\ln \frac{2 + \lambda'}{\lambda'} - \frac{2}{2 + \lambda'} \right) - 8 + 8\lambda' \ln \frac{2 + \lambda'}{- \lambda'} - \frac{8\lambda'}{2 + \lambda'} - \frac{200 m^4}{q_0^4 \lambda' (2 + \lambda')} \right\} . \tag{9.33}$$

The corresponding result for a magnetic dipole potential is

$$\sigma = (2\pi)^2 (2\pi)(16\pi G) e^2_{\mathrm{Ren}} (4\pi \mathcal{M}' \cos\theta)^2 \left\{ 8 q_0^2 \left(2 - 2\lambda' + \lambda'^2 \ln \frac{2 + \lambda'}{\lambda'} \right) \right.$$

$$\left. - 80 m^2 \left(2 + \lambda' \ln \frac{\lambda'}{2 + \lambda'} \right) + 200 \frac{m^2}{q_0^4} \ln \frac{2 + \lambda'}{\lambda'} \right\} . \tag{9.34}$$

9.3.2. The third order radiative corrections

The third order radiative corrections correspond to the diagrams of fig. 9.5.

$T_{\mu\nu\lambda\sigma}$ can be calculated by introducing the polarization tensor for the closed loop with the crossed photon and graviton lines. $T_{\mu\nu\lambda\sigma}$ has a finite part and a part which has a logarithmic divergence. The tensor can however, be regularised (Karplus and Neumann [50]). The calculation of $T_{\mu\nu\lambda\sigma}$ is straightforward but very lengthy. After squaring the matrix element and summing over the polarizations one obtains the total cross section in the low energy approximation as:

$$\sigma = (16\pi)^4 \frac{e^6 Z^2 G}{9} . \tag{9.35}$$

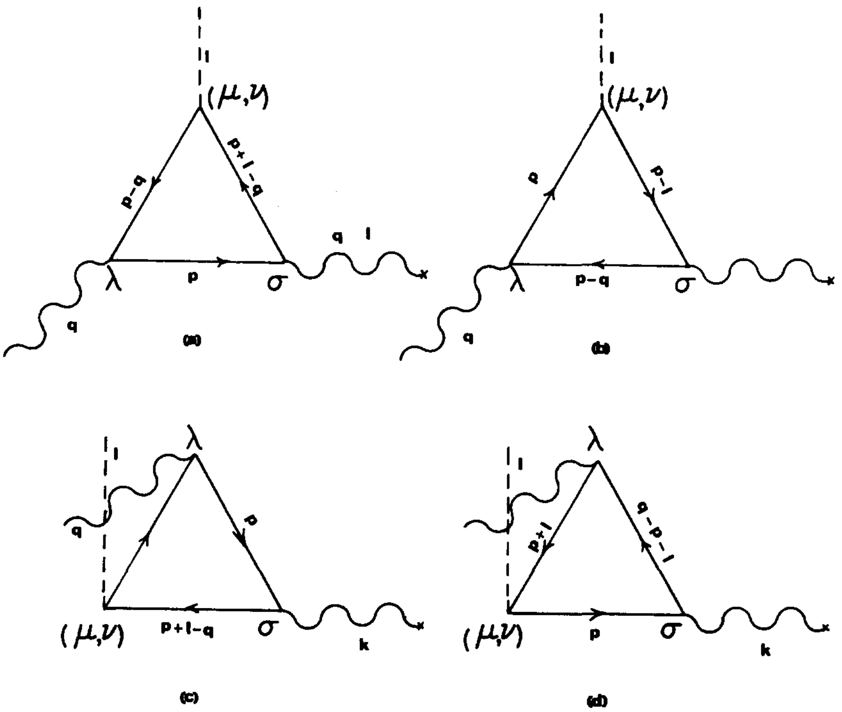

Fig. 9.5. The third order radiative corrections.

In the relativistic approximation $q_0 \gg m$ and for a screened Coulomb potential one obtains

$$\int \sum_f \sum_i |\langle \epsilon_{\mu\nu} | l_\mu T_{\nu\lambda\sigma} + l_\nu T_{\mu\lambda\sigma} | \frac{1}{|\boldsymbol{q} - \boldsymbol{l}|^2 + \lambda_1^2} | \epsilon_\lambda \eta_\sigma \rangle|^2 \, \mathrm{d}^3 l =$$

$$-\frac{i\pi^2}{2} \sum_{\text{Pol}} \left[\int \left| \epsilon_{\mu\nu} \left\{ (l_\mu q_\nu + l_\nu q_\mu) \frac{6}{9ql} (q_\sigma + l_\sigma) \epsilon_\lambda l_\lambda \eta_\sigma + 2 g_{\lambda\sigma} \left(1 + \frac{m^2}{ql}\right) \epsilon_\lambda \eta_\sigma \right. \right. \right. \tag{9.36}$$

$$\left. \left. \left. - (l_\mu g_{\nu\lambda} + l_\nu g_{\mu\lambda}) \left(1 - \frac{m^2}{ql}\right) 4(\eta l) \epsilon_\lambda + 2(\epsilon l) \left(1 + \frac{m^2}{ql}\right) (l_\mu g_{\nu\sigma} + l_\nu g_{\mu\sigma}) \eta_\sigma \right\} \frac{1}{|\boldsymbol{q} - \boldsymbol{l}|^2 + \lambda_1^2} \right|^2 \mathrm{d}^3 l \right],$$

$$|\boldsymbol{K}|^2 = |\boldsymbol{l} - \boldsymbol{q}|^2 = 2q_0^2 (1 - \cos\theta + \lambda_2), \qquad \lambda_2 = \lambda_1^2 / 2q_0^2 . \tag{9.37}$$

Integration over the angles and the summation over the initial polarizations yield the cross section:

$$\sigma^3 = 4^6 \sigma^1 \alpha^3 \left[1 + \frac{m^2}{q_0^2} \left(\frac{2}{2 + \lambda_2} + \ln \frac{\lambda_2}{2 + \lambda_2} \right) + \frac{m^4}{q_0^4 \lambda_2 (2 + \lambda_2)} + \frac{1}{18} - \frac{1 + \lambda_2}{36} \ln \frac{2 + \lambda_2}{\lambda_2} \right.$$

$$\left. - \frac{m^2}{q_0^2} \frac{2}{\lambda_2} + \ln \frac{\lambda_2}{2 + \lambda_2} \right] . \tag{9.38}$$

σ^1 is the first order cross section and α is the fine structure constant. For the magnetic field of a dipole one obtains

$$\sigma_d^3 = \sigma_d^1\, 16^2 \alpha^2 \left\{ \frac{m^2}{q_0^2} (2+\lambda_2) \ln \frac{2+\lambda_2}{\lambda_2} + \frac{18m^2}{q_0^2} + 8 + 4 \frac{m^4}{q_0^4} - \frac{1}{9} \ln \frac{2+\lambda_2}{\lambda_2} \right\}. \tag{9.39}$$

10. Synchrotron radiation

Pustovoit and Gertsenshtein [31] have considered the production of gravitational synchrotron radiation from charged particles spiraling in magnetic fields. Gravitational radiation is produced not only because of the quadrupole moment of the particle in motion but also because of the electromagnetic stresses caused by the charge. The trajectory of the particle is completely determined by the electromagnetic interaction since the gravitational interaction is much smaller. The ratio of the power due to the gravitational losses to that of the electric losses is independent of energy in the ultra relativistic case

$$\left(\frac{dE}{dt}\right)_{\text{grav.loss}} \Bigg/ \left(\frac{dE}{dt}\right)_{\text{electric.loss}} = \frac{13}{4} \frac{16\pi G M^2}{e^2}\,; \tag{10.1}$$

M is the mass and e the charge of the particle.

Another possibility of producing gravitons as high frequency radiation is by collisions between atoms and particles. A good example is the Coulomb scattering of electrons in the interior of a star. Unfortunately, the rate of photon emission is very small compared not only with the rate of photon emission, but also with the rate of neutrino–antineutrino pair emission calculated by Gandelman and Pinaev [30].

Gravitational synchrotron radiation from particles in stationary external gravitational fields has been extensively studied in recent times. It has in fact been suggested by Misner [55, 56] that a gravitational synchrotron effect might be responsible for the events observed by the Maryland–Argonne gravitational wave detectors. Because of the strong focussing effect and the proximity of the solar system to the galactic plane, the amount of radiation necessary to explain Weber's observations would be reduced substantially if a hypothetical source at the galactic center emitted gravitational synchrotron radiation into the galactic plane. Several models have been discussed in the literature involving radiation from relativistic particles in orbit around massive spherically symmetric or rotating black holes. None seems to yield a possible astrophysical source of synchrotron radiation. A comprehensive review of the subject has been published by Breuer [14].

11. Terrestrial sources of gravitational radiation

11.1. Induced emission of gravitons

Weber [84] has described the production of gravitational waves by large vibrating piezoelectric crystals and found that the production of gravitational radiation by microscopic systems is insignificantly small. The gravitational decay of excited states of nuclei or molecules takes place at too slow a rate to be of any significance as a source of experiment in the terrestial laboratory. It is however possible in principle that the gravitational radiation from such systems can be increased greatly by using induced emission at resonance. This process has been considered by Halpern and Laurent.

An apparatus working on the principle of induced emission would be the gravitational counterpart of the "laser" and has been called a "gaser". The "gaser" conceived by Halpern and Laurent [36] consists of a material the microscopical parts of which can be excited by external means. Besides the production of gravitons, induced emission would produce an avalanche of photons in the direction of the rod, which could however be removed, by means of mirrors. The induced emission would therefore be specific in the sense that gravitons would increase the emission of gravitons and not that of photons.

The cross section for the process of induced emission at resonance is:

$$\sigma_{\text{geom}} = \lambdabar^2 , \tag{11.1}$$

where λ denotes the wavelength of the emitted radiation. If this condition could be realized the gaser would be as effective as the laser. However, the extreme weakness of the gravitational interaction and the unavoidable competition from perturbations prevent such a possibility. More realistic limits can be obtained as follows:

Let $\delta\omega_1$ denote the line width due to the electromagnetic interaction and $\delta\omega_2$ the spread in frequency due to all disturbances. The two effects can roughly be taken into account by writing the cross sections as

$$\sigma = \lambdabar^2/\tau\delta\omega_1 \qquad (\delta\omega_1 > \delta\omega_2)$$

$$\sigma = \lambdabar^2/\tau\delta\omega_2 \qquad (\delta\omega_2 > \delta\omega_1) . \tag{11.2}$$

τ is the lifetime of the excited state as calculated from the gravitational interaction only. For a quadrupole transition

$$1/\tau = \chi^2 \omega^5 d^4 M^2 \tag{11.3}$$

where $\chi^2 \cdot 2 \cdot 8\pi G/c^4 = 4.14 \times 10^{-48}\ \text{g}^{-1}\ \text{cm}^{-1}\ \text{s}^2$. G is the gravitational constant, M is the mass of the radiating particle and d is a length of the order of magnitude of the orbit of the particle,

$$\omega = kc \tag{11.4}$$

and

$$\delta\omega_1 = \omega^5 d^4 e^2 \ ; \qquad \delta\omega_2 = \Delta\omega \ , \tag{11.5}$$

where Δ is a measure of the frequency spread. Also the mean free path is given by

$$l = 1/\sigma N \ , \tag{11.6}$$

where N is the number of microscopic systems/cm^3. By using the expression for $1/\tau$, one obtains

$$\bar{l} = \frac{e^2}{\chi^2 M^2} \cdot \frac{1}{N\lambdabar^2} \quad \text{when} \quad \left(\frac{\lambdabar}{d}\right)^4 < \frac{e^2}{\Delta} \tag{11.7}$$

and

$$\bar{l} = \frac{\Delta}{\chi^2} \frac{\lambda\!\!\!^{-2}}{Nd^4 M^2} \quad \text{when} \quad \left(\frac{\lambda\!\!\!^{-}}{d}\right)^4 > \frac{e^2}{\Delta} . \tag{11.8}$$

The mean free path $\bar{l}$, the characteristic length of the gaser, is defined as the distance over which the number of gravitons double. The minimum of l is given when

$$\lambda\!\!\!^{-}_{\text{min}} = d(e^2/\Delta)^{1/4} . \tag{11.9}$$

This gives

$$\bar{l}_{\text{min}} = e\sqrt{\Delta}/(k^2 M^2 N d^2) . \tag{11.10}$$

In natural units for nuclei $e \approx 10^{-1}$, $M \approx 10^{-13}$ cm^{-1}, $N \approx 10^{23}$ cm^{-3}, $d \approx 10^{-12}$ cm, $\chi \approx 10^{-32}$ cm. Then

$$\bar{l}_{\text{min}} \approx 10^{38}\sqrt{\Delta}\ \text{cm} . \tag{11.11}$$

Such a large number excludes of course nuclei as the source of radiation in a gaser. The smallest value for Δ is in fact similar to that of the Mössbauer experiment ($\Delta \approx 10^{-14}$).

Therefore (11.10) suggests that larger systems than nuclei should be tried as sources. If big molecules or systems comparable to our microscopic systems are used, the wavelength of the radiation is of the order of that for vibrational transitions (Schiff [71]),

$$\lambda\!\!\!^{-} \approx (mM)^{1/2}\, d^2 \tag{11.12}$$

where m is the electron mass. Also

$$M \approx \rho d^3 , \quad \rho \approx 10^{36}\ \text{cm}^{-4}\ (\text{corresponding to 1 g cm}^{-3}) . \tag{11.13}$$

Therefore:

$$\lambda\!\!\!^{-} \approx \sqrt{\rho m d^7} . \tag{11.14}$$

Comparison of (11.14) and (11.9) gives:

$$\lambda\!\!\!^{-} > \lambda\!\!\!^{-}_{\text{min}} \quad \text{if} \quad d \geqslant (e/m\rho)^{1/5}\,(1/\Delta)^{1/10} . \tag{11.15}$$

If $\Delta \approx 10^{-14}$, $\rho \approx 10^{36}$ cm^{-4}

$$\lambda\!\!\!^{-} > \lambda\!\!\!^{-}_{\text{min}} \quad \text{if} \quad d \geqslant 10^{-8}\ \text{cm} . \tag{11.16}$$

The above condition for d is fulfilled for all molecules.

The condition

$$\bar{l} = \frac{\Delta}{\chi^2} \frac{\lambda\!\!\!^{-2}}{Nd^4 M^2} \quad \text{if} \quad (d/\lambda\!\!\!^{-})^{-4} > \frac{e^2}{\Delta} \tag{11.17}$$

can therefore be used.

Insertion of (11.14) and the approximate relation $N \approx d^{-3}$ into (11.17) gives

$$\bar{l} \approx \frac{m}{\chi^2} \frac{\Delta}{\rho} \approx 10^{38}\ \Delta\ \text{cm (for molecules)} . \tag{11.18}$$

This excludes also vibrational and rotational transitions in molecules as sources for the radiation in a gaser. For molecules the attainable Δ, is of course, much greater than it is for nuclei.

11.2. Super radiating states

Nagibarov and Kopvillem [57] discuss the possibility of creating a super radiating gravitational state (SGS) from quantum systems that already are in a super non-radiating electromagnetic state (NES). The ratio of gravitational and electromagnetic power $I^{(g)}$ and $I^{(e)}$ radiated by free particles in free space in interatomic and internuclear transitions are respectively

$$\eta = I^g/I^e \sim 10^{-42} \qquad \text{(interatomic transitions)} \tag{11.19}$$

$$\eta = I^g/I^e \sim 10^{-36} \qquad \text{(internuclear transitions)}\,. \tag{11.20}$$

It is shown by Dicke [94] that the use of lasers increases η by a factor $N_c\lambda^3 Q$ as a result of a decrease in the spontaneous electromagnetic radiation in the resonator; and by a factor $\tau_p\tau^{-1}N$ as a result of retaining the photons in the resonator where N_c is the number of resonator modes, Q is the resonator figure of merit, λ is the photon wavelength, τ_p and τ are the photon lifetime in resonator and duration of particle phosphorescence in free space and N is the total number of particles. Therefore η could be increased by $10^{15}-10^{20}$ times. In addition, $I^{(g)}$ increases by about N times because of the coherence of the gravitational waves. A further increase in η can be obtained by using the directional property of radiation from large systems. The intensity of the coherent part of the spontaneous emission of such systems is

$$I^g = \int I_0^g(\boldsymbol{K}) \sum_{p\neq q} \left[\exp\left\{\mathrm{i}\left(\boldsymbol{k} - \sum_{\zeta,\theta} a_{\zeta\theta}\, k_{\zeta\theta}\right)\cdot \boldsymbol{r}_p\right\} \exp\left\{-\mathrm{i}\left(\boldsymbol{k} - \sum_{\zeta,\theta} a_{\zeta\theta}\, k_{\zeta\theta}\right)\cdot \boldsymbol{r}_q\right\}\right] \mathrm{d}\Omega \tag{11.21}$$

where $I_0^{(g)}(K)$ is the spontaneous emission intensity in a unit solid angle $\mathrm{d}\Omega$ in the direction of the wave vector $\boldsymbol{K}$; $\boldsymbol{r}_p$ and $\boldsymbol{r}_q$ are the radius vectors of the radiating particles, $a_{\zeta\theta}$ are ≈ 1, and $\boldsymbol{k}_{\zeta\theta}$ are the wave vectors of the exciting electromagnetic pulses (ζ is the serial number of pulse sequence, and θ numbers the wave vectors within the limits of one pulse in the case of multi quantum excitation). To satisfy the equality $\boldsymbol{K} = \Sigma_{\zeta,\theta}\, a_{\zeta\theta}\, \boldsymbol{k}_{\zeta\theta}$ in the exponential of (11.21), two quantum excitation has to be used.

It is possible to excite the systems simultaneously by two laser pulses with:

$$\boldsymbol{k}_{11}(\mathrm{e}) = k_{11}^{\mathrm{e}}\boldsymbol{n}_0\,, \qquad \boldsymbol{k}_{12}(\mathrm{e}) = -k_{12}^{\mathrm{e}}\boldsymbol{n}_0 \tag{11.22}$$

with carrier frequencies

$$\nu_{11} = \frac{\nu_0}{2(1+1/\sqrt{\mathrm{e}})}\,, \qquad \nu_{12} = \frac{\nu_0}{2(1-1/\sqrt{\mathrm{e}})}\,. \tag{11.23}$$

Two quantum electromagnetic excitation is physically equivalent to a certain effective frequency

$$\nu_0 = \nu_{11} + \nu_{12}\,, \qquad k\boldsymbol{n}_0 = \boldsymbol{k}^g\,. \tag{11.24}$$

From (11.21) it follows that

$$I^{(i)} \sim I_0^{(i)} k^{(i)} \nu_0 N^2 (1/L\nu)^2 \sin^2\phi \tag{11.25}$$

$$\phi \sim \nu_0^{-1} E_{11} E_{12} \mu^2 \Delta t \tag{11.26}$$

where E_{11} and E_{12} are the electric field intensities of the electromagnetic wave, μ the dipole matrix element, Δt the duration of the exciting pulse, and L the length of the sample in the $\boldsymbol{n}_0$ direction.

The equation (11.25) is similar in form to (11.21), but is obtained taking into account the difference between photon and graviton velocities in matter with the super radiating electromagnetic state (SES) completely excluded, thus increasing η further by

$$N(1/L\nu_0)^2 \text{ times (for } N = 10^{19}, L = 1 \text{ cm}, \nu_0 = 310 \text{ sec}^{-1}), \qquad \eta = 10^9 \text{ times}. \tag{11.27}$$

There is no electromagnetic radiation at frequency ν_0 at all in the $\boldsymbol{n}_0$ direction.

The occurrence of super radiating electromagnetic generation is avoided by the excitation of gravitational echoes in the two quantum mode. The problem of reception has also been discussed by Nagibarov and Kopvillem. It is best to receive pulses of coherent gravitational waves with the aid of analogous quantum systems. An SES is first excited which decays after a time $T_2^* \ll T_2$ as a result of the inhomogeneities of the local fields. In the same way a system is obtained in which energy and information on the phase and on the wave vectors of the exciting pulse are stored and this makes directional reception possible. At an instant of time $T_2^* < t < T_2$ this system is subject to pulses of gravitational and laser beams, in which the directions of the wave vectors and the frequencies are such that an SES is produced in the system. After a time $2t$ the system begins to generate, in the required direction, coherent electromagnetic waves. In such a method of reception, the gravitational pulse serves as the means that permits at the required instant of time and in the required direction, the release of the energy stored in the receiver. Though there is a difference in the angular distributions of the radiation intensities of the mass and electric quadrupoles, it is still possible to excite high frequency radiation by means of an alternating electric field. In regular lattices containing excited nuclei, SES and NES can spontaneously occur for γ quanta and consequently SGS can occur at the frequency of the γ quanta relative to the nucleonic mass quadrupoles. The mass quadrupole can be excited by means of an electric dipole transition, making it possible to let the mass quadrupole to emit at maximum intensity.

11.3. Continuous generation of a gravitational beam

Nagibarov and Kopvillem [58] have considered the use of superradiant states of quantum systems for the generation of coherent gravitational continuous waves in the optical region. It is the graviton analog of stimulated optical and acoustic emission.

It is well known that in order to generate gravitational waves it is necessary to excite oscillations of mass quadrupole moments. The intensity of the radiation of the gravitational waves by an isolated quadrupole is proportional to the sixth power of the circular frequency of the oscillations

$$I_g \sim \omega_g^6 . \tag{11.28}$$

Excitation of oscillations of electric multipoles by an external electromagnetic field leads also to oscillations of the mass multipoles. Therefore, lasers can be used for the excitation of mass quadrupoles in the stimulated emission regime. To excite gravitational waves in the range $10^{12} \leqslant \omega_g \leqslant 10^{14}$ rad/sec, the rotational and vibrational levels of the molecule are used. In the range

$10^{14} \leqslant \omega_g \leqslant 10^{17}$ rad/sec the electronic levels of atoms and molecules are used and in the γ-ray frequencies internuclear transitions are used.

The generation of gravitational radiation, in the considered range of frequencies ω_g, is effected by a system of $N \gg 1$ mass quadrupoles, the linear dimensions of the system being $\gg \lambda_g$ (wavelength of the gravitons). Due to interference effects;

$$I_g \sim \lambda_g^2 \omega_g^6 . \tag{11.29}$$

Since for the reception of GR it is necessary to record at least one quantum, the efficiency of the discussed method of generation of reception of GR increases with frequency like ω_g^3.

In media in which n (the refractive index for e.m. waves) > 1, upon excitation of GR in the single quantum regime

$$I_g \sim \lambda_g^4 \omega_g^6 \tag{11.30}$$

because of the difference between the wave vectors k_e of the photons and k_g of the gravitons with the same frequency. To avoid an additional decrease of I_g by a factor λ_g^2, it is necessary to excite the emission of GR in the two quantum regime by using two lasers, which gives

$$k_e^{\text{effec}} = k_g . \tag{11.31}$$

The above equality in single quantum excitations can apparently also be attained in inverted systems. In dense gases, $n = 1$, $k_e = k_g$.

A serious obstacle to the experimental realization of GR generation is the powerful e.m. radiation which necessarily accompanies GR.

To decrease the intensity I_{em} of the electromagnetic radiation the following prescriptions are useful.

1) The system should be excited with the help of two lasers for which effectively $k_e = k_g$; then the intensity of e.m. radiation at frequency ω_g decreases by a factor λ^2 compared with the case of single quantum excitations.

2) One should excite systems in electromagnetic resonators which have no mode at frequency ω_g, at least in the direction of k_g; this in general suppresses the coherent spontaneous e.m. radiation with $k_e = k_g$.

3) The direction of effective $k_e = k_g$ should coincide with one of the minima of the intensity of the spontaneous emission of the isolated electric quadrupole. The method for the reception of GR is thè inverse of that proposed above. A quantum statistical theory can also be developed for the generation of GR with the aid of lasers at the electronic levels of atoms and molecules, at the rotational and vibrational levels and at the Landau levels. I_g are listed in table 11.1. The parameter of the receivers and the intensities of the light ray at the receiver output are listed in table 11.2.

Table 11.1.
GR generation power in the stimulated induction regime (from Nagibarov and Kopvillem [58]).

Nature of levels $\|1\rangle$ and $\|2\rangle$	ω_g (rad/sec)	m_0 (g)	Q_0 (cm^2)	V (cm^3)	$V^{-1}N$ (cm^{-3})	S (cm^2)	$I_{coh}(\omega_g)/\sin^2\theta$ (erg/sec)
	10^{16}	10^{-27}	10^{-16}	10^6	10^{21}	10^3	10^{-12}
	10^{13}	10^{-22}	10^{-15}	10^6	10^{23}	10^3	10^{-10}

Table 11.2
Power at the receiver output in the derection of a gravitational ray in various reception methods (from Nagibarov and Kopvillem [58].

Methods of GR reception	Nature of levels $\|1\rangle$ and $\|2\rangle$	Receiver parameters							Power at receiver output (erg/sec)
		ω_g	G (erg) of η,ζ,S,t	$v^0_{S,t}$	ω_{ph}	$\omega_\zeta-\omega_{ph}+i\Gamma_\zeta$	$\Delta\omega$	T_2^*	
1) With photon pumping	Π_{el}	10^{16}	10^{-10}	10^{-5}	10^{11}	10^7	10^7	10^{-6}	10^{-13}
	Π_{rv}	10^{13}	10^{-10}	10^{-3}	10^8	10^7	10^7	10^{-6}	10^{-11}
2) With photon pumping	Π_{el}	10^{16}	10^{16}	10^7		10^{-18}			10^{-7}
	Π_{rv}	10^{13}	10^{13}	10^7		10^{-18}			10^{-3}
		ω^2	$\Delta\omega_2$	T_2^*	$\omega_\alpha-\omega_{ph}$	$\omega_\alpha-\omega_{pr}$	$I_{coh}(\omega_{ph})$		
3) Phonon graviton	Π_{el}	5×10^{15}	10^9	10^{-8}	10^5	10^{15}	10^{-18}		10^{-3}
	Π_{rv}	5×10^{12}	10^9	10^{-8}	10^2	10^{12}	10^{-18}		10^5

11.4. Stimulated generation of coherent gravitational radiation

A system of N identical particles with a discrete spectrum is considered. $|1\rangle$ is the ground state of the isolated particle. $|2\rangle$ is one of its excited states with energy $E_2 - E_1 = w_0$. The unperturbed spectrum of such a subsystem is described by the operator

$$\mathcal{H}_0 = w_0 \sum_{j=1}^{N} R_3^j \, . \tag{11.32}$$

Its interaction with the external generators with effective wave vector k and effective frequency ω can be represented in the form:

$$\mathcal{H}_r = A_k R_{k_+} + A_k^+ R_{k_-} \tag{11.33}$$

$$R_{k_\pm} = \sum_j R_\pm^j e^{\pm i k r_j} \, , \quad R_\pm^j = R_1^j + iR_2^j \, ; \quad [R_+^j, R_-^j] = 2R_3^j \tag{11.34}$$

$$[R_3^j, R_+^j]_- = R_+^j \, , \quad [R_3^j, R_-^j] = -R_-^j \, . \tag{11.35}$$

R_d^j is the component of the operator of the effective spin $R = \frac{1}{2}$, r_j is the radius vector of particle j. A_k is an operator describing the interactions with the external generator.

The intensity of the stimulated gravitational emission (SGI) is given by:

$$I(w_g) = \int I(k_g)\, d\Omega \tag{11.36}$$

$$I(k_g) = I_0(k_g) \frac{N}{2}\left[1 - \cos\theta_1 \tanh \frac{w_g}{2k_B T} + \frac{\sin^2\theta}{2} \tanh^2\left(\frac{w_g}{2k_B T}\right) N^{-1} \sum_{j\neq 1}^{N} \exp\{i(k_g - k_1 - k_2)\cdot r_{jl}\}\right] \tag{11.37}$$

where $d\Omega$ is the solid angle element, $r_{jl} = r_j - r_l$, k_B is Boltzmann's constant, T is the temperature,

$$\cos\theta_1 = (1 + (T_2\Delta w)^2 + w_r^2 T_1 T_2)^{-1}\ (1 + (T_2\Delta w)^2) \tag{11.38}$$

$$\sin\theta_2 = (1 + (T_2\Delta w)^2 + w_r^2 T_1 T_2)^{-1}\, |w_r|\, T_2^2\,\Delta w \tag{11.39}$$

$$\Delta w = w_0 - |w_g|\ ; \qquad |w_g| = |w_1 \pm w_2| \tag{11.40}$$

$$w_r = |\langle 1|\mathcal{H}_r^j|2\rangle|\ , \qquad \mathcal{H}_r = \sum_j \mathcal{H}_r^j \tag{11.41}$$

$$I_0(k_g) = \frac{k_1 w_g^6}{4\pi c^5}\left[\tfrac{1}{4}\left|\sum_{\mu,\delta}\langle 1|D_{\mu\delta}^{m}|2\rangle\,\eta_\mu\eta_\delta\right|\left[2 + \tfrac{1}{2}\sum_{\mu,\delta}\left|\langle 1|D_{\mu\delta}^{m}|2\rangle\right|^2\right]\right.$$

$$\left. - \tfrac{1}{2}\sum_{\mu,\delta,\chi}\{\langle 1|D_{\mu\delta}^{m}|2\rangle\langle 2|D_{\mu\chi}^{m}|1\rangle + \langle 2|D_{\mu\delta}^{m}|1\rangle\langle 1|D_{\mu\chi}^{m}|2\rangle\,\eta_\delta\eta_\chi\}\right] \tag{11.42}$$

$$D_{xx}^{m} = \sqrt{\tfrac{1}{6}}(D_2^{m} + D_{-2}^{m}) - \tfrac{1}{3}D_0^{m}\ , \qquad D_{zz}^{m} = \tfrac{2}{3}D_0^{m} \tag{11.43}$$

$$D_{yy}^{m} = -\sqrt{\tfrac{1}{6}}(D_2^{m} + D_{-2}^{m}) - \tfrac{1}{3}D_0^{m}\ ; \qquad D_{xy}^{m} = D_{yx}^{m} = \frac{-\mathrm{i}}{\sqrt{6}}(D_2^{m} - D_{-2}^{m}) \tag{11.44}$$

$$D_{yz}^{m} = D_{zy}^{m} = \frac{\mathrm{i}}{\sqrt{6}}(D_1^{m} + D_{-1}^{m})\ , \qquad D_{xz}^{m} = D_{zx}^{m} = \frac{1}{\sqrt{6}}(D_{-1}^{m} - D_1^{m})$$

$$D_0^{m} = \tfrac{1}{2}\alpha[3L_z^{j2} - L^j(L^j+1)]\ , \qquad \alpha_0 = m_0 Q_0 (L^j(2L^j-1))^{-1} \tag{11.45}$$

$$m_0 Q_0 = \langle L^j L^j|\int\rho_j(3z'^2 - r'^2)\,\mathrm{d}v'|L^j L^j\rangle\ , \qquad L_\pm^j = L_x^j + \mathrm{i}L_y^j\ ; \tag{11.46}$$

k_1 is the gravitational constant; $\mu, \delta, \chi = x, y, z$; η_μ are the direction cosines of the vector k_g, $\mathrm{d}v' = \mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}z'$, ρ_g is the density of the electron mass at point x', y', z' of the electron shell of the particle j, m_0 the electron mass, L_ξ^j are the components of the effective orbital motion of the particle.

In (11.37) the coherent part of the gravitational radiation describes the factor containing

$$\sum_{j\neq l}\exp\{\mathrm{i}(\boldsymbol{k}_g - \boldsymbol{k}_1 - \boldsymbol{k}_2)\,r_{jl}\}\ .$$

At $\boldsymbol{k}_g = \boldsymbol{k}_1 + \boldsymbol{k}_2$, this function has a sharp maximum. To obtain the maximum possible $I(w_g)$ it is necessary that this direction coincides with one of the maxima of $I_0(k_g)$. The power of the coherent part of the SGI is given by

$$I_{coh}(w_g) = I_0(n_0 k_g)(\lambda_g^2 N^2/4S)\sin^2\theta_2\ \tanh^2(w_g/2k_B T) \tag{11.47}$$

$4S$ is the area of the face perpendicular to $\boldsymbol{k}_1 + \boldsymbol{k}_2 = \boldsymbol{n}_0\ |\boldsymbol{k}_1 + \boldsymbol{k}_2|$, $S/\lambda_g l_0 \gg 1$ (l_0 = length of sample, λ_g = wavelength of the generated gravitons, n_0 coincides with one of the maxima of $I(k_g)$),

$$w_1 = (w_g/2n)(n+1)\ ; \qquad w_2 = (w_g/2n)(n-1) \tag{11.48}$$

$$(\sin\theta_2)_{\max} = |w_r| T_2 (1 + w_r^2 T_1 T_2)^{-1/2} \text{ (when } \Delta w = (1 + w_r^2 T_1 T_2)^{1/2}/T_2) . \tag{11.49}$$

If the generator power is such that

$$|w_r| \gg 1/\sqrt{T_1 T_2}, \qquad (\sin\theta_2)_{\max} = \sqrt{T_2/T_1} . \tag{11.50}$$

The transition $|1\rangle \to |2\rangle$ is realized as a result of absorption of two quanta, when

$$w_1 + w_2 = w_e . \tag{11.51}$$

Then it follows that

$$|w_r| = \left| \sum_\alpha \frac{\langle 2|\mathcal{H}_1^j|\alpha\rangle\langle\alpha|\mathcal{H}_2^j|1\rangle}{w_\alpha - w_2 + i\Gamma_\alpha} + \frac{\langle 2|\mathcal{H}_2^j|\alpha\rangle\langle\alpha|\mathcal{H}_1^j|1\rangle}{w_\alpha - w_1 + i\Gamma_\alpha} \right| . \tag{11.52}$$

$\mathcal{H}_1$ and $\mathcal{H}_2$ are respectively the Hamiltonians of the interactions of the first and second electromagnetic generators with impurity j, and Γ_α is the width of level α.

If the transition $|1\rangle \to |2\rangle$ is a result of absorption of a quantum w_1 and creation of a quantum w_2 $(w_g = w_1 - w_2)$, then

$$|w_r| = \left| \sum_\alpha \frac{\langle 1|\mathcal{H}_1^j|\alpha\rangle\langle\alpha|\mathcal{H}_2^j|2\rangle}{w_\alpha + w_2 + i\Gamma_\alpha} + \frac{\langle 2|\mathcal{H}_2^j|\alpha\rangle\langle\alpha|\mathcal{H}_1^j|1\rangle}{w_\alpha - w_1 + i\Gamma_\alpha} \right| . \tag{11.53}$$

Also in this case

$$\sin\tilde{\alpha}/\sin\beta = (w_1 - w_g)/w_1 . \tag{11.54}$$

$\tilde{\alpha}$ is the angle between $\boldsymbol{n}_0$ and $\boldsymbol{k}_1$; β the angle between $\boldsymbol{n}_0$ and $\boldsymbol{k}_2$

$$\beta = \cos^{-1}\left[\frac{w_g}{2(w_1 - w_g)} + \frac{n}{2w_g}(w_1 + w_g)\right] \tag{11.55}$$

$$\frac{w_2}{w_1} = \tan\tilde{\alpha} , \qquad \tilde{\alpha} = \sin^{-1}\left(\frac{w_1 - w_T}{w_1}\sin\beta\right) . \tag{11.56}$$

If the GR is excited by a system with an equidistant spectrum and an effective spin $R > \frac{1}{2}$

$$I_{\mathrm{coh}}(w_g) = 4 I_0(\boldsymbol{k}_g)\frac{\lambda_g^2}{9} R^2 (R+1)^2 \frac{N^2}{S}\sin^2\theta_2 \tanh^2\left(\frac{w_g}{2k_B T}\right) . \tag{11.57}$$

When $R = \frac{1}{2}$, (11.47) is obtained.

For the case of GR generation using Landau levels

$$\sin^2\theta_2 = \left(\frac{P_0 E}{2\pi} T_2^2 \Delta w\right)^2 \left(1 + (T_2\Delta w)^2 + \left(\frac{P_0 E}{2\pi}\right)^2 T_2^2\right)^{-2} , \qquad P_0 = (4\pi c e/H_z)^{1/2} . \tag{11.58}$$

e is the electron charge and H_z is the Z component of the constant external magnetic field. The matrix elements of the quadrupole moment between the states whose Landau quantum numbers n, differ by 1, are

$$\approx \frac{2}{eH_z}\{(n_1+1)(n_1+l+1)\}^{1/2} \tag{11.59}$$

where l is the orbital quantum number and E the intensity of the alternating electric field.

The Brans–Dicke theory [13] predicts that the pulsations of the mass density should lead to generation of scalar gravitational waves. For a system of N particles pulsating coherently with frequency w_g,

$$I_{\mathrm{coh}}(w_g) = \frac{4k_1}{9}\,\frac{\chi|A|^2 w_g^6}{1+6x}\,\frac{\lambda_g^2 N^2}{4S}\sin^2\theta_2\,\tanh^2(w_g/2k_B T) \tag{11.60}$$

$$k_g = k_1 + k_2\;;\quad k_2 = k_1(1+4x)/(1+6x) \tag{11.61}$$

$$A = \langle 1|\int \mathrm{d}v'(r'^2)\,\rho_g|2\rangle\,. \tag{11.62}$$

k_2 is the gravitational constant with allowance for the scalar field. k can be either >0 or <0, or $|k| \sim 1$ (Sexl [72], Dicke and Goldenberg [34]). The oscillations of the quantity A can also be excited by lasers through oscillations of the charge density of the electron cloud, of the nuclei in the molecule, or of the nucleons in the nucleus.

Nagibarov and Kopvillem discuss also methods of detecting rays of coherent GR.

11.5. Lattice vibrations in solids

The quantized model of a simple cubic lattice serves to study the gravitational radiation resulting from lattice vibrations. The interaction of electromagnetic radiation with solid proves the existence of optical transitions and the Brillouin effect. In both cases an extended crystal contributes coherently to the emission. The gravitational analog of these processes were not considered in the works of Mironowsky [54] and Weber [82]. Mironowsky apparently did not take into account the conservation of crystal momentum, which applies even to a continuum as the limiting case of a crystal lattice, and his result has to be modified.

Halpern and Desbrandes [35] have considered a simple model of a crystal lattice and investigated its gravitational radiation. The lattice is quantized but the gravitational field remains classical. The work is closely patterned on that of Halpern and Laurent [36]. The emitting bodies have extensions that are large compared with the wavelength and hence the multipole approximation cannot be applied. The gravitational radiation emitted by the quantum transitions of solids has been considered by Halpern in the semi classical, linear approximation and the procedure used is analogous to that of the transitions of microscopic systems discussed by Halpern and Laurent [36].

The linearized theory relates the classical gravitational field $G^{\alpha\mu}$ to the energy momentum density of matter $T^{\alpha\mu}(x)$ as;

$$\Box G^{\alpha\mu} = 16\pi G T^{\alpha\mu}\,,$$

where

$$G^{\alpha\mu} = \sqrt{-g}\,g^{\alpha\mu}\,,\quad g = \det(g_{\alpha\mu})\,. \tag{11.63}$$

When $r = |\boldsymbol{x} - \boldsymbol{x}'| \gg \lambda$, the wavelength of the GR, as well as the extension of the source, then terms of higher powers in $1/r$ can be neglected and the following result is obtained,

$$G^{\alpha\mu}(x) = \eta^{\alpha\mu} + 2G\int \frac{x'_\alpha x'_\mu}{r} T^{00}_{R,00}(x')\,\mathrm{d}^3x' . \tag{11.64}$$

T^{00}_{R} denotes the value of $T^{00}(x')$ retarded with respect to the point x'. Therefore the quadrupole approximation can be replaced by the retarded quadrupole approximation.

There are N atoms per periodocity interval for the lattice. For the nth particle the mean displacement is

$$\bar{x}(n) = \bar{R}_n + \bar{u}(n) . \tag{11.65}$$

Assuming central forces,

$$V(\bar{x}_1 \dots \bar{x}_N) = \tfrac{1}{2} \sum_{n,m} U_{\alpha\mu}(\bar{R}_n - \bar{R}_m)\, u^\alpha(n)\, u^\mu(m) ,$$
$$U_{\alpha\mu}(\bar{R}_m - \bar{R}_\mu) = -F_\mu(n) , \tag{11.66}$$

where $F_\mu(n)$ is the μ component of the force acting on the nth particle as a result of a unit displacement from equilibrium position of the mth particle in the α direction. The equation of motion of the system is given by

$$m\ddot{u}^\alpha(n) = -\sum_m U_{\alpha\mu}(R_n - R_m)\, u^\mu(m) . \tag{11.67}$$

The solutions of (11.67) constitute lattice waves.

The angular frequencies of the normal modes of the system are given by

$$w(\lambda) = w_0 \sin(a\pi/\lambda) , \qquad w_0 = \sqrt{4\gamma/m} ,$$

a is the lattice constant,

$$U_{\alpha\alpha'}(c\mu/t) = -\gamma\delta_{\alpha\alpha'}\delta_{\mu\alpha} . \tag{11.68}$$

μ/t ($\mu = 1, 2, 3$) are three mutually perpendicular vectors parallel to an equal in length to each one of the edges of an elementary cube of the lattice and $c = \pm 1$. The gravitational waves propagate even in the crystal with a velocity almost equal to that of light because their interaction with matter is extremely weak.

The Hamiltonian for the quantized system can be expressed in terms of Hermitian operators a_λ and a^+_λ as:

$$H = \sum_{\bar{k},\lambda} w_\lambda(\bar{k})\,\{a^+_\lambda(\bar{k})\, a_\lambda(\bar{k}) + \tfrac{1}{2}\}\,(|k| = |\bar{p} + \bar{p}'| = w + w' = w'') . \tag{11.69}$$

A normalized state of n quanta (phonons) of polarization λ and momentum $\bar{p}$ as well as n' phonons of polarization λ' and momentum $\bar{p}'$ is represented by

$$(n!\, n'!)^{-1/2} (a^+_\lambda(\bar{p}))^n\, (a^+_{\lambda'}(p))^{n'} |0\rangle = |n(\lambda\bar{p})\, n'(\lambda'\bar{p})\rangle , \tag{11.70}$$

where $|0\rangle$ is the state of zero phonons.

The contribution due to the rest masses is $m(\Sigma_n\, x_\alpha(n) x_\mu(n)/\boldsymbol{r})_{R,00}$ and the matrix element then becomes

$$\langle\langle (n-1)(\lambda\bar{p})(n'-1)(\lambda'\bar{p}') \left| \sum_n \frac{u_\alpha(n)u_\mu(n)}{r} \right| n(\lambda,\bar{p})n'(\lambda'\bar{p}')\rangle\rangle \,. \tag{11.71}$$

$x_\alpha(n)$ can be replaced by $\bar{u}_\alpha(n)$ in (11.71) because only the displacements $\bar{u}$ contribute to (11.65) after the time derivatives are taken. Also $1/r$ can be replaced by R the constant distance between the field point and the centre of the crystal. Therefore (11.71) becomes

$$\frac{1}{2R}\,(nn')^{1/2}\,(w(\bar{p})w'(p'))^{-1/2}\,\mathcal{E}^{\alpha}_{\lambda}\mathcal{E}^{\mu}_{\lambda'}\sum_n \exp\{\mathrm{i}(\bar{p}+\bar{p}')\cdot\bar{R}_n - \mathrm{i}(w+w')t\}\,. \tag{11.72}$$

The crystal momentum is conserved for the factor $\exp\{\mathrm{i}(\bar{p}+\bar{p}')\cdot R\}$ compensates the retardation effect. The whole crystal can contribute to the emission. The flux density in the neighborhood of the Z axis is

$$t^3_0(x) \approx \frac{G(w+w')^6 nn'}{32\pi R^2 w'^2} \approx \frac{2GEE'w^2}{\pi R^2} \tag{11.73}$$

with $E = nw$, $E' = n'w' \approx n'w$.

The total rate of radiation emitted is

$$\approx R^2\psi^2 t^3_0 \approx GEE'\,w/L \approx G\rho\rho' wq^2 L \tag{11.74}$$

where $\rho = E/V$ is the energy density and $V = qL$ where L is the length of the crystal in Z direction. ψ is the solid angle and is $\approx\frac{1}{2}\pi\cdot w''L$. The above results are valid in the harmonic approximation.

12. Results of astrophysical interest

In this section are collected those results which are more interesting from the astrophysical point of view. They are mainly classified according to the particular astrophysical object to which they refer. It might be worth mentioning that the results obtained by Papini and Valluri [64–66] are obtained from application of the results obtained in section 9.3.

a) *The Sun*

The emission power of gravitons by photoproduction from the Sun in the infrared region has been calculated (Papini and Valluri [64–67]) to be 5×10^{12} erg/sec corresponding to a flux at the Earth of 1.7×10^{-2} gravitons/cm^2 sec. Such a value is subject to variation depending on the data available for the sizes of the chromosphere, the corona and the magnetic field. This result does not differ very much from the values 6×10^{14} erg/sec and 5×10^{15} erg/sec given by Weinberg and Carmeli as indicated by eqs. (5.42) and (5.50) respectively. Bremsstrahlung and photoproduction in the sun therefore constitute the strongest source of GR in our planetary system. Classical quadrupole radiation for the Jupiter Sun system follows in intensity with 7.6×10^{11} erg/sec in an entirely different frequency range. Quite in general, thermal collisions may provide the most important source of gravitational radiation in the universe.

b) *Quasars*

Estimates for emission power from bremsstrahlung are not yet available and might be difficult to make since quasars are still not well understood objects. Estimates for photoproduction by 3C273 in particular have been given by Papini and Valluri [67] for various regions of the spectrum. They depend on presently available data for the quasar's magnetic field and suffer from the same uncertainties. Again emission peaks at infrared frequencies with an upper limit of $\sim 5 \times 10^{32}$ erg/sec. The corresponding flux at Earth, assuming the distance of 3C273 to be 500 Mpc, is $\sim 3 \times 10^{-26}$ erg/cm^2 sec, indeed negligible, and much smaller than the fluxes of $\sim 10^{-12}$ erg/cm^2 sec due to broad band bursts produced by large explosions in quasar and galactic nuclei as conjectured by Press and Thorne [69]. A considerable amount of GR, perhaps $\sim 10^{28}$ erg/sec, could also be emitted by photoproduction in the gravitational field of a quasar (Boccaletti et al. [10]). This figure is considerably higher than the gravitational luminosity, $\sim 10^{25}$ erg/sec, of 10^9 supernovae (Vladimirov [79]). 3C273 in the above optical range would emit at the rate of $\sim 5.5 \times 10^{25}$ erg/sec (Papini and Valluri [67]). Since there may be one quasar every 100 galaxies, the quasar contribution to the overall graviton density in the universe may be non negligible.

c) *Seyfert galaxies and galactic centre*

Photoproduction in the infrared can account for up to $\sim 3 \times 10^{25}$ erg/sec for NGC1068 and $\sim 8 \times 10^{22}$ erg/sec for the galactic centre (Papini and Valluri [67]). The corresponding fluxes at Earth amount to $\sim 2 \times 10^{-27}$ erg/cm^2 sec and $\sim 2 \times 10^{-27}$ erg/cm^2 sec respectively, again smaller than the outbursts predicted by Press and Thorne [69]. Over long periods of time their contribution to the graviton background in the universe may be sizeable. Much higher is the estimate 10^{38} erg/sec given by Mironovskii [53,54] for gravitational luminosity of a galaxy due to the motion of its double stars. The gravitons in this case have extremely low frequencies. Graviton photoproduction in static external gravitational fields cannot explain the red shift of light as a "tired light" phenomenon. As shown in section 9.1, the results of Boccaletti et al. [8] indicate that even if the external gravitational field is that of a galaxy the cross section for the process is only 10^{-1} cm^2 and the energy loss of a photon is only $\sim 10^{-50}$ eV.

d) *Pulsars and neutron stars*

Unlike the classical problem that admits radiation only if the axis of the magnetic dipole moment of the star and its rotation axis are at an angle, in the quantum problem there can be emission of GR also with complete alignment. Estimates for NPO532 have been given by Papini and Valluri [65], for photo-production in the magnetosphere of the star, in different frequency ranges. They find $P \sim 3.6 \times 10^{20}$ erg/sec for radio frequencies, $P \sim 3 \times 10^{21}$ erg/sec in the optical range and 4.1×10^{25} erg/sec in the infrared region and estimates for X-ray and γ-ray regions are 6.2 $\times 10^{32}$ erg/sec and 4.1×10^{31} erg/sec respectively.

Shortly after birth pulsars possess external electric fields which are then gradually neutralized by the near plasma charge separation (Pacini [61]). For hot neutron stars the surface luminosity can be $\sim 10^{38}$ erg/sec (Tsuruta and Cameron [75]) and is peaked in the soft X-ray region. By assuming an electric field at the surface $\sim 10^{12}$ V/cm, one obtains

$$P \sim 2.8 \times 10^{21} \text{ erg/sec}.$$

Magnetic fields of intensity upto 10^{15} gauss can exist within neutron stars (Vandakurov [78]). The power emitted by a pulsar with $L_S \sim 10^{38}$ erg/sec is $P \sim 6.2 \times 10^{34}$ erg/sec and possibly higher values soon after birth and the radiation is peaked in the soft X-ray range. The power emitted via photoproduction can therefore exceed the quadrupole contribution for pulsars of ellipticity $< 10^{-3}$. It is interesting to speculate that photons produced in the interior of the star in the presence of strong magnetic fields may also be converted into gravitons that could then escape and therefore play a role in the cooling of the star. It also appears that the gravitational luminosity of a neutron star can be larger than both gravitational (10^{15} erg/sec) and electromagnetic (10^{33} erg/sec) luminosities of our sun. Comparable in efficiency with photoproduction is gravitational bremsstrahlung in neutron-neutron scattering. Boccaletti estimates infact $P \sim 10^{27}$ erg/sec at frequencies 10^{22} Hz for a canonical neutron star as indicated in eq. (5.56). One should therefore expect the presence in the universe of a background of GR distributed over the complete spectrum and due to the overall population of neutron stars. It is difficult at this stage to gain a perspective on the general graviton density in the universe. Attempts to estimate it have been made by Boccaletti et al. [10] taking into account various processes that can both produce and annihilate gravitons. For an universe with an age of 10^{18} sec they find that the situation is far from equilibrium and that the gravitons are still accumulating. They conclude that the mean density of gravitons in the universe may be $\sim 10^{-19}$ erg/cm^3, smaller therefore than the cosmological background of electromagnetic radiation ($\sim 10^{-14}$ erg/cm^3). This conclusion may however be premature not only because information as to the number, distribution and efficiency of strong sources in the universe is still missing, but also because the efficiency of the quantum linear processes so far studied can indeed be at least as high as that of the classical ones. Bremsstrahlung and photoproduction in particular may significantly increase the estimates given in the literature.

References

[1] A. Ashtekar and R. Geroch, Rep. Progr. Phys. 37 (1974) 1211.
[2] R.D. Amado, Phys. Rev. 132 (1963) 485.
[3] B.M. Barker, M.S. Bhatia and S.N. Gupta, Phys. Rev. 182 (1969) 1387.
[4] B.M. Barker, R.D. Haracz and S.N. Gupta, Phys. Rev. 149 (1966) 1027.
[5] B.M. Barker, R.D. Haracz and S.N. Gupta, Phys. Rev. 158 (1967) 1498.
[6] B.M. Barker, R.D. Haracz and S.N. Gupta, Phys. Rev. 162 (1967) 1750.
[7] A.M. Bincer, Phys. Rev. 118 (1960) 855.
[8] D. Boccaletti, V. De Dabbata, C. Gualdi and P. Fortini, N. Cim. A48 (1967) 58.
[9] D. Boccaletti, V. De Dabbata, C. Gualdi and P. Fortini, N. Cim. B60 (1969) 320.
[10] D. Boccaletti, V. De Sabbata, C. Gualdi and P. Fortini, N. Cim. B54 (1968) 134.
[11] D. Boccaletti, V. De Sabbata, C. Gualdi and P. Fortini, N. Cim. B11 (1972) 289.
[12] D. Boccaletti, Lett. N. Cim. 4 (1972) 927.
[13] C. Brans and R.H. Dicke, Phys. Rev. 124 (1961) 925.
[14] R.A. Breuer, Gravitational Perturbation Theory and Synchrotron Radiation (Springer-Verlag, 1975).
[15] M. Carmeli, Phys. Rev. 158 (1967) 1248.
[16] M. Davis, R. Ruffini, W.H. Press and R.H. Price, Phys. Rev. Lett. 27 (1971) 1966.
[17] S. Deser and P. van Nieuwenhuizen, Phys. Rev. Lett. 32 (1974) 245.
[18] S. Deser and P. van Nieuwenhuizen, Phys. Rev. D10 (1974) 401.
[19] S. Deser and P. van Nieuwenhuizen, Lett. N. Cim. 11 (1974) 218.
[20] S. Deser and P. van Nieuwenhuizen, Phys. Rev. D10 (1974) 411.
[21] B.S. DeWitt, Phys. Rev. 162 (1967) 1239.
[22] B.S. DeWitt, Phys. Reports 19C (1975) 295.
[23] S.D. Drell and H. Pagels, Phys. Rev. B140 (1965) 397.
[24] I. Duck, Nucl. Phys. B1 (1967) 96.
[25] F.J. Dyson, Phys. Rev. 75 (1949) 1736.
[26] L. Fadeev and V. Popov, Phys. Lett. 25B (1967) 29.

[27] M. Fierz, Helv. Phys. Acta 12 (1939) 3.
[28] M. Fierz and W. Pauli, Proc. Roy. Soc. 173 (1939) 211.
[29] V.A. Fock and D. Ivanenko, Compt. Rend. 188 (1929) 1470.
[30] G.M. Gandelman and V. Pinaev, Sov. Phys. JETP 10 (1960) 764.
[31] M.E. Gertsenshtein and W.I. Pustovoit, Sov. Phys. JETP 14 (1962) 84.
[32] D. Gross and R. Jackiw, Phys. Rev. 166 (1968) 1287.
[33] S.N. Gupta, Proc. Phys. Soc. 65A (1952) 161, 608.
[34] H.M. Goldenberg and R.H. Dicke, Phys. Rev. Lett. 18 (1967) 313
[35] L. Halpern and R. Desbrandes, Ann. Inst. Henri Poncaré 11 (1969) 309.
[36] L. Halpern and B. Laurent, N. Cim. 33 (1964) 728.
[37] L. Halpern, N. Cim. 25 (1962) 1239.
[38] M.G. Hare and G. Papini, Nucl. Phys. B34 (1971) 200.
[39] M.G. Hare and G. Papini, Phys. Rev. D4 (1971) 684.
[40] S.W. Hawking, Phys. Rev. Lett. 26 (1971) 1344.
[41] S.W. Hawking, Comm. Math. Phys. 43 (1975) 199.
[42] D. Ivanenko and A. Sokolov, Vestnik. Moscow State Univ. 8 (1947) 103.
[43] D. Ivanenko and A. Sokolov, Quantum Field Theory (1952) p. 678.
[44] D. Ivanenko, Theories Relativ. Gravit. (Royamont, 1959).
[45] D. Ivanenko and A. Brodsky, Acad. Nauk. 92 (1953) 731.
[46] R. Jackiw, Phys. Rev. 168 (1968) 1623.
[47] J.M. Jauch and R. Rohrlich, The theory of Photons and Electrons (Cambridge, Mass., 1955) p. 330
[48] I.Y. Kobzarev and L.B. Okun, Phys. JETP Lett. 16 (1963) 1343.
[49] M.P. Korkina, Ukr. Fiz. Zh. 5 (1960) 762.
[50] R. Karplus and M. Neumann, Phys. Rev. 80 (1950) 380.
[51] L.D. Landau and E.M. Lifshitz, The Classical Theory of Fields (Addison-Wesley, 1962) Sect. 66–71, 104.
[52] S. Mandelstam, Ann. of Phys. 19 (1962) 25.
[53] V.N. Mironowskii, Sov. Astron. 9 (1966) 752.
[54] V.N. Mironowskii, Astron. Zh. 42 (1965) 977.
[55] C.W. Misner, Phys. Rev. Lett. 28 (1972) 996.
[56] C.W. Misner, Bull. Am. Phys. Soc. 17 (1972) 472.
[57] V.R. Nagibarov and U.Kh. Kopvillem, Sov. Phys. JETP Lett. 5 (1967) 360.
[58] V.R. Nagibarov and U.Kh. Kopvillem, Sov. Phys. JETP 29 (1969) 112.
[59] V.R. Nagibarov and U.Kh. Kopvillem, Sov. Phys. JETP Lett. 2 (1965) 329.
[60] V.R. Nagibarov and U.Kh. Kopvillem, Izv. Vuzov. Fiz. 9 (1967) 66.
[61] F. Pacini, Riv. B. Cim. 2 (1972) 498.
[62] H. Pagels, Phys. Rev. B144 (1966) 1250, 1261, 1268.
[63] A. Papapetrou and E. Corinaldesi, Proc. Roy. Soc. A209 (1951) 248, 259.
[64] G. Papini and S.R. Valluri, Can. J. Phys. 53 (1975) 2306.
[65] G. Papini and S.R. Valluri, Can. J. Phys. 53 (1975) 2312.
[66] G. Papini and S.R. Valluri, Can. J. Phys. 53 (1975) 2315.
[67] G. Papini and S.R. Valluri, Can. J. Phys. 54 (1976) 76.
[68] R.G. Parson, Phys. Rev. 168 (1968) 1562.
[69] W.H. Press and K.S. Thorne, Ann. Rev. Astron. Astroph. 10 (1972) 344.
[70] L. Rosenfeld, Ann. of Phys. 5 (1930) 113.
[71] L.I. Schiff, Quantum Mechanics (N.Y., 1949) p. 289.
[72] R.U. Sexl, Phys. Lett. 20 (1966) 376.
[73] L. Spitzer Jr., Physics of Fully Ionized Gases (Interscience Publishers, 1956) Chap. 5.
[74] G. 't Hooft and M. Veltman, Ann. Inst. Herni Poincaré 20A (1974) 69.
[75] S. Tsuruta and A.G.W. Cameron, Can. J. Phys. 44 (1966) 1863.
[76] W.G. Unruh, Phys. Rev. D10 (1974) 3194.
[77] P. Van Nieuwenhuizen, M. Grossmann Meeting, ICTP, Trieste (1975).
[78] Y.N. Vandakurov, Ap. Lett. 5 (1970) 267.
[79] Y.S. Vladimirov, Sov. Phys. JETP 18 (1964) 176.
[80] Y.S. Vladimirov, Sov. Phys. JETP 16 (1963) 65.
[81] J.C. Ward, Phys. Rev. 77, 78 (1950) L293, L182.
[82] J. Weber, Relativity, Groups and Topology, eds. C. DeWitt and B.S. DeWitt (N.Y., 1964).
[83] J. Weber and H. Hinds, Phys. Rev. 128 (1962) 2414.

[84] J. Weber, General Relatively and Gravitational Waves (London, 1961).
[85] S. Weinberg, Phys. Rev. B133 (1964) 1318.
[86] S. Weinberg, Phys. Rev. B134 (1964) 882.
[87] S. Weinberg, Brandeis Lectures (1964) 409–48.
[88] S. Weinberg, Phys. Rev. B140 (1965) 516.
[89] J.A. Wheeler, Relativity Space Time and Geometrodynamics (Princeton, 1961).
[90] J.A. Wheeler and D. Brill, Rev. Mod. Phys. 29 (1957) 465.
[91] J.A. Wheeler and D. Brill, Rendiconti Scuola, Varenna, Corso No. 12 (1960).
[92] D.R. Yennie, H. Suura and S.C. Frautschi, Ann. of Phys. 13 (1961) 379.
[93] F.J. Zerilli, Phys. Rev. D2 (1970) 2141.
[94] R.H. Dicke, Phys. Rev. 93 (1954) 99.